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BOUNDARY BEHAVIOR OF INVARIANT GREEN'S POTENTIALS ON THE UNIT BALL IN $\mathbf{C}^n$

K. T. HAHN AND DAVID SINGMAN

ABSTRACT. Let $p(z) = \int_B G(z,w)\,d\mu(w)$ be an invariant Green's potential on the unit ball B in $\mathbf{C}^n$ $(n \geq 1)$, where G is the invariant Green's function and μ is a positive measure with $\int_B (1-|w|^2)^n\,d\mu(w) < \infty$.

In this paper, a necessary and sufficient condition on a subset E of B such that for every invariant Green's potential p,

$$\liminf_{z \to e} (1-|z|^2)^n p(z) = 0, \qquad e = (1,0,\ldots,0) \in \partial B,\ z \in E,$$

is given. The condition is that the capacity of the sets $E \cap \{z \in B \mid |z-e| < \varepsilon\}$, $\varepsilon > 0$, is bounded away from 0. The result obtained here generalizes Luecking's result, see [L], on the unit disc in $\mathbf{C}$.

1. Introduction. Let E be a subset of B, the unit ball in $\mathbf{C}^n$, $n \geq 1$. An invariant Green's potential is a function on B of the form

$$p(z) = \int_B G(z,w)\,d\mu(w),$$

where G is the invariant Green's function (see 2.11b) and μ is a positive measure such that $\int_B (1-|w|^2)^n\,d\mu(w) < \infty$. In this paper we give a necessary and sufficient condition on E such that, for every invariant Green's potential p,

$$\liminf_{\substack{z \to e \\ z \in E}} (1-|z|^2)^n p(z) = 0,$$

where $e = (1,0,\ldots,0)$. It is that the capacity of the sets $E \cap \{z \in B \mid |z-e| < \varepsilon\}$, $\varepsilon > 0$, be bounded away from 0. Here, capacity refers to the capacity with respect to the potential theory based on the Laplace-Beltrami operator on the ball with respect to the Bergman metric. See (4.1). This solves a problem posed in [HS], where the result was proved in the special case of $E = \{(z,z') \in \mathbf{C} \times \mathbf{C}^{n-1} \mid \operatorname{Im} z = 0,\ (\operatorname{Re} z)^2 + |z'|^2 < 1\}$. Our result generalizes a result of Luecking [L] on the unit disc in $\mathbf{C}$.

2. Preliminaries. For $z, w \in \mathbf{C}^n$ let $\langle z,w\rangle = \sum_{\alpha=1}^n z_\alpha \bar{w}_\alpha$, $|z|^2 = \langle z,z\rangle$. For $0 < r \leq 1$, let $B_r = \{z \in B \mid |z| < r\}$, $S_r = \{z \in B \mid |z| = r\}$, $B = B_1$, and $S = S_1$. Let σ be the rotation-invariant positive Borel measure on S with $\sigma(S) = 1$. Put $e = (1,0,\ldots,0)$. For each $a, z \in B$ let

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|a|^2}\,Q_a(z)}{1 - \langle z,a\rangle},$$

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where $P_a(z) = \langle z, a\rangle a/|a|^2$ and $Q_a(z) = z - P_a(z)$. Then one has

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2} \qquad [\mathbf{R}, \text{Theorem 2.2.2}]. \tag{2.1}$$

The Bergman metric on the ball B is given by

$$ds_B^2(z) = \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar\beta}\, dz_\alpha\, d\bar z_\beta, \tag{2.2a}$$

where

$$g_{\alpha\bar\beta} = \frac{n+1}{(1-|z|^2)^2}\{(1-|z|^2)\delta_{\alpha\beta} + \bar z_\alpha z_\beta\} \tag{2.2b}$$

[**St**, p. 23]. The corresponding invariant volume element [**K**, p. 17] is thus

$$d\lambda(z) = \frac{n+1}{(1-|z|^2)^{n+1}}\, dm(z), \tag{2.2c}$$

where dm denotes the restriction of Lebesgue measure to B. For each $f \in L^1(\lambda)$, $a \in B$, λ satisfies

$$\int_B f \circ \varphi_a\, d\lambda = \int_B f\, d\lambda \tag{2.3}$$

[**R**, Theorem 2.2.6]. The inverse of $(g_{\alpha\bar\beta})$ is $(g^{\alpha\bar\beta})$, where

$$g^{\alpha\bar\beta} = \frac{1-|z|^2}{n+1}(\delta_{\alpha\beta} - \bar z_\alpha z_\beta).$$

The Laplace-Beltrami operator of the metric is

$$\begin{aligned}\Delta &= 4\sum_{\alpha,\beta=1}^{n} g^{\alpha\bar\beta}\frac{\partial^2}{\partial \bar z_\alpha \partial z_\beta}\\ &= \frac{4}{n+1}(1-|z|^2)\sum_{\alpha,\beta=1}^{n}(\delta_{\alpha\beta} - \bar z_\alpha z_\beta)\frac{\partial^2}{\partial \bar z_\alpha \partial z_\beta}\end{aligned}$$

[**St**, p. 27].

A C^2 function defined on an open subset of B that is annihilated by Δ will be called *harmonic*. The set of functions harmonic on open subsets of B forms a Brelot harmonic space [**H**, Théorème 34.1]. We will make some use of the definitions and results available in such a general setting. For details see [**B**].

Let $\Omega \subset B$ be open. A function u is said to be *superharmonic* on Ω if (i) $u\colon \Omega \to (-\infty, \infty]$, (ii) u is lower semicontinuous, (iii) for each $a \in \Omega$, there exists $r(a) > 0$ such that for all $0 < r \le r(a)$,

$$u(a) \ge \int_S u(\varphi_a(r\varsigma))\, d\sigma(\varsigma), \tag{2.4}$$

and (iv) none of the integrals in (iii) is ∞ [**U**, Definition 1.15].

It was observed in [**UT**, Proposition 1.6] that if u is superharmonic on Ω then (2.4) holds for all $r > 0$ such that $\varphi_a(\overline{B_r}) \subset \Omega$.

