Quiz 10 - Wednesday - Sections 3.5, 4.1

3.5 18) Want to find $x$ and $y$ that minimize $C$.

$C = \text{cost of the box}$

$C = (\text{cost of top + bottom}) + (\text{cost of sides})$

$= 2x^2 + 2x^2 + (1)(4xy)$

$\uparrow \quad \uparrow \quad \uparrow$

$\text{bottom} \quad \text{top} \quad \text{area of}$

$\text{side}$

$= 4x^2 + 4xy$

Constraint: \hspace{1cm} $250 = x^2y \quad \Rightarrow \quad y = \frac{250}{x^2}$

$C = 4x^2 + 4x \left( \frac{250}{x^2} \right)$

$= 4x^2 + \frac{1000}{x}$

\[ \begin{cases} 
C = 4x^2 + 1000x^{-1} \\
C' = 8x - 1000x^{-2} = 8x - \frac{1000}{x^2}
\end{cases} \]

Find critical numbers:

$C' = 8x - \frac{1000}{x^2} = 0$

$8x = \frac{1000}{x^2}$

$8x^3 = 1000 \Rightarrow x^3 = \frac{1000}{8} = 125$

$\Rightarrow \quad x = 5$

Are we done? No. We need to find $C$. 

$\text{MINIMIZE THIS}$
\[ C(5) = 4(5)^2 + \frac{1000}{5} = 100 + 200 = 300 \]

No. We need at least $300 to make the box.

Exponents:

\[ f(x) = b^x \quad b > 0, \ b \neq 1 \]
The natural exponential base

The natural exponential base is the number $e$ defined by

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$\approx 2.71828\ldots$$

**Compound interest**

- $P$ = principal
- $r$ = interest rate (annual)
- $P_0 = P$
- $P_k = \text{balance after } k \text{ years}$
- $P_1 = P(1+r)$
- $P_2 = P(1+r)(1+r) = P(1+r)^2$
- $P_3 = P(1+r)^3$
- \ldots
- $P_m = P(1+r)^m$
Say you compound quarterly.

\[ P_0 = P \]
\[ P_{\frac{1}{4}} = P \left(1 + \frac{r}{4}\right) \]
\[ P_{\frac{2}{4}} = P \left(1 + \frac{r}{4}\right)^2 \]
\[ P_{\frac{3}{4}} = P \left(1 + \frac{r}{4}\right)^3 \]
\[ P_1 = P \left(1 + \frac{r}{4}\right)^4 \]

\[ P = 1000 \]
\[ r = 0.06 \]

\[ P_1 = 1000 \left(1 + \frac{0.06}{4}\right)^4 \]
\[ = 1000 \left(1.0616778\ldots\right) \]
\[ = \$1061.36 \]

Monthly:

\[ P_0 = P \]
\[ P_{\frac{1}{12}} = P \left(1 + \frac{r}{12}\right) \]
\[ P_{\frac{2}{12}} = P \left(1 + \frac{r}{12}\right)^2 \]
\[ \vdots \]
\[ P_1 = P \left(1 + \frac{r}{12}\right)^{12} \]

\[ P_1 = 1000 \left(1 + \frac{0.06}{12}\right)^{12} \]
\[ = 1000 \left(1.0616778\ldots\right) \]
\[ = \$1061.68 \]
If we compound \( b \) times per year, then after \( t \) years the balance is:

\[
B(t) = P \left( 1 + \frac{r}{b} \right)^{bt}
\]

We see that if \( b \) is large, then \( B(t) \) gets larger as well.

What if we let \( b \to \infty \)?

\[
B(t) = P \left[ \left( 1 + \frac{r}{b} \right)^b \right]^{rt} \rightarrow \text{Say } r = 0.06
\]

\[
2.7101715 \ldots
2.7142155 \ldots
2.7174666 \ldots
2.718182 \ldots
\]

Continuous compounding:

\[
B(t) = Pe^{rt}
\]

\( e \)
4.2. Logarithmic Functions

If \( x \) is a positive number, then the logarithm of \( x \) to the base \( b (b > 0, b \neq 1) \), denoted \( \log_b x \), is the number \( y \) such that \( b^y = x \); that is,

\[
y = \log_b x \quad \text{if and only if} \quad b^y = x \quad \text{for} \quad x > 0
\]

Example
Evaluate \( \log_{10} 1,000 \).

\[
\log_{10} 1000 = y
\]
\[
y = 3
\]

Example
Solve the equation \( \log_4 x = \frac{1}{2} \).

