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Trees as Brelot spaces

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Abstract

A Brelot space is a connected, locally compact, noncompact Hausdorff space together with the choice of a sheaf of functions on this space which are called *harmonic*. We prove that by considering functions on a tree to be functions on the edges as well as on the vertices (instead of just on the vertices), a tree becomes a Brelot space. This leads to many results on the potential theory of trees. By restricting the functions just to the vertices, we obtain several new results on the potential theory of trees considered in the usual sense. We study trees whose nearest-neighbor transition probabilities are defined by both transient and recurrent random walks. Besides the usual case of harmonic functions on trees (the kernel of the Laplace operator), we also consider as "harmonic" the eigenfunctions of the Laplacian relative to a positive eigenvalue showing that these also yield a Brelot structure and creating new classes of functions for the study of potential theory on trees.

Keywords: Harmonic space; Superharmonic; Flux

1. Introduction

In recent years there has been considerable attention to discretizations of many classical problems in harmonic analysis, potential theory, and geometry (e.g., see [10–12,14,17,20, 22]). While trying to answer several questions from potential theory in the environment of

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trees, we became aware that trees, identified with their set of vertices, behave like Brelot spaces, but miss one fundamental property: connectedness. In this article, we show that viewing them as connected graphs, that is, their elements are either vertices or points on the edges, trees can be endowed with a metric structure for which all topological requirements of Brelot spaces hold. In addition, by extending harmonic functions on trees (intended as functions on the vertices) along the edges linearly, we obtain a class of harmonic functions satisfying the three axioms of Brelot spaces. Using the tools of Brelot theory, we then derive many properties that hold on trees that were not previously known and relate these results to the potential theoretic aspects of trees that had been developed by Cartier in [11] under the more restrictive assumption that the transition probabilities be transient.

Before giving the specific findings of this research, we give a brief overview on trees and on Brelot spaces.

1.1. Trees

A *tree* is a locally finite connected graph with no loops, which, as a set, is identified with the collection of its vertices. Two vertices v and w of a tree are called *neighbors* if there is an edge connecting them, in which case we use the notation $v \sim w$. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \ldots]$ such that $v_k \sim v_{k+1}$. A *geodesic path* is a path $[v_0, v_1, \ldots]$ such that $v_{k-1} \neq v_{k+1}$ for all k. An infinite geodesic path is also called a *ray*. If u and v are any vertices, we denote by $[u, v]$ the unique geodesic path joining them. A vertex is said to be *terminal* if it has a single neighbor.

Definition 1.1. Given a finite subtree S of T, the *interior* of S is the set $\mathring{S}$ consisting of all vertices $v \in S$ such that every vertex of T which is a neighbor of v belongs to S. The *boundary* of S in T is defined as the set ∂S of all vertices $v \in S$ such that exactly one neighbor $\tilde{v}$ of v is in S. We say that S is a *complete* subtree of T if $S = \mathring{S} \cup \partial S$.

A tree T may be endowed with a metric d as follows. If u, v are vertices, $d(u, v)$ is the number of edges in the unique geodesic path from u to v. Given a root e, the *length* of a vertex v is defined as $|v| = d(e, v)$.

Given two neighboring vertices v, w, the *sector* determined by the edge $[v, w]$ is defined as

$$S(v, w) = \{u \in T\colon\ w \text{ is in the geodesic path joining } v \text{ to } u\}.$$

Fixing e as a root of the tree, the *predecessor* u^- of a vertex u, with $u \neq e$, is the next to the last vertex of the geodesic path from e to u. An *ancestor* of u is any vertex in the geodesic path from e to u^-. A *descendant* of a vertex v is a vertex w such that v is an ancestor of w. We call *children* of a vertex v the vertices u such that $u^- = v$. We call *siblings* vertices with the same predecessor.

The *boundary* ∂T of T is the set of equivalence classes of rays under the relation $\simeq$ defined by the shift, $[v_0, v_1, \ldots] \simeq [v_1, v_2, \ldots]$, together with the set of terminal vertices. For any nonterminal vertex u, we denote by $[u, \omega)$ the (unique) ray starting at u in the class ω, or the geodesic path from u to ω if ω is a terminal vertex. Then ∂T can be identified with the set of rays starting at u together with the terminal vertices. Furthermore, ∂T is a compact space under the topology generated by the sets

$$I_v = \{\omega \in \partial T\colon\ v \in [e, \omega)\}$$

which we call *intervals*.

We now define a metric on T whose completion is $T \cup \partial T$. If $v \sim w$, define $\rho(v, w) = 1/m^2$ where $m = \max\{|v|, |w|\}$. If v and w are any vertices and $[v, w]$ is the geodesic path $[v_1, \ldots, v_N]$, with $v_k \sim v_{k+1}$, let

$$\rho(v, w) = \sum_{k=1}^{N} \rho(v_k, v_{k+1}).$$

(Of course, $\rho(v, v) = 0$ for each $v \in T$.) Observe that $\rho(v, w) < \pi^2/3$. The completion of T with respect to this metric is $T \cup \partial T$, which is sequentially compact and hence compact, as a completion of a bounded countable space. On ∂T this metric topology is the same as the topology defined earlier. For an alternate proof of the compactness of $T \cup \partial T$ see [11].

A *distribution* is a finitely additive complex measure on finite unions of the sets I_v. Let us denote by $\mathcal{D}$ the space of finite-valued distributions on ∂T.

Each $\omega \in \partial T$ induces an orientation on the edges of T: $[u, v]$ is *positively oriented* if $v \in [u, \omega)$. For $\omega \in \partial T$, and $v \in T$, define the *horocycle index* $k_\omega(v)$ as the number of positively oriented edges minus the number of negatively oriented edges in the geodesic path from e to v.

Given a tree T, let p be a *nearest-neighbor transition probability* on the vertices of T, that is, $p(v, u) > 0$, if v and u are neighbors, $p(v, u) = 0$, if v and u are not neighbors. It is convenient to set $p(v, v) = -1$, so that for each vertex v, we have $\sum_u p(v, u) = 0$.

Two trees T and T' with transition probabilities p and p', respectively, are said to be *isomorphic* if there exists a bijection φ from the vertices of T to the vertices of T' such that $p(\varphi(v), \varphi(u)) = p(v, u)$ for all $v, u \in T$.

As is customary, a function on a tree T will mean a function on its set of vertices. The *Laplacian* of a function $f : T \to \mathbb{C}$ is defined as

$$\Delta f(v) = \sum_{u \in T} p(v, u) f(u) \qquad \text{for all nonterminal vertices } v \in T.$$

Definition 1.2. A function f on T is said to be *harmonic* at v if $\Delta f(v) = 0$. A real-valued function s on T is said to be *superharmonic* (respectively, *subharmonic*) at v if $\Delta s(v) \leqslant 0$ (respectively, $\Delta s(v) \geqslant 0$). A *potential* is a positive superharmonic function which does not have any positive harmonic minorants. A superharmonic function s on T is said to be *admissible* if there is a finite set K and a harmonic function h on $T \setminus K$ such that $h(x) \leqslant s(x)$ for all $x \notin K$.

A harmonic function defined off a finite set of vertices does not necessarily extend to a harmonic function on the whole tree as the following example shows.

Let $v_0 \in T$ with $|v_0| = 1$ and define h on T except e by letting $h(v) = 1$ if v is a descendant of v_0 or $v = v_0$ and $h(v) = 0$ otherwise. Then h is harmonic except at the set of all v with $|v| \leqslant 1$, but in order for h to be harmonic at v_0, $h(e)$ would have to be 1. For h to be harmonic at $u \neq v_0$, $|u| = 1$, $h(e)$ would have to be 0. Thus h cannot be extended to a harmonic function on T.