We wish to see that the above definition agrees with the definition of superharmonic in a Brelot space [**H**, p. 427, Definition A]. If u is C^2, then both definitions of superharmonic are equivalent to $\Delta u \leq 0$ [**H**, Proposition 34.1; **U**, p. 505 or **UT**, Proposition 1.18]. Since it is clear that the limit of an increasing sequence of Brelot superharmonic functions is either identically ∞ or Brelot superharmonic, the result will follow from

PROPOSITION 2.1. *Let u be superharmonic on Ω. Let ω be an open, relatively compact subset of Ω. Then there is an increasing sequence of $C^\infty(\omega)$ functions, superharmonic on ω, with limit u.*

Note. This was proved for $\Omega = B$ in [**UT**, Theorem 1.25]. We include the proof for the reader's convenience.

PROOF. We shall see that the proof depends only on the values of u on some compact neighborhood of ω. Thus, since constants are harmonic, we may assume for the remainder of the proof that u is nonnegative.

For $f, g \geq 0$, define, for $a \in B$,

$$f * g(a) = \int_B f(z)g(\varphi_a(z))\,d\lambda(z)$$

[**U**, Definition 2.1].

Let ω_1 be a relatively compact subset of Ω with $\bar{\omega} \subset \omega_1$. There exists $r_1 > 0$ such that

$$\varphi_a(\overline{B_{r1}}) \subset \omega_1 \quad (\text{all } a \in \omega). \tag{2.5}$$

Since $\varphi_a^{-1} = \varphi_a$, this says z is in ω_1 whenever $|\varphi_a(z)| \leq r_1$.

Let $\chi \geq 0$ be radial and C^∞ with support in B_{r_1} and suppose $\int_B \chi\,d\lambda = 1$. Since

$$u * \chi(a) = \int_B u(z)\chi(\varphi_a(z))\,d\lambda(z) = \int_{B_{r_1}} u(\varphi_a(z))\chi(z)\,d\lambda(z), \tag{2.6}$$

the value of $u * \chi$ on ω depends only on the value of u on ω_1. The first equality in (2.6) shows $u * \chi \in C^\infty(\omega)$. Integrating the second in polar coordinates shows $u * \chi \leq u$ on ω.

Fix $a \in \omega$. Let ω_2 be a relatively compact open subset of Ω such that $\bar{\omega}_1 \subset \omega_2$. Choose $0 < r_2 < r_1$ such that $\varphi_b(\underline{B_{2r_2}}) \subset \omega_2$ (all $b \in \omega_1$). Let $\varepsilon > 0$. Since $(u * \chi) \circ \varphi_a$ is uniformly continuous on $\overline{B}_{r2}$, there exists $0 < \delta < r_2$ such that

$$|(u * \chi)(\varphi_a(r\varsigma)) - (u * \chi)(\varphi_a(s\varsigma))| < \varepsilon \tag{2.7}$$

for all $0 < r < r_2$, $\varsigma \in S$, and $|s - r| < \delta$. Fix any $r > 0$ with $r < r_2$. Let $h \geq 0$ be C^∞, radial, $\int_B h\,d\lambda = 1$ with support in $\{z | r - \delta < |z| < r + \delta\}$. Then, with a similar proof as above,

$$u * h(\varsigma) \leq u(\varsigma) \quad (\text{all } \varsigma \text{ in } \omega_1). \tag{2.8}$$

Thus, by [**U**, Proposition 2.2],

$$\begin{aligned}(u * \chi) * h(a) &= u * (\chi * h)(a) = u * (h * \chi)(a) = (u * h) * \chi(a)\\ &= \int_{B_{r_1}} (u * h)(\varphi_a(z))\chi(z)\,d\lambda(z)\\ &\leq \int_{B_{r_1}} u(\varphi_a(z))\chi(z)\,d\lambda(z)\\ &= u * \chi(a).\end{aligned} \tag{2.9}$$

The inequality in (2.9) follows from (2.5) and (2.8). From (2.7),

$$\begin{aligned} &|(u*\chi)*h(a) - \int_S (u*\chi)(\varphi_a(r\varsigma))\,d\sigma(\varsigma)| \\ &\leq \int_{|s|=r-\delta}^{r+\delta} \int_S |(u*\chi)(\varphi_a(s\varsigma)) - (u*\chi)(\varphi_a(r\varsigma))| \frac{s^{2n-1}}{(1-s^2)^{n+1}} h(s)\,d\sigma(\varsigma)\,ds \\ &< \varepsilon. \end{aligned}$$

Thus

$$\int_S (u*\chi)(\varphi_a(r\varsigma))\,d\sigma(\varsigma) < \varepsilon + (u*\chi)*h(a) \leq \varepsilon + (u*\chi)(a) \tag{2.10}$$

by (2.9). Since ε is arbitrary, (2.10) implies $u*\chi$ is superharmonic on ω.

Let $\{r_j\}$ be a sequence of real numbers decreasing to 0. Choose, for each j, $\chi_j \geq 0$, radial, C^∞, $\int_B \chi_j = 1$ with support in $\{z | r_{j+1} < |z| < r_j\}$. Then we have seen $\{u*\chi_j\}$ is a sequence of functions superharmonic on ω. The fact that they increase to u on ω was shown in [**U**, Lemma 2.13]. This completes the proof.

Let S^+ denote the space of nonnegative superharmonic functions on B. An element of S^+ which majorizes no positive harmonic functions is called a *potential* [**H**, p. 427]. These are precisely of the form

$$G\mu(z) = \int_B G(z,w)\,d\mu(w), \tag{2.11a}$$

where

$$G(z,w) = \frac{n+1}{2n} \int_{|\varphi_z(w)|}^1 \frac{(1-t^2)^{n-1}}{t^{2n-1}}\,dt \tag{2.11b}$$

is the Green's function on B and μ is a positive measure for which $G\mu(z) \not\equiv \infty$ [**U**, Theorem 2.16]. It should be pointed out that although the definition of Green's function of B is given in [**U**], the actual computation of the Green's function of B was carried out in [**HM**] as a special case of more general classical Cartan domains.