\[
4^{\frac{1}{2}} = x
\]
\[
X = 2
\]
Properties of Logarithms

Let \( b(b > 0, b \neq 1) \) be any logarithmic base. Then,

\[ \log_b 1 = 0 \quad \text{and} \quad \log_b b = 1 \iff b^1 = b \]

and if \( u \) and \( v \) are any positive numbers, then

- **The equality rule:** \( \log_b u = \log_b v \) if and only if \( u = v \)
- **The product rule:** \( \log_b (uv) = \log_b u + \log_b v \)
- **The power rule:** \( \log_b u^r = r \log_b u \) for any real number \( r \)
- **The quotient rule:** \( \log_b \left( \frac{u}{v} \right) = \log_b u - \log_b v \)
- **The inversion rule:** \( \log_b b^u = u \)

\[ \left( \log_b b^u \right) = u \quad \log_b (b) = u \]
Properties of Logarithms

Example
Use logarithm rules to rewrite each of the following expressions in terms of $\log_3 2$ and $\log_3 5$.

a. $\log_3 270 = \log_3 (27 \cdot 10) = \log_3 (3 \cdot 3 \cdot 3 \cdot 2 \cdot 5)$
   
   $= \log_3 (3) + \log_3 (3) + \log_3 (3) + \log_3 (2) + \log_3 (5)$

b. $\log_3 \left( \frac{64}{125} \right) = 3 + \log_3 (2) + \log_3 (5)$
   
   $= \log_3 (64) - \log_3 (125)$

   $= \log_3 (2^6) - \log_3 (5^3)$

   $= 6 \log_3 (2) - 3 \log_3 (5)$
Properties of Logarithms

Example
Use logarithm rules to simplify each of the following expression.

a. \( \log_3(x^3y^{-4}) = \log_3(x^3) + \log_3(y^{-4}) \)
   \[ = 3 \log_3(x) - 4 \log_3(y) \]

b. \( \log_7(x^3\sqrt{1 - y^2}) \)
   \[ = \log_7(x^3) + \log_7((1 - y^2)^{1/2}) \]
   \[ = 3 \log_7(x) + \frac{1}{2} \log_7(1 - y^2) \]
   \[ = 3 \log_7(x) + \frac{1}{2} \log_7((1 - y)(1 + y)) \]
   \[ = 3 \log_7(x) + \frac{1}{2} \log_7(1 - y) + \frac{1}{2} \log_7(1 + y) \]

\[ \log_3(x^3y^{-4}) = 5 \]
\[ 3 \log_3(x) - 4 \log_3(y) = 5 \]
\[ 3 \log_3(x) - 5 = 4 \log_3(y) \]
\[ \frac{1}{4} (3 \log_3(x) - 5) = \log_3(y) \]
\[ y = 3^{(\frac{1}{4} (3 \log_3(x) - 5))} \]
\[ (\log_3(y) = t \iff 3^t = y) \]

\[ 3^5 = x^3y^{-4} \]
\[ y^{-4} = \frac{243}{x^3} \]
\[ \bullet 243 = x^3y^{-4} \]
\[ y^4 = \frac{243x^3}{243} \]
The Natural Logarithm

The logarithm \( \log_e x \) is called the natural logarithm of \( x \) and is denoted by \( \ln x \); that is,

\[
y = \ln x \quad \text{if and only if} \quad e^y = x
\]

Properties of the Natural Logarithm

For positive numbers \( u \) and \( v \),

- The equality rule: \( \ln u = \ln v \) if and only if \( u = v \)
- The product rule: \( \ln(uv) = \ln u + \ln v \)
- The power rule: \( \ln u^r = r \ln u \) for any real number \( r \)
- The quotient rule: \( \ln \left(\frac{u}{v}\right) = \ln u - \ln v \)
- Special values: \( \ln 1 = 0 \) and \( \ln e = 1 \)
The Natural Logarithm

The Inverse Relationship between $e^x$ and $\ln x$
\[ e^{\ln x} = x \text{ for } x > 0 \quad \text{and} \quad \ln e^x = x \text{ for all } x \]

Example
Solve the following equations.

a. $-2 \ln x = 3$
\[ \ln (x) = \frac{-3}{2} \]
\[ e^{\frac{-3}{2}} = e^{\ln (x)} \]
\[ x = e^{\frac{-3}{2}} \]

b. $\ln x = 2(\ln 3 - \ln 5)$
\[ \ln(e^x) = \ln(\frac{1}{3}) \]
\[ -x = \ln(\frac{1}{3}) \]
\[ x = -\ln(\frac{1}{3}) \]

\[ e^y = -3 \]
\[ \ln(-3) = y \]
\[ \ln(x) \text{ is undefined for } x \leq 0 \]
\[ \text{Domain of } \ln(x) \text{ is } x > 0. \]

\[ \ln(x) = 2(\ln 3 - \ln 5) \]
\[ = 2 \ln(\frac{3}{5}) \]
\[ = \ln \left( \left( \frac{3}{5} \right)^2 \right) \]
\[ e^\ln(x) = e^{\ln \left( \left( \frac{3}{5} \right)^2 \right)} \]
\[ x = \left( \frac{3}{5} \right)^2 = \frac{9}{25} \]