By a *homogeneous tree* of degree $q + 1$ (with $q \in \mathbb{N}$) we mean a tree T all of whose vertices have $q + 1$ neighbors and, unless otherwise specified, whose associated nearest-neighbor transition probability is $p(v, u) = 1/(q + 1)$ if v and u are neighbors. If T is homogenous of degree $q + 1$, the Poisson kernel is then given by

$$P_\omega(v) = q^{k_\omega(v)} \quad \text{for } v \in T,\ \omega \in \partial T,$$

since it satisfies the following properties analogous to those that hold in the classical case [22]:

(i) For any $\omega \in \partial T$, $v \mapsto P_\omega(v)$ is a harmonic function on T.
(ii) If $\mu \in \mathcal{D}$, then the function defined by the Poisson integral

$$f(v) = \int_{\partial T} P_\omega(v)\, \mathrm{d}\mu(\omega)$$

is well-defined and harmonic on T. Conversely, every harmonic function f on T has such an integral representation for some unique $\mu \in \mathcal{D}$.

Let T be a tree with a nearest neighbor transition probability p. If $\gamma = [v_0, \ldots, v_n]$ is a path, set $p(\gamma) = \prod_{j=1}^n p(v_{j-1}, v_j)$. For $v, w \in T$, let $\Gamma_{v,w}$ be the set of all finite paths from v to w, and let $\Gamma'_{v,w}$ be the set of finite paths of positive length from v to w that visit w after the first step only once, that is,

$$\Gamma'_{v,w} = \bigl\{[v_0, \ldots, v_n] \in \Gamma_{v,w}\colon\ v_j \neq w \text{ for } 0 < j < n,\ n \geqslant 1\bigr\}.$$

Define the Green function G of T as $G(v, w) = \sum_{\gamma \in \Gamma_{v,w}} p(\gamma)$, and the function $F(v, w) = \sum_{\gamma \in \Gamma'_{v,w}} p(\gamma)$. Probabilistically, $G(v, w)$ is the expected number of times the associated random walk starting at v visits w, and $F(v, w)$ is the probability that a random walk starting at v will ever reach w in positive time. In [11] it is shown that if $G(v, w)$ is finite for some vertices v and w, then it is finite for all pairs of vertices in T. This means that the associated random walk is transient. This is equivalent to saying that F restricted to the diagonal of $T \times T$ is always less than 1. If G is infinite, the random walk is recurrent, since F is identically 1. It is well known (see Appendix by Picardello and Woess of [18]) that if there exists $\delta > 0$ such that $\delta < p(v, w) < \frac{1}{2} - \delta$ for all $v \sim w$, then the random walk is transient.

Proposition 1.1 [11]. *Let v, w be distinct vertices and let $[v_0, \ldots, v_n]$ be the geodesic path from $v = v_0$ to $w = v_n$. Then*

(a) $G(v,w) = F(v,w)G(w,w)$;
(b) $G(v,v) = 1/(1 - F(v,v))$;
(c) $F(v,w) = \prod_{k=0}^{n-1} F(v_k, v_{k+1})$;
(d) $F(v,v) = \sum_{u\sim v} p(v,u)F(u,v)$.

Proposition 1.2 [15]. (a) *If s is positive superharmonic on a sector $S(v,w)$, where $v, w \in T$ are neighboring vertices, then*

$$F(w,v)s(v) \leqslant s(w).$$

(b) *If s is positive superharmonic on T, then for any pair of neighboring vertices v and w,*

$$F(v,w) \leqslant \frac{s(v)}{s(w)} \leqslant \frac{1}{F(w,v)}.$$

In recent years, eigenfunctions of the Laplacian on homogeneous trees have been studied (cf. [12,20]). Let T be a tree and let μ_1 denote the averaging operator of the vertices at distance 1, that is

$$\mu_1 f(v) = \sum_{w\sim v} p(v,w) f(w).$$

Observation 1.1. As an operator on any Banach space of functions, μ_1 has norm less than or equal to 1.

Definition 1.3. Let T be a tree and λ be a nonzero complex number. A function f on T is said to be λ-*harmonic* (respectively, λ-*superharmonic*) if $\mu_1 f = \lambda f$ (respectively, $\mu_1 f \leqslant \lambda f$).

Thus the λ-harmonic functions are the eigenfunctions of the Laplacian relative to the eigenvalue $\lambda - 1$. In particular, the ordinary harmonic functions are the 1-harmonic functions.

In [20], the authors showed that in the homogeneous case the correspondence

$$f \mapsto \int_{\partial T} P_\omega(v)^\zeta \,\mathrm{d}\mu(\omega)$$

maps harmonic functions to λ-harmonic functions, where $P_\omega(v) = P_\omega(e,v)$ and $\lambda = (q^\zeta + q^{\zeta-1})/(q+1)$. In [12], a different operator from the space of harmonic functions to the space of λ-harmonic functions, local in nature, was introduced and studied.

A λ-harmonic function of considerable interest on homogeneous trees is given by the *spherical function* φ_λ (cf. [14]). It is the only radial λ-harmonic function (that is, the value at a vertex v depends only on $|v|$) satisfying the condition $\varphi_\lambda(e) = 1$.

The values of φ_λ are given by $\varphi_\lambda(v) = q_n(\lambda)$ for any v of length n, where $q_n(\lambda)$ is the polynomial of degree n in λ satisfying the recursive relation

$$q_{n+1}(\lambda) = \frac{q+1}{q}\lambda q_n(\lambda) - \frac{1}{q}q_{n-1}(\lambda), \quad \text{for } n \geqslant 1, \tag{1}$$

with the initial conditions $q_0(\lambda) = 1$ and $q_1(\lambda) = \lambda$ (cf. [14]). If $\lambda > 2\sqrt{q}/(q+1)$, then the roots of the quadratic equation corresponding to the recurrence relation are both positive and given by

$$\frac{(q+1)\lambda \pm \sqrt{(q+1)^2\lambda^2 - 4q}}{2q}.$$

Let α denote the smaller one and let β denote the other. Thus

$$\varphi_\lambda(v) = q_n(\lambda) = A\beta^n + B\alpha^n, \tag{2}$$

where $A = (\lambda - \alpha)/(\beta - \alpha)$, $B = (\beta - \lambda)/(\beta - \alpha)$. Therefore $\varphi_\lambda(v) > 0$ for all $\lambda > 2\sqrt{q}/(q+1)$, and hence for all $\lambda > 1$.

1.2. Brelot spaces

The field of potential theory goes back to the nineteenth century, with the work following the research done by Gauss in 1840. Since then, many axiomatic treatments of the theory have been formulated. For a survey of the different developments of potential theory and a historical context, see [8]. We shall now give the main outline of the axiomatic theory of harmonic and superharmonic functions developed by Brelot (see [7]).

Definition 1.4. A *Brelot space* is a connected locally compact but not compact Hausdorff space Ω together with a harmonic structure in the following sense. For each open set $U \subset \Omega$ there is an associated real vector space of real-valued continuous functions on U (which are called *harmonic functions on U*) satisfying the following three axioms.