Let $0 < c < 1$ be fixed. It is shown in [**HS**] that there are constants c_1 and $c_2 = c_2(c)$ such that

$$G(z,w) \geq c_1(1-|\varphi_z(w)|^2)^n \quad (\text{all } z, w \text{ in } B), \tag{2.12a}$$

$$G(z,w) \leq c_2(1-|\varphi_z(w)|^2)^n \quad (\text{if } |\varphi_z(w)| \geq c). \tag{2.12b}$$

It follows from this and (2.1) that $G\mu$ is a potential if and only if

$$\int_B (1-|w|^2)^n\,d\mu(w) < \infty. \tag{2.13}$$

For $E \subset B$ and any $v \in S^+$, the *reduced function* and its *regularization* are defined by

$$R_v^E(z) = \inf\{w(z) | w \in S^+,\ w \geq v \text{ on } E\}$$

and

$$\hat{R}_v^E(z) = \liminf_{\varsigma \to z} R_v^E(\varsigma),$$

respectively [**H**, p. 433 or **B**, Definition 9]. Then $\hat{R}_v^E$ is harmonic on $B\backslash\bar{E}$ and superharmonic on B. Moreover, we have the following relations:

$$\hat{R}_v^E \le R_v^E \le v \quad \text{on } B, \tag{2.14a}$$

$$R_v^E = v \quad \text{on } E, \tag{2.14b}$$

and

$$\hat{R}_v^E = R_v^E \quad \text{on } B\backslash\bar{E} \text{ and the interior of } E. \tag{2.14c}$$

Thus, $\hat{R}_v^E = R_v^E$ in case E is open. If E is relatively compact in B, then $\hat{R}_v^E$ is a potential.

A set $E \subset B$ is *polar* if there is a potential which is ∞ on E [**H**, p. 434 or **B**, Definition 21]. The polar sets defined here coincide with the corresponding notion in the classical potential theory [**H**, Théorème 36.1].

The most important result concerning polar sets is the Cartan-Brelot convergence theorem which states that if $\{v_k\}$ is a decreasing sequence in S^+, then

$$\left(\lim_{k\to\infty} v_k(z)\right)^{\wedge} = \liminf_{\varsigma\to z}\left(\lim_{k\to\infty} v_k(\varsigma)\right)$$

is superharmonic and equals $\lim_{k\to\infty} v_k(z)$ except at most on a polar set [**H**, p. 436, Theorem 27]. The Topological Lemma of Choquet [**D**, Lemma A, VIII.3] implies

$$\hat{R}_v^E(z) = R_v^E(z), \tag{2.15}$$

except perhaps on a polar set. The proof of the convergence theorem makes use of the *Domination Principle* [**H**, p. 436, Axiome D, and Corollaire to Théorème 36.2] which states that if $v \in S^+$, $G\mu$ is a finite potential satisfying $v \ge G\mu$ on the support of μ, then $v \ge G\mu$ holds on B. As a consequence, a set $E \subset B$ is polar if and only if $\hat{R}_1^E \equiv 0$ or equivalently $R_1^E(z) = 0$ for one $z \in B$.

For any $E \subset B$,

$$R_1^E(z) = \inf\{R_1^U(z) | U \text{ open, } U \supset E\} \tag{2.16a}$$

[**H**, p. 434]. Hence, by the Topological Lemma of Choquet, there is a decreasing sequence of open sets $\{U_k\}$, $U_k \supset E$, such that, for all $z \in B$,

$$\hat{R}_1^E(z) = \left(\lim_{k\to\infty} R_1^{U_k}(z)\right)^{\wedge}. \tag{2.16b}$$

For a Borel set $E \subset B$,

$$\hat{R}_1^E(z) = \sup\{\hat{R}_1^K(z) | K \text{ compact, } K \subset E\} \tag{2.17a}$$

[**H**, p. 434, Théorème 8]. Hence, by [**D**, Theorem A, VIII.2], there is an increasing sequence of compact sets $\{K_j\}$ contained in E such that, for all $z \in B$,

$$\hat{R}_1^E(z) = \lim_{j\to\infty} \hat{R}_1^{K_j}(z). \tag{2.17b}$$

3. Energy. Let μ and ν be two positive measures on B such that $G\mu$ and $G\nu$ are the corresponding potentials. Define the *mutual energy* of μ and ν by

$$[\mu, \nu] = \int_B G\mu(z)\, d\nu(z). \tag{3.1}$$

The *energy* of μ (or $G\mu$) is $\|\mu\| = [\mu,\mu]^{1/2}$. Fubini's theorem and the symmetry of G (deduced from (2.1)) implies

$$[\mu,\nu] = [\nu,\mu]. \tag{3.2}$$

LEMMA 3.1. *Let $\{G\mu_j\}$, $\{G\nu_j\}$ be increasing sequences of potentials with limits $G\mu$ and $G\nu$, respectively. Then $\{[\mu_j,\nu_j]\}$ is an increasing sequence with limit $[\mu,\nu]$.*

PROOF. Since

$$\begin{aligned}\int_B G\mu_j\,d\nu_j \le \int_B G\mu_{j+1}\,d\nu_j &= \int_B G\nu_j\,d\mu_{j+1}\\ \le \int_B G\nu_{j+1}\,d\mu_{j+1} &= \int G\mu_{j+1}\,d\nu_{j+1},\end{aligned}$$

the sequence $\{[\mu_j,\nu_j]\}$ is indeed increasing. For each $k>0$,

$$\begin{aligned}\lim_{j\to\infty}[\mu_j,\nu_j] \ge \liminf_{j\to\infty}\int_B G\mu_k\,d\nu_j &= \liminf_{j\to\infty}\int_B G\nu_j\,d\mu_k\\ = \int_B G\nu\,d\mu_k &= \int_B G\mu_k\,d\nu.\end{aligned}$$

Letting $k\to\infty$ gives $\lim_{j\to\infty}[\mu_j,\nu_j] \ge [\mu,\nu]$. The opposite inequality is obvious.

An improvement of Lemma 3.1 will be given later in Corollary 3.4.

PROPOSITION 3.2. *Let $G\mu$ and $G\nu$ be two potentials. Then*

$$[\mu,\nu] \le \|\mu\|\,\|\nu\|. \tag{3.3}$$

PROOF. Let $u = G\mu$ and $v = G\nu$. Suppose first that u and v are C^∞ and μ and ν are supported by B_{r_0}, $0<r_0<1$. Let $\{\psi_j\}$ and $\{\varphi_j\}$ be sequences in $C_c^\infty(B)$, $\{\psi_j\}$ increasing to 1 on B and $\varphi_j \equiv 1$ on a neighborhood of the support of ψ_j. Then

$$\begin{aligned}-\int_B \psi_j u\,d\nu = \int_B v\Delta(\psi_j u)\,d\lambda &= \int_B \varphi_j v\Delta(\psi_j u)\,d\lambda\\ = \int_B \psi_j u\Delta(\varphi_j v)\,d\lambda &= \int_B \psi_j u\Delta v\,d\lambda.\end{aligned}$$

