Quiz 14 Wednesday 5/4  S.2
Final Exam - Monday 5/16.
   cumulative
   About half will be on Ch. 5.
   About half on previous sections
   Chapters 2 & 3 emphasized

5.3 Definite Integral, Fund. Thm. of Calc.

Example: \( s(x) \) - position at time \( x \)
\[ s'(x) \] - velocity or speed.

Just knowing \( s'(x) \) allows us to find
our \underline{relative position} or the distance we
\underline{travel} between two points of time

\[ s(x) = \int s'(x) \, dx \quad \text{or} \quad s(x) + C \]

\[ \int s'(x) \, dx = s(x) + C \]
Suppose \( s'(x) = 3 \) (meters per second)

Graphically

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 1 & 2 & 3 & 4 & 5 \\
\hline
s(x) & \text{Area = 12} \\
\hline
\end{array}
\]

Between \( x = 1 \), \( x = 5 \) how far have I travelled?

\[ D = 3(5-1) = 3 \cdot 4 = 12 \text{ m} \]

Another way to answer this:

\[
s'(x) = 3 \\
s(x) = 3x + C \\
\text{Distance travelled} = s(5) - s(1) \\
= (3(5) + C) - (3(1) + C) \\
= 3(5) - 3(1) = 3(5-1) = 12 .
\]
We have a connection between

**Area under the graph of** $S'(x)$ **AND Displacement of** $S(x)$ **between 2 values of** $x$.

$S'(x) = 3x$

\[ \text{Area} = \frac{1}{2}(4)(12) + (4)(3) = 24 + 12 = 36 \]

$S(x) = \frac{3}{2}x^2 + C$

Distance travelled = $S(5) - S(1)$

\[ = \left( \frac{3}{2} \cdot 5^2 + C \right) - \left( \frac{3}{2} \cdot 1^2 + C \right) = \frac{3}{2} \cdot 25 - \frac{3}{2} \cdot 1 = \frac{3}{2} \cdot 24 = 36 \text{ m} \]
\[ s'(x) = 3x^2 \]

Is the area also 124?

Total distance travelled between \( x=1, x=5 \):

\[ s(x) = x^3 + C \]

Distance \[ = s(5) - s(1) \]
\[ = (5^3 + C) - (1^3 + C) \]
\[ = 125 - 1 = 124 \text{ m} \]
The Definite Integral

The area under the graph of $f(x)$ between $x=a$, $x=b$.

The Fundamental Theorem of Calculus
If the function $f(x)$ is continuous on the interval $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \leq x \leq b$.

Example (#4)
Evaluate $\int_1^4 (5 - 2t) \, dt$.

$$\int_1^4 (5 - 2t) \, dt = \left[ 5t - t^2 \right]_1^4 = \left( 5(4) - (4)^2 \right) - \left( 5(1) - (1)^2 \right)$$

$$= 20 - 16 - 5 + 1 = 0$$
The Definite Integral

Area as a Definite Integral
If \( f(x) \) is continuous and \( f \geq 0 \) on the interval \( a \leq x \leq b \), then the region under the curve \( y = f(x) \) over the interval \( a \leq x \leq b \) has area given by the definite integral \( \int_a^b f(x) \, dx \).

Example (#38)
Find the area of the region that lies under the curve \( y = \sqrt{x}(x + 1) \) over the interval \( 0 \leq x \leq 4 \).

\[
A = \int_0^4 \sqrt{x}(x+1) \, dx = \int_0^4 x^{\frac{3}{2}}(x+1) \, dx = \int_0^4 (x^{\frac{3}{2}} + x^{\frac{1}{2}}) \, dx
\]

\[
= \left[ \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} \right]_0^4 = \frac{2}{5}(4)^{\frac{5}{2}} + \frac{2}{3}(4)^{\frac{3}{2}} - 0
\]
Rules of Definite Integrals

Let $f$ and $g$ be continuous on $a \leq x \leq b$. Then

- The constant multiple rule:
  \[
  \int_{a}^{b} kf(x) \, dx = k \int_{a}^{b} f(x) \, dx \quad \text{for constant } k
  \]

- The sum rule:
  \[
  \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx
  \]

- The difference rule:
  \[
  \int_{a}^{b} [f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx
  \]
Rules of Definite Integrals

Let $f$ and $g$ be continuous on $a \leq x \leq b$. Then

- $\int_{a}^{a} f(x) \, dx = 0$
- $\int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx$
- The subdivision rule:

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$
Rules of Definite Integrals

Example (#36)
Let $f(x)$ and $g(x)$ be continuous on $-3 \leq x \leq 1$ and satisfy
\[
\int_{-3}^{1} f(x) \, dx = 0 \quad \int_{-3}^{1} g(x) \, dx = 4
\]
Evaluate $\int_{-3}^{1} [2f(x) + 3g(x)] \, dx$. 
Rules of Definite Integrals

Example (#32)
Let \( g(x) \) be continuous on \(-3 \leq x \leq 2\) and satisfies

\[
\int_{-3}^{2} g(x) \, dx = -2 \quad \int_{-3}^{1} g(x) \, dx = 4
\]

Evaluate \( \int_{1}^{2} g(x) \, dx \).
Area \approx \left[\sum_{i=1}^{n} f(x_i) \Delta x\right] \Delta x

\Delta x \to 0
\implies \int_{a}^{b} f(x) \, dx
Area \approx (x_1-x_0)f(x_1^*) + (x_2-x_1)f(x_2^*)
+ \cdots + (x_8-x_7)f(x_8^*)

\rightarrow \text{Riemann sum.}

\text{Approximate area under curve.}

\text{Exact area} = \lim_{\text{width of intervals} \to 0} \left(\text{Riemann sums}\right) = \int_a^b f(x) \, dx
The Definite Integral

Let $f(x)$ be a continuous function on $a \leq x \leq b$. Subdivide the interval $a \leq x \leq b$ in $n$ equal parts, each of width $\Delta x = \frac{b - a}{n}$, and choose a number $x_k$ from the $k$th subinterval. Form the sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

called the Riemann Sum.

Then the definite integral of $f$ on the interval $a \leq x \leq b$, denoted by $\int_a^b f(x) \, dx$, is the limit of the Riemann sum as $n \to +\infty$; that is,

$$\int_a^b f(x) \, dx = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$
Take one rectangle and blow it up:
\[ f(x) = F'(x) \]

Area of rect
\[ \approx F'(x^*) (x_3 - x_2) \]
\[ = f(x^*) (x_3 - x_2) \]
\[ = F(x_3) - F(x_2) \]