Axiom 1. (i) *If U_0 is an open subset of U, the restriction to U_0 of any function harmonic on U is harmonic on U_0.*

(ii) *A function defined on an open set U which is harmonic on an open neighborhood of each point of U is harmonic on U.*

Definition 1.5. An open set U is called *regular* if it is relatively compact in Ω and for any real-valued continuous function f on ∂U, there exists a unique harmonic function h_f^U on U approaching f at each point of ∂U. Furthermore, h_f^U is nonnegative whenever f is nonnegative.

Axiom 2. *There exists a base of regular domains for the open sets of Ω.*

In particular, Ω is locally connected.

Axiom 3 (Harnack's property). *Any increasing directed family of harmonic functions defined on a domain U has upper envelope* (*supremum*) *which is either identically $+\infty$ or is harmonic on U.*

Remark 1.1. If Ω is second countable, Axiom 3 is equivalent to the corresponding statement for increasing sequences rather than directed families.

Definition 1.6. Let Ω be a Brelot space, U an open subset of Ω, $x_0 \in \partial U$. A *barrier for U at x_0* is a positive harmonic function h defined in the intersection of U and an open neighborhood of x_0 such that

$$\lim_{x \in U,\, x \to x_0} h(x) = 0.$$

If such a barrier exists, we say that x_0 is a *regular boundary point* of U.

Definition 1.7. A compact subset K of a Brelot space Ω is *outer regular* if every point of ∂K has a barrier for $\Omega \setminus K$.

Definition 1.8. Given a regular open set U, for any $x \in U$, the map $f \mapsto h_f^U(x)$ is a positive linear functional on the space of the continuous functions on ∂U. By the Riesz representation theorem, there exists a positive Radon measure ρ_x^U on ∂U, called *harmonic measure relative to U and x* such that

$$h_f^U(x) = \int_{\partial U} f \, \mathrm{d}\rho_x^U.$$

Let Ω be a Brelot space and let U be a regular domain in Ω. Assume g is a lower semi-continuous function bounded below on ∂U. Then, for each $x \in U$ define

$$\int_{\partial U} g \, \mathrm{d}\rho_x^U = \sup \int_{\partial U} f \, \mathrm{d}\rho_x^U,$$

where the supremum is taken over all continuous functions f on ∂U such that $f \leqslant g$. By Axiom 3, $x \mapsto \int g \, \mathrm{d}\rho_x^U$ is either harmonic or identically $+\infty$.

Definition 1.9. Let U_0 be an open subset of a Brelot space Ω. A function $s : U_0 \to (-\infty, \infty]$, is said to be *superharmonic* if

(1) it is lower semi-continuous;
(2) for any regular domain U with closure contained in U_0,

$$s(x) \geqslant \int_{\partial U} s \, \mathrm{d}\rho_x^U \quad \text{for each } x \in U;$$

(3) s is not identically ∞ on any connected component of U_0.

The *harmonic support* of a superharmonic function s is the complement of the largest open set where s is harmonic.

Observation 1.2. Condition (2) of Definition 1.9 says that $s \geqslant h^U_{s|\partial U}$ on any regular domain U whose closure is contained in U_0 provided that $s|\partial U$ is continuous.

Definition 1.10. A subset A of a Brelot space Ω is said to be a *polar set* if there exists a positive superharmonic function on Ω whose restriction to A is identically ∞. A set A is *locally polar* if there exists a superharmonic function on Ω which is identically ∞ on A.

In a Brelot space, the minimum principle for superharmonic functions holds:

Theorem 1.1 [7, p. 71]. *A nonnegative superharmonic function on a domain U in a Brelot space is either identically zero or positive everywhere on U.*

Any nonnegative superharmonic function which has a harmonic minorant has a greatest harmonic minorant (see [7, p. 87]).

Definition 1.11. A superharmonic function s on a Brelot space Ω is said to be *admissible* if there is a compact set K and a harmonic function h on $\Omega \backslash K$ such that $h(x) \leqslant s(x)$ for all $x \notin K$.

Clearly, positive superharmonic functions and superharmonic functions of compact harmonic support are admissible.

Definition 1.12. A nonnegative superharmonic function on an open subset U of a Brelot space is called a *positive potential* (or briefly, a *potential*) if its greatest harmonic minorant on U is identically zero.

Definition 1.13. A *BH space* is a Brelot space whose sheaf of harmonic functions contains the constants. A *BP space* is a BH space on which there is a positive potential. A BH space on which no positive potentials exist is called a *BS space*.

Definition 1.14. A BP space is said to satisfy the *axiom of proportionality* if any two potentials with the same one-point harmonic support are proportional.

Theorem 1.2 [13, p. 139]. *In a Brelot space without potentials all positive superharmonic functions are harmonic and proportional. In particular, in a BS space, every positive superharmonic function must be constant.*

Thus, a Brelot space which possesses positive superharmonic functions which are not harmonic, has potentials.

Theorem 1.3 [3, p. 66]. *In a BP space a superharmonic function s is admissible if and only if there exist a potential p and a harmonic function h on the whole space such that $s = p + h$.*

Theorem 1.4 ([16, Theorem 16.1] and [1, Theorem 3.6]). *If Ω is a Brelot space with positive potentials and a countable base of neighborhoods or a BS space, then for any $x \in \Omega$, there exists a superharmonic function with harmonic support at $\{x\}$.*

It should be pointed out that both of these results assume an additional condition known as Axiom 3′. It was subsequently shown by Mokobodski, Loeb and Walsh that Axiom 3′ holds in any Brelot space (see [9]).

As a consequence of Theorems 1.4 and 1.3, in a Brelot space Ω with potentials and a countable base of neighborhoods, for any $x \in \Omega$ there exists a potential with harmonic support at $\{x\}$.

1.3. Outline of results

In Section 2 we shall consider functions on trees to be defined both on vertices and on edges, and in Theorem 2.1 we shall show that infinite trees then have a harmonic structure in the sense of Brelot. Since a finite tree is compact, it is not even a candidate to be a Brelot space. We shall refer to the properties of functions defined only at vertices as being in the sense of Cartier. Notationally we let $f : T \to \mathbb{R}$ be a function only on the vertices and $g : \widetilde{T} \to \mathbb{R}$ be a function on the simplicial complex (that is, on the vertices and edges). Any f may be extended (e.g., linearly) to a g and any g may be restricted to an f.

Harmonic (respectively, superharmonic) functions on a tree in the Cartier sense can be extended to harmonic (respectively, superharmonic) functions in the Brelot sense. For harmonic functions, this extension (linear) is unique, but for superharmonic functions there are nonlinear extensions (Proposition 2.2). Conversely, harmonic (superharmonic) functions in the Brelot sense restrict to harmonic (superharmonic) functions in the Cartier sense. We show (Proposition 2.3) that superharmonic functions are necessarily finite-valued and continuous. Moreover (Theorem 2.2), any superharmonic function in the Brelot sense whose restriction to the set of vertices is harmonic in the Cartier sense must be harmonic in the Brelot sense.

In Section 3, we study the trees which under the harmonic structure of Section 2 are BP spaces. These are the trees for which the random walk of the transition probabilities is transient. The restriction to the vertices of a potential in the Brelot sense is a potential in the Cartier sense. Conversely, the linear extension of a potential in the Cartier sense is a potential in the Brelot sense (Proposition 3.2). In Theorem 3.2, we prove that corresponding to each function u which is harmonic in the Cartier sense outside a finite set of vertices there exists a harmonic function h on the entire tree such that $u - h$ is bounded. As a consequence, we show (Corollary 3.1) that a superharmonic function in the Cartier sense on a BP tree T is admissible if and only if it has a harmonic minorant on T. Moreover, we obtain (Corollary 3.2) a characterization of admissible superharmonic functions on T analogous to one proved by Cartier (Theorem 3.1) for positive superharmonic functions. We use these results to give an example of a nonadmissible superharmonic function.