Here the first equality follows from Theorem 2.5 of [U] and the third from Proposition 2.4 of [U]. Letting $j\to\infty$, we obtain

$$[\mu,\nu] = -\int_B u\Delta v\,d\lambda. \tag{3.4}$$

Let $r_0<r<1$. By Green's identity [**S**, 92.5],

$$\int_{S_r} u\frac{\partial v}{\partial n}\,d\tau - \int_{B_r} u\Delta v\,d\lambda = \int_{B_r}(\operatorname{grad} u, \operatorname{grad} v)\,d\lambda,$$

where $d\tau$ is the volume element determined by the metric (2.2) on $\partial B_r = S_r$, and

$$(\operatorname{grad} u, \operatorname{grad} v) = 4\operatorname{Re}\sum_{\alpha,\beta} g^{\alpha\bar\beta}\frac{\partial u}{\partial \bar z_\alpha}\frac{\partial v}{\partial z_\beta}.$$

We claim

$$\lim_{r\to1}\int_{\partial B_r} u\frac{\partial v}{\partial n}\,d\tau = 0. \tag{3.5}$$

For $|z| = r$,

$$\tag{3.6} \begin{aligned} u(z) &= \int_{B_{r_0}} G(z,w)\,d\mu(w) \le c_2 \int_{B_{r_0}} (1-|\varphi_z(w)|^2)^n\,d\mu(w) \\ &= c_2 \int_{B_{r_0}} \frac{(1-|z|^2)^n(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}\,d\mu(w) = O(1-r^2)^n. \end{aligned}$$

The inequality in (3.6) follows from (2.12b), while the last equality is a consequence of u being a potential.

The derivative of v in the outward normal direction along S_r is given by

$$\tag{3.7} \begin{aligned} \frac{\partial v}{\partial n} &= 2\,\mathrm{Re}\left(\sum_{\alpha,\beta} g^{\alpha\bar\beta}\frac{\partial v}{\partial z_\beta}\frac{\partial f}{\partial \bar z_\alpha}\right) \Bigg/ \left(\sum_{\alpha,\beta} g^{\alpha\bar\beta}\frac{\partial f}{\partial z_\beta}\frac{\partial f}{\partial \bar z_\alpha}\right)^{1/2} \\ &= \frac{(\operatorname{grad} f, \operatorname{grad} v)}{\|\operatorname{grad} f\|}, \end{aligned}$$

where $f(z) = \sum_\alpha z_\alpha \bar z_\alpha - r^2$.

After some calculations, we obtain

$$\tag{3.8a} \|\operatorname{grad} f\|^2 = \frac{4}{n+1} r^2 (1-r^2)^2,$$

$$\tag{3.8b} (\operatorname{grad} f, \operatorname{grad} v) = \frac{4}{n+1}(1-|z|^2)^2\,\mathrm{Re}\sum_{\alpha=1}^n z_\alpha \frac{\partial v}{\partial z_\alpha}.$$

So, from (3.7) and (3.8),

$$\tag{3.9} \frac{\partial v}{\partial n} = \frac{2}{\sqrt{n+1}}\frac{1-r^2}{r}\,\mathrm{Re}\left(\sum_{\alpha=1}^n z_\alpha \frac{\partial v}{\partial z_\alpha}\right).$$

But a straightforward computation shows

$$\mathrm{Re}\left(\sum_{\alpha=1}^n z_\alpha \frac{\partial v}{\partial z_\alpha}\right) = O(1-r^2)^{n-1}.$$

Thus,

$$\tag{3.10} \partial v/\partial n = O(1-r^2)^n.$$

For $z \in S_r$, $d\lambda(z)$ is reduced to

$$\tag{3.11} d\tau(z) = \frac{n+1}{(1-r^2)^{n+1}} r^{2n-1}\,d\sigma(z).$$

So, from (3.6), (3.11) and (3.12),

$$\int_{S_r} u\frac{\partial v}{\partial n}\,d\tau = O(1-r^2)^{n-1} \to 0 \quad \text{as } r \to 1 \text{ if } n \ge 2.$$

This proves the claim in (3.5).

Now

$$\begin{aligned}
|(\operatorname{grad} u, \operatorname{grad} v)| &= \left| 4\operatorname{Re} \sum_{\alpha,\beta} g^{\alpha\bar{\beta}} \frac{\partial v}{\partial z_\beta} \frac{\partial u}{\partial \bar{z}_\alpha} \right| \\
&\le \left(\sum_{\alpha,\beta} 4 g^{\alpha\bar{\beta}} \frac{\partial u}{\partial z_\beta} \frac{\partial u}{\partial \bar{z}_\alpha} \right)^{1/2} \left(\sum_{\alpha,\beta} 4 g^{\alpha\bar{\beta}} \frac{\partial v}{\partial v_\beta} \frac{\partial v}{\partial \bar{z}_\alpha} \right)^{1/2} \\
&= \|\operatorname{grad} u\| \, \|\operatorname{grad} v\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{S_r} u \frac{\partial v}{\partial n}\, d\tau - \int_{B_r} u \Delta v \, d\lambda \le \int_{B_r} \|\operatorname{grad} u\| \, \|\operatorname{grad} v\| \, d\lambda \\
&\le \left(\int_{B_r} \|\operatorname{grad} u\|^2 \, d\lambda \right)^{1/2} \left(\int_{B_r} \|\operatorname{grad} v\|^2 \, d\lambda \right)^{1/2} \\
&= \left(\int_{S_r} u \frac{\partial u}{\partial n}\, d\tau - \int_{B_r} u \Delta u \, d\lambda \right)^{1/2} \left(\int_{S_r} v \frac{\partial v}{\partial n}\, d\tau - \int_{B_r} v \Delta v \, d\lambda \right)^{1/2}.
\end{aligned}$$

Letting $r \to 1$ and using (3.4) and (3.5) we obtain

$$[\mu, \nu] \le \|\mu\| \, \|\nu\|. \tag{3.12}$$

Thus the proposition is true in this case.

