\[ \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4} \]

\[ \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{x - 4} \cdot \frac{1}{\sqrt{x} + 2} = \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2} \]
Difference quotient: \[ \frac{f(x+h) - f(x)}{h} \]

Idea: \( x = \text{time (on a clock)} \)

\( f(x) = \text{reading on trip odometer} \)

<table>
<thead>
<tr>
<th>Odometer</th>
<th>Clock</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.8 mi</td>
<td>3:05</td>
</tr>
<tr>
<td>23.0 mi</td>
<td>3:15</td>
</tr>
<tr>
<td>12.2 mi</td>
<td></td>
</tr>
</tbody>
</table>

\[ \text{Speed} = \frac{12.2}{\frac{1}{6}} = 73.2 \text{ miles/hour} \]

Average speed over the 10 min period.

Q: How fast was I going exactly at 3:10?
Want instantaneous speed at 3:10

\[ x = \text{time in minutes after 3:05} \]

\[ f(x) = \text{odometer reading} \]

\[ 3:10 \leftrightarrow x = 5 \]

\[ \text{speed} = \frac{f(5+h) - f(5)}{h} \quad \text{if } h \text{ is small.} \]

Inst. speed \( \leftrightarrow "h=0" \)

Makes no sense so take \( \lim_{h \to 0} \).
2.1 The Derivative

The derivative of a function
The derivative of the function \( f(x) \) with respect to \( x \) is the function \( f'(x) \) given by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

The process of computing the derivative is called differentiation, and we say that \( f(x) \) is differentiable at \( x = c \) if \( f'(c) \) exists.

Example
Find the derivative of the function \( f(x) = x^2 - 2x \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - 2(x + h) - (x^2 - 2x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - 2x - 2h - (x^2 - 2x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}
\]

\[
f'(x) = \lim_{h \to 0} (2x + h - 2)
\]

\[
f'(x) = 2x - 2
\]
\[
\frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}
\]
\[
= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}
\]
\[
= \frac{2xh + h^2 - 2h}{h}
\]
\[
= h(2x + h - 2)
\]
\[
= \frac{h}{h}
\]
\[
= 2x + h - 2
\]
\[
(\text{if } h \neq 0)
\]
\[
= \lim_{h \to 0} (2x + h - 2) = 2x - 2
\]
\[
f'(x) = 2x - 2
\]
Slope as a Derivative

The slope of the tangent line to the curve \( y = f(x) \) at the point \((c, f(c))\) is \( m_{\text{tan}} = f'(c) \).

Example

Find the equation of the tangent line to the curve \( y = x^2 - 2x \) at the point where \( x = -1 \).

\[
\frac{f(x+h) - f(x)}{h} \quad \text{slope of line through } (x, f(x)) \text{ and } (x+h, f(x+h))
\]
Equation of line: slope \leftarrow f'(\(-1\))
any point \n\leftarrow (-1, f(-1))

\begin{align*}
f'(x) &= 2x - 2 \\
f'(-1) &= 2(-1) - 2 = -4 \\
f(-1) &= (-1)^2 - 2(-1) = 1 + 2 = 3 \\
slope &= -4 \\
point &= (-1, 3) \\
y - 3 &= -4(x - (-1)) \\
y &= 3 - 4(x + 1) \\
y &= -4x - 1
\end{align*}
\( f(x) = 2 - 7x \quad x = -1 \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{2 - 7(x+h) - (2 - 7x)}{h}
\]

\[
= \lim_{h \to 0} \frac{2 - 7x - 7h - 2 + 7x}{h}
\]

\[
= \lim_{h \to 0} \frac{-7h}{h} = \lim_{h \to 0} -7 = -7
\]

Derivative of a linear function is the slope.

\[ y = 2 - 7x \]
$\text{#10} \quad f(x) = \frac{1}{x^2}, \quad x = 2$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}
= \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right)
= \lim_{h \to 0} \frac{1}{h} \left( \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right)
= \lim_{h \to 0} \frac{1}{h} \frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2}
= \lim_{h \to 0} \frac{1}{h} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$

Slope of tangent line at $x = 2$ is

$f'(2) = \frac{-2}{(2)^3} = \frac{-2}{8} = -\frac{1}{4}$
\[ y = \frac{1}{x^2} \]

\[ f(2) = \frac{1}{4} \]

\[ \text{slope} = -\frac{1}{4} \]