In Theorem 3.3 we use the Green function introduced by Cartier to construct the potentials (in the Brelot sense) of point harmonic support. We deduce that the axiom of proportionality holds for BP trees. In Proposition 3.4, we give an integral representation of potentials in the spirit of Hervé. In Corollary 3.3, we show that given any tree, the Green potential of a measure is either identically infinity, or is finite everywhere. In Theorem 3.5, we show that the ratio of the values of any Green function G_x evaluated at any two points is bounded away from zero with lower bound depending only on the two points and not on x.

In Section 4, after defining the flux at infinity of a harmonic function and recalling many results that hold on general BS spaces, we interpret these on BS trees. Our work on BS spaces, as well as some of our work on BP spaces, is influenced by the approach of V. Anandam (see [1–5]).

In Proposition 4.2, we construct an unbounded function H harmonic except at $\{e\}$, constant on siblings, such that $H(e) = 0$ and $\Delta H(e) = 1$. In Corollary 4.1, corresponding to a (Cartier) harmonic function f on the complement of a complete finite set of vertices K, we give the explicit constructions of the unique number α and a harmonic function h on T such that $f - h - \alpha H$ is bounded off K. The constant α is called the *flux of f at infinity* (with respect to H). In Theorem 4.4, we give an explicit formula for calculating the flux at infinity of such a function f. We then show that (Theorem 4.5) every positive superharmonic function on the complement of a finite set K of vertices is increasing along each ray in its domain. Furthermore, if the function takes on the same value at two neighboring vertices, then it must be constant on the sector determined by the corresponding edge.

In Section 5, we a give a condition for a random walk on a tree to be transient in terms of a certain function on the boundary. Specifically, in Theorem 5.1 we show that if the random walk is transient, then this function is finite-valued somewhere. On the other hand (Theorem 5.2), if this function is finite on an interval, then the random walk is transient. We provide an example of a tree for which this function is finite at one boundary point, yet, the random walk is recurrent.

In Section 6, we give other harmonic structures on trees by replacing the Laplacian operator with the operator $L = \Delta - a^2 I$ $(a > 0)$. These structures yield Brelot spaces whose sheaf of harmonic functions does not contain the constants. They are, however, always endowed with potentials. As in Section 3, (Proposition 6.1) harmonic and superharmonic functions, interpreted as eigenfunctions of the Laplacian relative to the eigenvalue a^2, can be extended to harmonic (respectively, superharmonic) functions in the Brelot sense. Furthermore, in Proposition 6.2 we show that superharmonic functions are always finite-valued. In Proposition 6.4 and Observation 6.2, we construct the potentials of harmonic point support on homogeneous trees. Finally, (Proposition 6.6) we give a formula for the Green function on any tree (which turns out to be finite everywhere), and (Theorem 6.1) show that the axiom of proportionality holds.

2. The Brelot structure on a tree

Notation. Let T be a tree with infinitely many vertices. Consider the space $\widetilde{T}$ which is the tree viewed as a 1-dimensional simplicial complex, with the terminal vertices removed. That is, for all u, v nonterminal vertices with $u \sim v$, consider the set $[u, v] = \{(1-t)u + tv\colon 0 \leqslant t \leqslant 1\}$. If v is a terminal vertex and $u \sim v$, then set $[u, v] = \{(1-t)u + tv\colon\ 0 \leqslant t < 1\}$.

Then $\widetilde{T} = \bigcup_{u \sim v} [u, v]$. Of course, we identify $(1-t)u + tv$ with $tv + (1-t)u$ and $0u + v$ with v.

We now state and prove our main result.

Theorem 2.1. *$\widetilde{T}$ may be given the structure of a Brelot space.*

Proof. Put a metric on $\widetilde{T}$ by extending the metric d on T as follows. Let $u, v, u', v' \in T$, with $u \sim v$ and $u' \sim v'$, and let $x = (1-t)u + tv$, $y = (1-t')u' + t'v'$, with $0 \leqslant t \leqslant 1$, $0 \leqslant t' \leqslant 1$. If $[u, v] = [u', v']$, we may assume that $u = u', v = v'$ and then define $d(x, y) = |t - t'|$. If the edges are different, assume that u and u' are the nearest of the four pairs (u, u'), (u, v'), (v, u'), (v, v'), then define

$$d(x, y) = t + d(u, u') + t'.$$

Under this metric, $\widetilde{T}$ is a locally compact, but not compact, connected and locally connected Hausdorff space. In addition, $\widetilde{T}$ has a countable base.

We now define harmonicity on $\widetilde{T}$.

Let U be an open subset of $\widetilde{T}$, f a function on U, and $x \in U$. If $x \notin T$, then $x = (1-t_0)u + t_0 v$ for some $u, v \in T$, $0 < t_0 < 1$. Then we define f to be harmonic at x if there exist $a, b \in \mathbb{R}$ and $\epsilon > 0$ such that $f((1-t)u + tv) = (1-t)a + tb$ for all t, with $|t - t_0| < \epsilon$. If $x = v \in T$, v not a terminal vertex, then f is harmonic at v if there exists $\epsilon > 0$ such that for all $t \in (0, \epsilon)$,

$$f(v) = \sum_{u \sim v} p(v, u) f\big((1-t)v + tu\big).$$

We say that a function f continuous on U is harmonic on U if f is harmonic at each $x \in U$. Since harmonicity is defined locally, the first axiom of Brelot is satisfied.

A harmonic function f on T can be extended to a harmonic function on $\widetilde{T}$ by linearity, i.e., $f((1-t)v + tw) = (1-t)f(v) + tf(w)$, for all $v \sim w$, $0 < t < 1$.

We now show that there is a base of regular domains. For $x \in \widetilde{T}$, consider the sets $B_\epsilon(x) = \{y \in \widetilde{T}\colon d(x, y) < \epsilon\}$, where if $x = v \in T$, then $0 < \epsilon \leqslant 1$, and if $x = (1-t_0)u + t_0 v$, with $0 < t_0 < 1$, then $\epsilon \leqslant \min\{1 - t_0, t_0\}$. Let $x = (1-t_0)u + t_0 v$, with $u, v \in T$, $0 < t_0 < 1$. Then the boundary of $B_\epsilon(x)$ consists of the points $(1 - t_0 - \epsilon)u + (t_0 + \epsilon)v$ and $(1 - t_0 + \epsilon)u + (t_0 - \epsilon)v$. If f is defined on the boundary of $B_\epsilon(x)$, then f can be extended linearly (and hence, harmonically) inside. Next, let $x = v \in T$, where v is not terminal. Then for $0 < \epsilon < 1$, the boundary of $B_\epsilon(x)$ is the set of points $\{(1-\epsilon)v + \epsilon u\colon$ for all neighbors u of $v\}$. If f is defined on $\partial B_\epsilon(x)$, then let

$$f(v) = \sum_{u \sim v} p(v, u) f\big((1-\epsilon)v + \epsilon u\big)$$

and extend f linearly inside. Thus, $B_\epsilon(x)$, $0 < \epsilon \leqslant 1$, is regular and the second axiom of Brelot is satisfied.

Next we show that Harnack's property holds. Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of harmonic functions on some connected open set $U \subset \widetilde{T}$ and let

$$\lim_{n\to\infty} f_n(x) = f(x) \leqslant \infty.$$

Furthermore, let us assume $f(x_0) < \infty$, for some $x_0 \in U$.