Suppose that $G\mu$ and $G\nu$ are potentials whose measures are compactly supported, say in $B_{1-\varepsilon}$ for some $\varepsilon > 0$. Thus $G\mu$ and $G\nu$ are harmonic on $B - \bar{B}_{1-\varepsilon}$. Let $\{r_j\}$ be a sequence of numbers strictly decreasing to 0. Let $\{\psi_j\}$ be C^∞ functions with support in $B_{r_j} - \bar{B}_{r_{j+1}}$, radial, $\psi_j \ge 0$ and $\int_B \psi_j \, d\lambda = 1$. Let $\delta > 0$ be so small that for $|z| > 1 - \delta$, $\phi_z(B_{r_j}) \subset B - \bar{B}_{1-\varepsilon}$. Then precisely as in the proof of Proposition 2.1, $\{G\mu * \psi_j\}$ and $\{G\nu * \psi_j\}$ are harmonic on $B - \bar{B}_{1-\delta}$, they are $C^\infty(B)$ and they increase respectively to $G\mu$ and $G\nu$ [U, Lemma 2.1]. Thus they are C^∞ potentials with compact support. From (3.12) we have

$$[G\mu * \psi_j, G\nu * \psi_j] \le \|G\mu * \psi_j\| \, \|G\nu * \psi_j\|.$$

Letting $j \to \infty$ and applying Lemma 3.1 gives

$$[\mu, \nu] \le \|\mu\| \, \|\nu\|. \tag{3.13}$$

Finally, the proof of the proposition will be completed for arbitrary potentials $G\mu$ and $G\nu$ by considering the restrictions of μ and ν to $B_{1-1/n}$ and applying (3.13) and Lemma 3.1.

COROLLARY 3.3. *Let μ and ν be two positive measures on B. Then* $\|\mu+\nu\| \le \|\mu\| + \|\nu\|$.

PROOF.

$$\begin{aligned}
\|\mu + \nu\|^2 &= \|\mu\|^2 + \|\nu\|^2 + 2[\mu, \nu] \\
&\le \|\mu\|^2 + \|\nu\|^2 + 2\|\mu\| \, \|\nu\| = (\|\mu\| + \|\nu\|)^2.
\end{aligned}$$

COROLLARY 3.4. *Let* $\{G\mu_k\}$ *be an increasing sequence of potentials such that* $\sup_k ||\mu_k|| < \infty$. *Then there is a potential* $G\mu$ *such that* $\lim_{k\to\infty} G\mu_k = G\mu$ *and* $\lim_{k\to\infty} ||\mu_k|| = ||\mu||$.

PROOF. Let $0 < r < 1$. B_r is regular for the Dirichlet problem on B_r. That is, for every continuous function f on S_r, there is a unique harmonic function $P_r f$ on B_r which tends to f on S_r and which is ≥ 0 if $f \geq 0$ [**R**, Lemma 5.5.4]. For $\xi \in B_r$, let ρ_ξ^r be the measure such that

$$\int_{S_r} f\, d\rho_\xi^r = P_r f(\xi)$$

[**H**, p. 426]. Then, by [**U**, Lemma 1.19],

$$\begin{aligned} ||\rho_\xi^r||^2 &= \int_{S_r} G\rho_\xi^r\, d\rho_\xi^r \leq G\rho_\xi^r(\xi) \\ &= \int_{S_r} G(\xi, w)\, d\rho_\xi^r(w) = P_r G(\xi, \cdot)(\xi). \end{aligned} \tag{3.14}$$

Since $P_r G(\xi, \cdot)$ decreases as r increases [**U**, Corollary 1.20], $\lim_{r\to 1} P_r G(\xi, \cdot)$ defines a harmonic function on B [**U**, Proposition 1.10] which minorizes $G(\xi, \cdot)$. It follows the limit as $r \to 1$ in (3.14) is 0. Thus $\lim_{r\to 1} ||\rho_\xi^r|| = 0$.

Let $u = \lim_{k\to\infty} G\mu_k$. Then

$$\int_{S_r} u\, d\rho_\xi^r = \lim_{k\to\infty} \int_{S_r} G\mu_k\, d\rho_\xi^r = \lim_{k\to\infty} [\mu_k, \rho_\xi^r] \leq \sup_{k>0} ||\mu_k||\, ||\rho_\xi^r||,$$

by Proposition 3.2. Thus

$$\lim_{r\to 1} \int_{S_r} u\, d\rho_\xi^r = 0 \quad \text{for each } \xi \in B. \tag{3.15}$$

It is easy to see that the limit in (3.15) defines the greatest harmonic minorant of u on B. It follows that u is a potential. The result now follows from Lemma 3.1.

LEMMA 3.5. *Let* E *be polar and let* $G\mu$ *be a potential of finite energy. Then* $\mu(E) = 0$.

PROOF. We may assume E is relatively compact. For each $k > 0$ let $A_k = \{z \in B | G\mu(z) \leq k\}$. Let μ_k be the restriction of μ to A_k. Since A_k is closed, $G\mu_k$ is harmonic on $B \backslash A_k$. For $z \in A_k$, $G\mu_k(z) \leq G\mu(z) \leq k$. Thus $G\mu_k$ is a finite potential and we may apply the Domination Principle to deduce $G\mu_k \leq k$ on B.

Let p be a potential that is ∞ on E. Let U be a relatively compact neighborhood of E. Then R_p^U is the potential of a measure ν with support in $\bar{U}$. Since

$$\int_E G\nu\, d\mu_k \leq \int G\nu\, d\mu_k = \int G\mu_k\, d\nu \leq k \cdot \nu(\bar{U}) < \infty$$

and $G\nu = \infty$ on E, $\mu_k(E)$ must be 0. Letting $k \to \infty$ gives

$$\mu[E \cap \{z \in B | G\mu(z) < \infty\}] = 0.$$

If $\mu[E \cap \{z \in B | G\mu = \infty\}]$ were positive, then

$$\infty = \int_E G\mu\, d\mu \leq \int G\mu\, d\mu = ||\mu||^2,$$

contradicting our assumption. Thus $\mu(E) = 0$.

PROPOSITION 3.6. *Let $\{G\mu_k\}$ be a decreasing sequence of potentials such that $||\mu_1|| < \infty$. If $G\mu = (\lim_{k\to\infty} G\mu_k)^\wedge$, then $||\mu|| = \lim_{k\to\infty} ||\mu_k||$.*

PROOF. The Cartan-Brelot convergence theorem implies $G\mu = \lim_{k\to\infty} G\mu_k$ except at most on a polar set. Using Lemma 3.5, the proof follows the same pattern as Lemma 3.1.

PROPOSITION 3.7. *Let $E \subset B$ be polar. Then there is a potential of finite energy that is ∞ on E.*

PROOF. Suppose first E is bounded. Since $\hat{R}_1^E \equiv 0$, there is a decreasing sequence of relatively compact open sets $\{U_k\}$, each containing E such that $(\lim_{k\to\infty} R_1^{U_k})^\wedge \equiv 0$ (2.16b). Let μ_k be the measure on $\bar{U}_k$ whose potential is $R_1^{U_k}$. (Recall $R_1^{U_k} = \hat{R}_1^{U_k}$ since U_k is open.) Then, by (2.14a),

$$||\mu_k||^2 = \int R_1^{U_k}\, d\mu_k \le \int d\mu_k < \infty.$$

Thus $\lim_{k\to\infty} ||\mu_k|| = 0$ (Proposition 3.6). Choose a subsequence $\{\mu_{k_j}\}$ such that $||\mu_{k_j}|| < 1/2^j$. Then Corollary 3.4 implies $\sum_{j=1}^{\infty} R_1^{U_{k_j}}$ is a potential of finite energy. Clearly it is ∞ on E (2.14b).

In general let $\{E_k\}$ be an increasing sequence of bounded sets with union E. Choose a potential $G\mu_k$ of finite energy that is ∞ on E_k. Put $\nu_k = \mu_k/2^k||\mu_k||$. Then $\sum G\nu_k$ is the required potential.

4. Capacity.

Let $E \subset B$. Define

$$c(E) = \begin{cases} \infty & \text{if } \hat{R}_1^E \text{ is not a potential,} \\ ||\mu||^2 & \text{if } \hat{R}_1^E = G\mu. \end{cases} \tag{4.1}$$

We call $c(E)$ the *capacity* of E.

PROPOSITION 4.1. (a) *Let $E \subset B$. There is a decreasing sequence of open sets $\{U_m\}$, each containing E such that $\lim_{m\to\infty} c(U_m) = c(E)$.*

(b) *If E is Borel, there is an increasing sequence of compact sets $\{K_m\}$, each contained in E such that $\lim_{m\to\infty} c(K_m) = c(E)$.*

PROOF. (a) If $c(E) = \infty$, there is nothing to show. Suppose then $c(E) < \infty$. The result will follow from (2.16b) and Proposition 3.6 if we show there is an open set U containing E having finite capacity.

Let $U_1 = \{z | \hat{R}_1^E(z) > \frac{1}{2}\}$. Since $R_1^E \equiv 1$ on E, (2.15) implies $E\backslash U_1$ is polar. By Proposition 3.7, there is a potential $G\nu$ of finite energy that is ∞ on $E\backslash U_1$. Put $U_2 = \{z | G\nu(z) > 1\}$. Then $U = U_1 \cup U_2$ is an open set containing E. Since

$$2\hat{R}_1^E(z) + G\nu(z) \ge 1 \quad (\text{all } z \in U),$$

we have

$$2\hat{R}_1^E(z) + G\nu(z) \ge R_1^U(z) \quad (\text{all } z \in B)$$

by definition of the reduced function. Since $c(E) < \infty$, $2\hat{R}_1^E$ has finite energy. Hence Corollary 3.3 implies so has R_1^U.

(b) The proof follows from (2.17b) and Lemma 3.1.

LEMMA 4.2. *For each $\xi \in B$ and $0 < r < 1$, the map*

$$f(w) = \int_S G(\varphi_\xi(rz), w)\, d\sigma(z)$$

is continuous on B.

PROOF. It is clearly continuous at points of $B \backslash \varphi_\xi(S_r)$. If now $w_0 \in \varphi_\xi(S_r)$, then for each $\varepsilon > 0$ the function

$$f_\varepsilon(w) = \int_S G(\varphi_\xi(r+\varepsilon)z, w)\, d\sigma(z)$$

is continuous at w_0. Since f_ε increases as ε decreases [**U**, Proposition 1.17, Corollary 1.20] f is lower semicontinuous at w_0. Repeating the argument with $-\varepsilon$ completes the proof.

LEMMA 4.3. *Let v be superharmonic on B. Then, for every $\xi \in B$,*

$$\lim_{r \to 0} \int_S v(\varphi_\xi(rz))\, d\sigma(z) = v(\xi).$$

PROOF. The lower semicontinuity of v implies

$$\liminf_{r \to 0} \int_S v(\varphi_\xi(rz))\, d\sigma(z) \geq v(\xi).$$

Super mean value property (2.4) implies

$$\limsup_{r \to 0} \int_S v(\varphi_\xi(rz))\, d\sigma(z) \leq v(\xi).$$

PROPOSITION 4.4. *Let E be a Borel, relatively compact subset of B. If $\hat{R}_1^E$ is the potential $G\mu$, then $||\mu||^2 = \mu(B)$.*

PROOF. Since $||\mu||^2 = \int_B G\mu\, d\mu \leq \int_B 1 \cdot d\mu = \mu(B)$, we may assume $||\mu|| < \infty$.
If E is closed, the result holds by Lemma 3.5 since, in general, $\mu(B \backslash \bar{E}) = 0$.

For a general E, choose $\{K_j\}$ as in (2.17b). Put $\hat{R}_1^{K_j} = G\mu_j$. Then, by Lemma 3.1,

$$\lim_{j \to \infty} \mu_j(B) = \lim_{j \to \infty} ||\mu_j||^2 = ||\mu||^2.$$

By passing to a subsequence we may assume $\{\mu_j\}$ converges weakly to a measure ν. We show $G\nu = \hat{R}_1^E$.

$$\begin{aligned} \int_S \hat{R}_1^E(\varphi_\xi(rz))\, d\sigma(z) &= \lim_{j \to \infty} \int_S G\mu_j(\varphi_\xi(rz))\, d\sigma(z) \\ &= \lim_{j \to \infty} \int_S \int_{\bar{E}} G(\varphi_\xi(rz), w)\, d\mu_j(w)\, d\sigma(z) \\ &= \lim_{j \to \infty} \int_{\bar{E}} \int_S G(\varphi_\xi(rz), w)\, d\sigma(z)\, d\mu_j(w). \end{aligned} \tag{4.2}$$

Lemma 4.2 shows that the inner integral in (4.2) is a continuous function of w. Thus the limit is

$$\int_{\bar{E}} \int_S G(\varphi_\xi(rz), w)\, d\sigma(z)\, d\nu(w) = \int_S G\nu(\varphi_\xi(rz))\, d\sigma(z).$$

Letting $r \to 0$ in the first and last term in this string of equalities and applying Lemma 4.3 gives $\hat{R}_1^E(\xi) = G\nu(\xi)$.