Claim 1. *If $x_0 \in (u, v)$, for some neighboring vertices u, v, then f is finite and harmonic on $[u, v] \cap U$.*

For, let $x \in [u, v] \cap U$, $x \neq x_0$. Let y be a point of $[u, v] \cap U$ on the opposite side of x_0 from x. Then there exists some $t \in (0, 1)$ such that $x_0 = (1 - t)x + ty$, so that $f_n(x_0) = (1-t)f_n(x) + tf_n(y)$. Since $\{f_n\}$ is increasing and $f(x_0) < \infty$, then $f(x) < \infty$. Also since the finite limit of linear functions is linear, f is harmonic on $[u, v] \cap U$.

Claim 2. *If $x_0 = u \in T$, then for all $v \sim u$, f is finite and harmonic on $[u, v] \cap U$.*

For, choose $\epsilon > 0$ such that $x_v = (1 - \epsilon)u + \epsilon v \in U$ for all $v \sim u$. Then

$$f_n(x_0) = \sum_{v\sim u} p(u, v) f_n(x_v).$$

As above, this implies that $f(x_v) < \infty$, and so by Claim 1, f is finite on $[u, v] \cap U$, and again by linearity, f is harmonic on $[u, v] \cap U$. By connectedness, f is finite and harmonic on U. Hence the third axiom is satisfied and $\widetilde{T}$ is a Brelot space. □

Observation 2.1. The harmonic functions on $\widetilde{T}$ correspond to the harmonic functions on T in the sense that their restriction to T are harmonic according to Definition 1.2, and conversely, every harmonic function on T (in the sense of Definition 1.2) extends linearly to a (unique) harmonic function on $\widetilde{T}$.

We now describe the harmonic measure on $\widetilde{T}$. Let $U = B_\epsilon(x_0)$ be a regular neighborhood of $x_0 \in \widetilde{T}$.

Case 1. If $x_0 \notin T$, then $x_0 = (1 - t_0)u + t_0 v$ $(0 < t_0 < 1)$, for some neighboring vertices u and v, and

$$\partial U = \bigl\{x_\pm = (1 - t_0 \pm \epsilon)u + (t_0 \mp \epsilon)v\bigr\}.$$

Then for any $x \in U$, $x = (1 - t)x_+ + tx_-$, for some $t \in (0, 1)$. Thus

$$\int f \,\mathrm{d}\rho_x^U = (1 - t) f(x_+) + t f(x_-)$$

for any function defined on ∂U.

Case 2. If $x_0 = v \in T$, then $\partial U = \{x_u = (1-\epsilon)v + \epsilon u \mid u \sim v\}$. If $x \in U$, then there is a neighbor u_0 of v such that

$$x = (1 - t\epsilon)v + t\epsilon u_0 = (1-t)v + t\big((1-\epsilon)v + \epsilon u_0\big), \quad \text{for some } 0 \leqslant t < 1,$$

so

$$\int f \, \mathrm{d}\rho_x^U = (1 - t\epsilon) \sum_{u \sim v} p(v,u) f(x_u) + t\epsilon f(x_{u_0}),$$

for any function defined on ∂U.

Observation 2.2. Any complete finite set of vertices on a tree T together with its edges is regular. Indeed, we can solve the Dirichlet problem on the interior vertices of any such set (cf. [10]) and then extend the unique solution linearly on the edges.

Next we study the superharmonic functions on $\widetilde{T}$.

Proposition 2.1. *Given a superharmonic function on T, its linear extension to $\widetilde{T}$ is superharmonic on $\widetilde{T}$.*

Proof. Let s be superharmonic on T (in the sense of Definition 1.2) and define $s((1-t)v + tu) = (1-t)s(v) + ts(u)$ for any pair of neighboring vertices v and u. On the interior of the edge $[u, v]$, s is linear hence harmonic and hence superharmonic. Furthermore, for each $\epsilon \in (0,1)$ and each vertex v,

$$\begin{aligned} \sum_{u \sim v} p(v,u) s\big((1-\epsilon)v + \epsilon u\big) &= \sum_{u \sim v} p(v,u)(1-\epsilon)s(v) + \sum_{u \sim v} p(v,u)\epsilon s(u) \\ &= (1-\epsilon)s(v) + \epsilon \sum_{u \sim v} p(v,u)s(u) \\ &\leqslant (1-\epsilon)s(v) + \epsilon s(v) = s(v). \qquad \square \end{aligned}$$

Linear extensions of superharmonic functions on T are the smallest superharmonic extensions to $\widetilde{T}$.

Proposition 2.2. *If s is a superharmonic function on T which is not harmonic, there are nonlinear superharmonic functions on $\widetilde{T}$ extending s.*

Proof. Let v be a vertex on T such that s is superharmonic but not harmonic at v. Let $L > 0$ be such that $s(v) = \sum_{u \sim v} p(v,u)s(u) + L$. Fix $u \sim v$. Define s linearly on $[v, w]$ for all $w \sim v$, $w \neq u$. On $[v, u] = \{u_t = (1-t)v + tu \colon 0 \leqslant t \leqslant 1\}$, let s be any nonlinear concave function starting at $s(v)$ and ending up at $s(u)$ and such that for any $t \in (0,1)$,

$$s(u_t) \leqslant (1-t)s(v) + ts(u) + \frac{L}{p(v,u)}.$$

Any such s is superharmonic at v. Thus s is superharmonic on $\widetilde{T}$, but nonlinear on the edge $[v,u]$. $\square$

Proposition 2.3. *If s is superharmonic on $\widetilde{T}$, then s is finite-valued and continuous. In particular, there are no polar sets in $\widetilde{T}$.*

Proof. First we show that s is finite-valued on all of $\widetilde{T}$. First, assume $s(v)=\infty$ for some $v \in T$. Then for each $u \sim v$ and each $t \in (0,1)$,

$$s\big((1-t)v+tu\big) \geqslant (1-t)s(v)+ts(u),$$

so s is identically infinity on $[v,u)$. On the other hand, $s(u) \geqslant \sum_{w\sim u} p(u,w)s(w)=\infty$, since v is a neighbor of u and $s(v)=\infty$. Thus, s is identically infinity on the whole edge $[v,u]$. By connectedness, $s=\infty$ on $\widetilde{T}$, contradicting superharmonicity.

Next, assume $s(x)=\infty$, for some x in the interior of the edge $[v,u]$. Then, for each $t \in (0,1)$, $s((1-t)v+tx) \geqslant (1-t)s(v)+ts(x)=\infty$. Similarly, $s((1-t)u+tx) \geqslant (1-t)s(u)+ts(x)=\infty$. So s is infinity on (v,u). Again by superharmonicity, for all ϵ sufficiently small, $s(v) \geqslant \sum_{w\sim v} p(v,w)s((1-\epsilon)v+\epsilon w)=\infty$. Thus, $s(v)=\infty$. By the first case, we get a contradiction, completing the proof of the finiteness.

Next, suppose s is not continuous at x, a point in the interior of the edge $[v,u]$. Since s is lower semi-continuous, $\liminf_{y\to x} s(y) \geqslant s(x)$. Then there exists some $\lambda \in \mathbb{R}$ such that

$$\limsup_{y\to x} s(y) > \lambda > s(x).$$

Thus there exists a sequence $\{y_n\}$ approaching x such that $s(y_n) \geqslant \lambda$ for all $n \in \mathbb{N}$. For y_n sufficiently close to x, let z_n be the symmetric point of y_n with respect to x in the interior of the edge $[v,u]$. Then $s(z_n)+s(y_n) \leqslant 2s(x)$. Thus

$$2s(x) < s(x)+\lambda \leqslant \liminf_{n\to\infty}\big[s(z_n)+s(y_n)\big] \leqslant 2s(x),$$

a contradiction. Hence s is must be continuous at x.