Let $\psi \in C_c^\infty(B)$ be identically 1 on a neighborhood of $\bar{E}$. Then, using Theorem 2.5 of [U],

$$\begin{aligned}\mu(B) &= \int_B \psi\, d\mu = -\int_B G\mu \Delta\psi\, d\lambda = -\int_B G\nu\Delta\psi\, d\lambda \\ &= \int_B \psi\, d\nu = \nu(B).\end{aligned}$$

Thus

$$\|\mu\|^2 = \lim_{j\to\infty} \|\mu_j\|^2 = \lim_{j\to\infty} \mu_j(B) = \nu(B) = \mu(B).$$

This concludes the proof.

5. Main results.

LEMMA 5.1. *Let μ be a finite measure on B and let*

$$d\nu(w) = \frac{d\mu(w)}{(1-|w|^2)^n}.$$

For each $z \in B$, define $S(z) = \varphi_z(B_{1/2})$. Then

$$\begin{aligned}\lim_{z\to e}(1-|z|^2)^n \int_{B\setminus S(z)} G(z,w)\, d\nu(w) \quad (5.1)\\ = \lim_{z\to e}\int_{B\setminus S(z)} G(z,w)\, d\mu(w) = 0\end{aligned}$$

PROOF. If $w \in B\setminus S(z)$, $|\varphi_z(w)| \geq 1/2$. Thus, by (2.12b),

$$\begin{aligned}(1-|z|^2)^n G(z,w) &\leq c_2(1-|z|^2)^n(1-|\varphi_z(w)|^2)^n \\ &= \frac{c_2(1-|z|^2)^{2n}(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}} \\ &\leq \frac{c_2(1-|z|^2)^{2n}(1-|w|^2)^n}{(1-|z|\,|w|)^{2n}}.\end{aligned}$$

Thus

$$\begin{aligned}&\limsup_{z\to e}(1-|z|^2)^n \int_{B\setminus S(z)} G(z,w)\, d\nu(w) \\ &\leq c_2 \limsup_{z\to e}(1-|z|^2)^{2n}\int_B \frac{d\mu(w)}{(1-|z|\,|w|)^{2n}} \\ &= 0\end{aligned}$$

by the bounded convergence theorem.

The last limit in (5.1) is 0, again by the bounded convergence theorem, since by (2.12b),

$$\chi_{B\setminus S(z)}(w) G(z,w) \leq c_2 \chi_{B\setminus S(z)}(w)(1-|\varphi_z(w)|^2)^n \leq (\tfrac{3}{4})^n c_2.$$

THEOREM 5.2. *Let E be a Borel subset of B with $e = (1,0,\ldots,0)$ as a limit point. The following are equivalent:*

(a) *For every potential* $G\nu$,

$$\liminf_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n G\nu(z)=0.$$

(b) $\inf_{\varepsilon>0} c(E\cap\{z\in B|\ |z-e|<\varepsilon\})>0$.

REMARK 5.3. For each positive integer m, let $U_m=\{z\in B|\ |z-e|<1/m\}$. Suppose for some m_0, $\hat{R}_1^{E\cap U_{m_0}}$ is a potential $G\mu$. For each $m>m_0$ let μ_m be the restriction of μ to $E\cap U_m$. Since $G\mu_m$ is harmonic on $B\backslash\bar{U}_m$, $\lim_{m\to\infty} G\mu_m$ is harmonic on B and minorizes $G\mu$ [U, Proposition 1.10]. Thus $\lim_{m\to\infty} G\mu_m$ is identically 0 on B. Proposition 3.6 therefore implies that either $c(E\cap U_m)=\infty$ for all m or $\lim_{m\to\infty} c(E\cap U_m)=0$. Thus (b) above is equivalent to $c(E\cap U_m)=\infty$ for all m.

PROOF OF THE THEOREM. Suppose first that (b) fails. We will show there is a potential $G\nu$ such that

$$\lim_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n G\nu(z)=\infty. \tag{5.2}$$

Remark 5.3 implies our assumption is equivalent to $c(E\cap U_m)<\infty$ for some m. Proposition 4.1(a) implies there is an open set containing $E\cap U_m$ and having finite capacity. Thus, for the purpose of finding ν to satisfy (5.2), we may assume E is open.

Choose $\{m_j\}$ increasing to ∞ such that $c(E\cap U_{m_j})<1/j^2 4^j$. Put

$$V_j=E\cap\{z\in B|1/m_{j+2}<|z-e|<1/m_j\}.$$

Consider the potential $G\mu_j=jR_1^{V_{m_j}}$. Then $||\mu_j||^2=j^2c(V_{m_j})\le 1/4^j$. Thus Corollary 3.3 and Corollary 3.4 show $\sum G\mu_j$ is a potential $G\mu$ of finite energy. Since $\mu=\sum\mu_j$, Proposition 4.4 implies $\mu(B)=\sum\mu_j(B)=\sum||\mu_j||^2<\infty$, so μ is finite. Clearly

$$\lim_{\substack{z\to e\\ z\in E}} G\mu(z)=\infty. \tag{5.3}$$

Put $d\nu(w)=d\mu(w)/(1-|w|^2)^n$. Let $S(z)=\varphi_z(B_{1/2})$. Lemma 5.1 and (5.3) imply

$$\lim_{\substack{z\to e\\ z\in E}}\int_{S(z)} G(z,w)\,d\mu(w)=\infty. \tag{5.4}$$

If $w\in S(z)$, $|\varphi_z(w)|\le 1/2$, hence

$$\begin{aligned}\frac{3}{4}\le 1-|\varphi_z(w)|^2&=\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}\le\frac{(1-|z|^2)(1-|w|^2)}{(1-|z|\ |w|)^2}\\ &\le\frac{(1-|z|^2)(1-|w|^2)}{(1-|w|)^2}\le\frac{4(1-|z|^2)}{1-|w|^2}.\end{aligned}$$

Thus

$$1-|z|^2\ge\tfrac{3}{16}(1-|w|^2)\qquad(|\varphi_z(w)|\le\tfrac12). \tag{5.5}$$

Since $|\varphi_z(w)|=|\varphi_w(z)|$,

$$1-|w|^2\ge\tfrac{3}{16}(1-|z|^2)\qquad(|\varphi_z(w)|\le\tfrac12). \tag{5.6}$$

Thus

$$\begin{aligned}
&\liminf_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n G\nu(z)\\
&\qquad \ge \liminf_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n \int_{S(z)} G(z,w)\,d\nu(w)\\
&\qquad \ge (\tfrac{3}{16})^n \liminf_{\substack{z\to e\\ z\in E}} \int_{S(z)} G(z,w)\,d\mu(w)\\
&\qquad = \infty
\end{aligned}$$

by Lemma 5.1 and (5.3). This completes the proof of (a)$\Rightarrow$(b).