Now let us assume s is not continuous at $v \in T$. Then there is a neighbor u of v such that $\limsup_{\epsilon\to 0} s(u_\epsilon) > s(v)$, where $u_\epsilon=(1-\epsilon)v+\epsilon u$. Let $\{\epsilon_n\}$ be a sequence of positive numbers approaching 0 such that $s(u_{\epsilon_n}) \geqslant \lambda > s(v)$. Since s is superharmonic,

$$s(v) \geqslant \sum_{w\sim v} p(v,w)s(w_{\epsilon_n}),$$

and so

$$\begin{aligned} s(v) &\geqslant \liminf_{n\to\infty}\sum_{w\sim v} p(v,w)s(w_{\epsilon_n}) \geqslant \sum_{w\sim v} p(v,w)\liminf_{n\to\infty} s(w_{\epsilon_n}) \\ &\geqslant \sum_{w\sim v,\, w\neq u} p(v,w)s(v)+p(v,u)\lambda > s(v), \end{aligned}$$

a contradiction. Thus s is continuous at v, completing the proof. $\square$

Proposition 2.4. *If s is superharmonic on $\widetilde{T}$, then its restriction to T is superharmonic on T.*

Proof. Given $v \in T$, since $B_1(v)$ is regular, we have

$$s(v) \geqslant \sum_{u \sim v} p(v,u)s(u).$$

Thus s is superharmonic on T. □

We wish to highlight the following result, whose proof is a simple application of the minimum principle for superharmonic functions.

Theorem 2.2. *If s is superharmonic on $\widetilde{T}$ and its restriction to T is harmonic on T, then s is harmonic on $\widetilde{T}$.*

Proof. Let h be the linear extension of s restricted to T. Since s is concave on each edge, $s \geqslant h$ on $\widetilde{T}$. Since h is harmonic on $\widetilde{T}$, $s - h$ is nonnegative superharmonic on $\widetilde{T}$. By Theorem 1.1, either $s - h$ is identically zero or positive everywhere. But $s = h$ on T. Thus $s = h$ on $\widetilde{T}$. □

In what follows we shall refer to a tree T as being a BP (respectively, BS) tree if $\widetilde{T}$ is a BP (respectively, BS) Brelot space.

3. Trees as BP spaces

Throughout this section we shall assume that T is a tree endowed with a nearest neighbor transition probability p whose associated random walk is transient (so that the Green function $G(v,w)$ is finite for each $v, w \in T$).

Proposition 3.1. *$\widetilde{T}$ is a BP space.*

In particular, an example of a BP space is a homogeneous tree of degree greater than 2.

Proof. Given $w \in T$, the function $G_w : T \to [0,\infty)$ defined by $G_w(v) = G(v,w)$ satisfies the conditions $\Delta G_w(v) = 0$ for $w \neq v$ and $\Delta G_w(w) = -1$ (see [11, Proposition 2.3]). Therefore, G_w is positive superharmonic on T, nonconstant, and harmonic on $T \setminus \{w\}$. Extending G_w linearly on all edges yields a nonconstant positive superharmonic function on $\widetilde{T}$. By Theorem 1.2, $\widetilde{T}$ cannot be a BS space. □

Proposition 3.2. *A potential on $\widetilde{T}$ restricted to T is a potential on T. Conversely, the linear extension to $\widetilde{T}$ of a potential on T is a potential on $\widetilde{T}$.*

Proof. Let p be a potential on $\widetilde{T}$. Thus p, and hence $p|T$, is superharmonic by Proposition 2.4. If $h : T \to [0,\infty)$ is a harmonic minorant of $p|T$, then h can be extended by linearity to a unique nonnegative harmonic function on $\widetilde{T}$ which must be a harmonic minorant of p. Thus h must be identically 0. Therefore $p|T$ is a potential. The converse is clear. $\square$

Proposition 3.3. *If s is superharmonic on $\widetilde{T}$ and its restriction to T is a potential on T, then s is a potential on $\widetilde{T}$.*

Proof. Let h be a nonnegative harmonic minorant of s on $\widetilde{T}$. Then $h|T$ is a nonnegative harmonic minorant of $s|T$, so $h|T = 0$. Thus $h = 0$. $\square$

We shall see that there are potentials on $\widetilde{T}$ that are not linear extensions of potentials on T (for example, the potentials on $\widetilde{T}$ with harmonic point support in the interior of an edge).

If f is a function on T, define the *Green potential of f* on T by

$$Gf(v) = \sum_{u \in T} G(v, u) f(u),$$

if the series converges absolutely, diverges to ∞, or diverges to $-\infty$. Clearly, the harmonic support of Gf is exactly the support of f.

Theorem 3.1 [11]. *Every positive superharmonic function s on T is of the form $s = h + Gf$, where h is a nonnegative harmonic function and f is a nonnegative function on T with support equal to the harmonic support of s. Furthermore, this representation is unique.*

As a consequence, we see that every potential p on T is of the form Gf for a unique nonnegative function f with support equal to the harmonic support of p.

The following result was proved by Nakai in a general BP space and can be found in [21] or [2, Theorem 1.20]. We state it and prove it in the tree setting, where it does not follow trivially from known facts about trees.

Theorem 3.2. *Let u be a function on T which is harmonic outside a finite subset K of T. Then there exists a function h harmonic on T such that $h - u$ is bounded off K.*

Proof. Fix a root e. Choose $n \in \mathbb{N}$ such that K is a proper subset of B_n, the open ball of radius n: $B_n = \{v \in T\colon\ d(e, v) < n\}$. Define U on B_n as the solution to the Dirichlet problem with boundary values u on ∂B_n and define $U(v) = u(v)$ for $d(e, v) \geqslant n$. Let $h(v) = U(v) + \sum_{|w|=n} \Delta U(w) G(v, w)$. Then h is harmonic on T. Recalling (a) of Proposition 1.1, we have

$$\bigl|h(v) - U(v)\bigr| \leqslant \sum_{|w|=n} \bigl|\Delta U(w)\bigr| G(w, w),$$

for any $v \in T$. Thus $h - u$ is bounded off K. $\square$

Corollary 3.1. *A superharmonic function on T is admissible if and only if it has a harmonic minorant on T.*

Proof. Let s be an admissible superharmonic function on T. Then there exists a function u on T harmonic outside a finite set K such that $u(v) \leqslant s(v)$ for all $v \in T \backslash K$. By Theorem 3.2, there exists h harmonic on T such that $|h - u|$ is bounded on T. Thus $h \leqslant u + c$ for some positive constant c, whence $h - c$ is a harmonic minorant of u on T. The converse follows immediately from the definition of admissible superharmonic function. $\square$

We can now generalize Theorem 3.1 as follows.