Suppose now (b) holds. Consider a potential $G\nu$. Put $d\mu(w) = (1-|w|^2)^n\, d\nu(w)$. Let

$$h(z) = \int_{S(z)} G(z,w)\,d\mu(w),$$

where $S(z) = \varphi_z(B_{1/2})$. According to Remark 5.3, $c(E\cap U_j) = \infty$ for all j. Thus, by Proposition 4.1(b), there are compact sets E_j contained in $U_j \cap E$ such that

$$\lim_{j\to\infty} c(E_j) = \infty. \tag{5.7}$$

Let $G\mu_j = \hat{R}_1^{E_j}$ and let $F_j = \bigcup_{z\in E_j} \varphi_z(B_{1/2})$. Then, since $|\varphi_z(w)| = |\varphi_w(z)|$,

$$\begin{aligned}
\int h(z)\,d\mu_j(z) &= \int_{E_j} h(z)\,d\mu_j(z)\\
&= \int_{E_j}\int_B \chi_{S(z)}(w) G(z,w)\,d\mu(w)\,d\mu_j(z)\\
&= \int_{E_j}\int_B \chi_{S(w)}(z) G(z,w)\,d\mu(w)\,d\mu_j(z)\\
&= \int_{F_j}\int_{E_j} G(z,w)\,d\mu_j(z)\,d\mu(w)\\
&= \int_{F_j} \hat{R}_1^{E_j}(w)\,d\mu(w) \le \mu(F_j)
\end{aligned} \tag{5.8}$$

and the latter goes to 0 as $j\to\infty$ since $\chi_{F_j}(z)\to 0$ as $|z|\to 1$ (2.1). This shows

$$\liminf_{\substack{z\to e\\ z\in E}} h(z) = 0,$$

for if h were bounded below by $\varepsilon > 0$ on $\bigcup_j E_j$,

$$\begin{aligned}
\int h\,d\mu_j &= \int_{E_j} h\,d\mu_j \ge \varepsilon\mu_j(E_j)\\
&= \varepsilon c(E_j) \quad \text{(Proposition 4.4)}
\end{aligned}$$

which tends to ∞ as $j \to \infty$ (5.7), contradicting the last inequality in (5.8). Thus

$$\begin{aligned}
&\liminf_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n G\nu(z)\\
&\quad=\liminf_{\substack{z\to e\\ z\in E}}(1-|z|^2)^n\int_{S(z)}G(z,w)\,d\nu(w)\quad\text{(Lemma 5.1)}\\
&\quad\le\left(\tfrac{16}{3}\right)^n\liminf_{\substack{z\to e\\ z\in E}}\int_{S(z)}G(z,w)\,d\mu(w)\quad(5.6)\\
&\quad=\left(\tfrac{16}{3}\right)^n\liminf_{\substack{z\to e\\ z\in E}}h(z)\\
&\quad=0.
\end{aligned}$$

This completes the proof.

6. Remarks. Let $0<\delta\le 1$. Put

$$E_\delta=\{(z,z')\in\mathbf{C}\times\mathbf{C}^{n-1}\,|\,\operatorname{Im}z=0,(\operatorname{Re}z)^2+|z'|^2<\delta\}.$$

In [**HS**] it is shown that the limit result of Theorem 5.2 holds with $E=E_1$. We show in this section directly that E_1 satisfies condition (b) of Theorem 5.2.

LEMMA 6.1. *Let φ be an automorphism of B and E a Borel subset of B. Then $c(\varphi(E))=c(E)$.*

PROOF. Suppose first that E is relatively compact. Since $v\circ\varphi\in S^+$ whenever $v\in S^+$ [**U**, Proposition 1.17], it follows from the definition that $\hat{R}_1^{\varphi(E)}(z)=\hat{R}_1^E(\varphi(z))$. Thus, if $\hat{R}_1^E=G\mu$,

$$\begin{aligned}
\hat{R}_1^{\varphi(E)}(z)=\hat{R}_1^E(\varphi(z))&=\int_B G(\varphi(z),w)\,d\mu(w)\\
&=\int_B G(z,\varphi(w))\,d\mu(w)
\end{aligned}$$

since $|\varphi_{\varphi(z)}(\varphi(w))|=|\varphi_z(w)|=G\nu(z)$, where $\nu=\mu\circ\varphi$. Proposition 4.4 implies $c(\varphi(E))=\nu(\varphi(E))=\mu(E)=c(E)$.

In general, let $E_r=E\cap B_r$. Then $c(\varphi(E))=\lim_{r\to1}c(\varphi(E_r))=\lim_{r\to1}c(E_r)=c(E)$. The first and third equalities follow from Lemma 3.1 and the fact that if $\{A_m\}$ is an increasing sequence of sets with union A then $\lim_{m\to\infty}\hat{R}_1^{A_m}=\hat{R}_1^A$.

LEMMA 6.2. *Let $\hat{R}_1^E$ be a potential. Then E is polar if and only if $c(E)=0$.*

PROOF. If E is polar, $\hat{R}_1^E\equiv 0$, hence $c(E)=0$.

Suppose $c(E)=0$. For every j there is an open set V_j containing E such that $c(V_j)<1/2^j$. Then $\sum R_1^{V_j}$ is superharmonic (Corollary 3.3 and Corollary 3.4) and is ∞ on E. This proves the Lemma.

We now show that E_1 satisfies (b) of Theorem 5.2. For each $\delta\in(0,1)$, E_δ is a $(2n-1)$-dimensional disc, hence is not polar. (Indeed if it were polar, the fact that a finite union of polar sets is polar would allow us to construct an open region F whose boundary was polar. But a superharmonic function which was ∞ on ∂F would be ∞ on F, contradicting (iv) of the definition of superharmonic.) Thus $c(E_\delta)>0$.

Let $\{t_k\}$ be a sequence increasing to 1. Put $z_k = (t_k, 0, \ldots, 0)$. Then $\varphi_{z_k}(E_\delta) \subset E_1$, $c(\varphi_{z_k}(E_\delta)) = c(E_\delta) > 0$ and since $\varphi_{z_k}(E_\delta)$ moves out to e as $k \to \infty$, we see E_1 satisfies (b) of Theorem 5.2.

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA 22030