Corollary 3.2. *A superharmonic function s which is not harmonic on T is admissible if and only if there exist a potential p and a harmonic function h on T such that $s = p + h$ on T.*

Proof. Let s be an admissible superharmonic function on T, s not harmonic. Then s has a harmonic minorant on T. Let h be the greatest harmonic minorant of s. Thus $s - h$ is a positive superharmonic function whose greatest harmonic minorant is 0, so $s - h$ is a potential. Conversely, if $s = p + h$ with p a potential and h harmonic, then h is a harmonic minorant of s. Thus s is an admissible superharmonic function which is not harmonic. $\square$

Example 3.1. Let T be a homogeneous tree of degree $q + 1$ ($q \geqslant 2$) with root e. Then $s(v) = -|v|$ is a nonadmissible superharmonic function on T. To see this, observe that $\Delta s(e) = -1$ and $\Delta s(v) = -(q - 1)/(q + 1)$ for $v \neq e$, so s is superharmonic. If s were admissible, then by Corollary 3.1, s would have a harmonic minorant h on T. Thus, $h(v) \leqslant -|v|$ for each $v \in T$. But since h is harmonic, for any positive integer n, $h(e)$ is the average of the values of h at the vertices of length n. Thus $h(e) \leqslant -n$, for all $n \in \mathbb{N}$, a contradiction.

We now give the construction of the greatest harmonic minorant of an admissible superharmonic function s on T. For each $n \in \mathbb{N}$, let B_n denote the open ball centered at e of radius n and let h_n be the solution to the Dirichlet problem in B_n with boundary values $s|\partial B_n$. Since B_n is regular by Observation 2.2, by (2) of Definition 1.9, $s \geqslant h_n$ on B_n for all $n \in \mathbb{N}$. Thus, using the notation of Definition 1.8, we obtain

$$h_n = h^{B_n}_{s|\partial B_n} \geqslant h^{B_n}_{h_{n+1}|\partial B_n} = h_{n+1},$$

since h_{n+1} is harmonic on the closure of B_n and $s \geqslant h_{n+1}$ on ∂B_n. Hence $D(s) = \lim_{n\to\infty} h_n$ exists and is harmonic on T. $D(s)$ is the greatest harmonic minorant of s.

We now construct potentials with harmonic point support on $\widetilde{T}$ and show that all potentials with the same harmonic point support on $\widetilde{T}$ are proportional, that is, $\widetilde{T}$ satisfies the axiom of proportionality.

Theorem 3.3. *Let $x \in \widetilde{T}$. If $x \in T$, then the potentials supported on $\{x\}$ are positive multiples of the linear extension of G_x, where G_x is the Green function $G_v(u) = G(u, v)$ if $x = v \in T$. If $x \notin T$, $x = (1-t)v + tw$, with v, w neighboring vertices, $0 < t < 1$, then G_x is defined by*

$$G_x(y) = \begin{cases} \gamma_t\big[p(v,w)(1-t)G_v(y) + p(w,v)tG_w(y)\big] & \\ & \text{if } y \notin (v,w), \\ (1-s)G_x(v) + sG_x(w) + s(1-t)\gamma_t & \text{if } y = (1-s)v + sw,\ 0 \leqslant s \leqslant t, \\ (1-s)G_x(v) + sG_x(w) + (1-s)t\gamma_t & \text{if } y = (1-s)v + sw,\ t \leqslant s \leqslant 1, \end{cases}$$

where $\gamma_t = (1-t)/(p(v,w)) + t/(p(w,v))$.

Proof. As remarked above, if $x = v \in T$, then the function G_v is superharmonic on T, harmonic on $T\backslash\{v\}$, and its greatest harmonic minorant on T is zero. Thus its linear extension on $\widetilde{T}$ is a potential with harmonic support at v. If p is a potential on $\widetilde{T}$ with harmonic support at v, then by Proposition 3.2 and Theorem 3.1 applied to $p|T$, $p = Gf$, for some nonnegative function f with support $\{v\}$. Thus $p = f(v)G_v$ on T. By Observation 2.1, $p = f(v)G_v$ on $\widetilde{T}$.

Let us now assume $x \notin T$, so that $x = (1-t)v + tw$, where $v \sim w$ and $t \in (0,1)$. Clearly, G_x is harmonic off $[v,w]$ and, by linearity, it is also harmonic on the segments (v,x) and (x,w). Furthermore, G_x is superharmonic at x, since for $\epsilon > 0$ sufficiently small, the average between $G_x((1-t+\epsilon)v + (t-\epsilon)w)$ and $G_x((1-t-\epsilon)v + (t+\epsilon)w)$ is given by

$$(1-t)G_x(v) + tG_x(w) + t(1-t)\gamma_t - \frac{\epsilon}{2}\gamma_t = G_x(x) - \frac{\epsilon}{2}\gamma_t < G_x(x).$$

We now show that G_x is harmonic at v (hence, by symmetry, at w). Let $\widetilde{G}_x$ be the linear extension of $G_x|[v,x]$ to $[v,w]$. Thus

$$\widetilde{G}_x(w) = G_x(w) + (1-t)\gamma_t.$$

To prove that G_x is harmonic at v, we need to show that

$$\sum_{u\sim v,\, u\neq w} p(v,u)G_x(u) + p(v,w)\widetilde{G}_x(w) = G_x(v),$$

since harmonicity of G_x in an ϵ-neighborhood of v is equivalent to harmonicity at the nearest neighbors of its linear extension. Using $\Delta G_v(v) = -1$ and $\Delta G_w(v) = 0$, we obtain

$$\begin{aligned} &\sum_{u\sim v,\, u\neq w} p(v,u)G_x(u) + p(v,w)\widetilde{G}_x(w) \\ &\quad = \gamma_t p(v,w)(1-t)\sum_{u\sim v,\, u\neq w} p(v,u)G_v(u) + \gamma_t p(w,v)t\sum_{u\sim v,\, u\neq w} p(v,u)G_w(u) \\ &\qquad + \gamma_t p(v,w)\big[(1-t)p(v,w)G_v(w) + tp(w,v)G_w(w) + (1-t)\big] \end{aligned}$$

$$= \gamma_t \big[p(v,w)(1-t)\big(\Delta G_v(v) + G_v(v)\big) + p(w,v)t\big(\Delta G_w(v) + G_w(v)\big) + p(v,w)(1-t) \big]$$
$$= \gamma_t \big[p(v,w)(1-t)G_v(v) + p(w,v)tG_w(v) \big] = G_x(v).$$

Thus G_x is positive superharmonic on $\widetilde{T}$, harmonic off x and since it is defined as a positive linear combination of potentials off (v,w), its restriction to T is a potential on T. By Proposition 3.3, G_x is a potential with harmonic support at $\{x\}$.

Let us assume p_x is a potential with harmonic support at $\{x\}$, with $x \in (v,w)$, $x = (1-t)x + ty$, $0 < t < 1$. Thus $p_x|T$ must be harmonic on $T\backslash\{v,w\}$ and so it must be of the form $\alpha G_v + \beta G_w$ for some $\alpha, \beta \geqslant 0$. Let $\widetilde{G}_v(w)$ and $\widetilde{G}_w(v)$ be the numbers such that

$$G_v(v) = \sum_{u\sim v,\, u\neq w} p(v,u)G_v(u) + p(v,w)\widetilde{G}_v(w), \tag{3}$$

$$G_w(w) = \sum_{u\sim w,\, u\neq v} p(w,u)G_w(u) + p(w,v)\widetilde{G}_w(v). \tag{4}$$

Now p_x is harmonic on (v,w) except at x. Let us define $\widetilde{G}_v$ in a small neighborhood of v by

$$\widetilde{G}_v(z) = \begin{cases} G_v(z) & \text{for } z \in (u,v),\ u \sim v,\ u \neq w, \\ (1-s)G_v(v) + s\widetilde{G}_v(w) & \text{for } z = (1-s)v + sw. \end{cases}$$

Then $\widetilde{G}_v$ is harmonic near v and thus $p_x(y) = \alpha\widetilde{G}_v(y) + \beta G_w(y)$ for $y = (1-s)v + sw$, $0 \leqslant s \leqslant t$. Similarly, letting

$$\widetilde{G}_w(z) = \begin{cases} G_w(z) & \text{for } z \in (u,w),\ u \sim w,\ u \neq v, \\ (1-s)\widetilde{G}_w(v) + sG_w(w) & \text{for } z = (1-s)v + sw, \end{cases}$$

we get $p_x(y) = \alpha G_v(y) + \beta\widetilde{G}_w(y)$ for $y = (1-s)v + sw$, $t \leqslant s \leqslant 1$. In particular, $p_x(x)$ must agree with the values from both definitions. So

$$\alpha\big[(1-t)G_v(v) + t\widetilde{G}_v(w)\big] + \beta\big[(1-t)G_w(v) + tG_w(w)\big]$$
$$= \alpha\big[(1-t)G_v(v) + tG_v(w)\big] + \beta\big[(1-t)\widetilde{G}_w(v) + tG_w(w)\big].$$

Thus

$$\alpha t\big[\widetilde{G}_v(w) - G_v(w)\big] = \beta(1-t)\big[\widetilde{G}_w(v) - G_w(v)\big]. \tag{5}$$

Since $\Delta G_v(v) = -1$, using (3) we get

$$-1 = \sum_{u\sim v} p(v,u)G_v(u) - \sum_{u\sim v,\, u\neq w} p(v,u)G_v(u) - p(v,w)\widetilde{G}_v(w)$$
$$= p(v,w)G_v(w) - p(v,w)\widetilde{G}_v(w) = -p(v,w)\big[\widetilde{G}_v(w) - G_v(w)\big].$$

Thus $\widetilde{G}_v(w) - G_v(w) = 1/p(v,w)$. Similarly $\widetilde{G}_w(v) - G_w(v) = 1/p(w,v)$. Hence (5) becomes

$$\frac{\alpha t}{p(v,w)} = \frac{\beta(1-t)}{p(w,v)},$$

whence

$$\beta = \frac{\alpha t p(w,v)}{(1-t)p(v,w)}.$$

So off (v,w), we have

$$\begin{aligned} p_x &= \alpha G_v + \beta G_w = \alpha\left[G_v + \frac{tp(w,v)}{(1-t)p(v,w)}G_w\right] \\ &= \frac{\alpha}{(1-t)p(v,w)}\left[(1-t)p(v,w)G_v + tp(w,v)G_w\right] \\ &= \frac{\alpha}{(1-t)p(v,w)\gamma_t}G_x. \end{aligned}$$

By harmonicity on $(v,w) - \{x\}$ and continuity at x, we have that

$$p_x = \frac{\alpha}{(1-t)p(v,w)\gamma_t}G_x$$

everywhere on $\widetilde{T}$. $\square$

Definition 3.1. Let Ω be a Brelot space, and let U be a domain in Ω. A *Green function* on U, if it exists, is a function $G_U : U \times U \to (0,\infty]$ satisfying the conditions:

(a) G_U is lower semi-continuous on $U \times U$ and continuous on $U \times U \backslash \{(x,x)\colon x \in U\}$;
(b) For each $y \in U$, $x \mapsto G(x,y)$ is a potential with harmonic support at $\{y\}$.

Remark 3.1. If f is a positive continuous function on U and G_U is a Green function on U, then $G'_U(x,y) = f(y)G_U(x,y)$, $x, y \in U$, is also a Green function on U.

Theorem 3.4 [16, Theorem 18.2]. *Let Ω be a Brelot space satisfying the axiom of proportionality. Then there exists a Green function G on Ω and every potential P on Ω admits a unique integral representation of the form*

$$P(x) = G\mu(x) = \int_\Omega G(x,y)\,\mathrm{d}\mu(y), \quad x \in \Omega,$$

where μ is a nonnegative measure on Ω.

The function $G\mu$ is called *the Green potential of* μ.

Conversely (cf. [19, p. 68]), the function on U defined by the above integral is a potential if it is finite at one point.

As a consequence, by splitting the potentials with harmonic point support at the vertices from those with harmonic point support inside edges, we obtain

Proposition 3.4. *Every potential in $\widetilde{T}$ admits a unique integral representation of the form*

$$P = \sum_{v \in T} a_v G_v + \sum_{v \sim w} G\lambda_{v,w}, \tag{6}$$

where $a_v \geqslant 0$, and $\lambda_{v,w}$ is a (nonnegative) measure on (v, w). Conversely, the function on $\widetilde{T}$ defined in (6) is a potential if it is finite at one point.

Corollary 3.3. *If T is any tree, the Green potential of a measure on $\widetilde{T}$ is either identically infinity, or is finite everywhere.*

Proof. By Proposition 3.4, the function P in (6) is a potential (hence, superharmonic) if it is finite at one point. By Proposition 2.3, any superharmonic function on $\widetilde{T}$ is finite-valued. Thus, if P is finite at one point, then P must be finite everywhere. □

In particular, this yields a noncombinatorial proof of Cartier's Proposition 2.3 that $P = \sum a_v G_v$ is either finite everywhere or infinite everywhere. Cartier does this by showing that if y and z are vertices and $[v_0, \ldots, v_n]$ the path from y to z, then $\rho(y, z) = \prod_{j=1}^{n} p(v_{j-1}, v_j)$ has the property that for any vertex x,

$$G_x(y) \geqslant \rho(y, z) G_x(z). \tag{7}$$

Thus, $P(y) \geqslant \rho(y, z) P(z)$, so that $P(y)$ finite implies that $P(z)$ is finite. We now prove that $\rho : T \times T \to (0, 1]$ can be extended to a function on $\widetilde{T} \times \widetilde{T}$, so that (7) holds for all $x, y, z \in \widetilde{T}$.

Theorem 3.5. *If T is any tree, then for any $x, y, z \in \widetilde{T}$, there exists a positive constant $\rho(y, z)$ independent of x such that $G_x(y) \geqslant \rho(y, z) G_x(z)$.*

Proof. As noted above in (7), if $y, z \in T$ then $G_x(y) \geqslant \rho(y, z) G_x(z)$ for all $x \in T$, where $\rho(y, z)$ is the product of the transition probabilities along the edges of the geodesic path from y to z. In particular, if $y \sim z$ then $\rho(y, z) = p(y, z)$. We shall define $\rho(y, z)$ for all $y, z \in \widetilde{T}$ so that (7) holds for all $x \in \widetilde{T}$.

We show first that for all $y, z \in T$, (7) is valid for all $x \in \widetilde{T}$. Suppose $x = (1 - t)v + tw$ where $0 < t < 1$ and $v \sim w$. Then

$$\begin{aligned} G_x(y) &= \gamma_t \big[p(v, w)(1 - t) G_v(y) + p(w, v) t G_w(y) \big] \\ &\geqslant \gamma_t \big[p(v, w)(1 - t) \rho(y, z) G_v(z) + p(w, v) t \rho(y, z) G_w(z) \big] \\ &= \rho(y, z) G_x(z), \end{aligned}$$

which proves the result in case $y, z \in T$.

Next suppose that $y \in \widetilde{T} - T$ and z is a vertex one of whose edges contains y. Thus $y = (1-s)z + sz'$ where $z' \sim z$ and $0 < s < 1$. Let

$$A_{z,z'} = \min\bigl\{p(z,z')G_z(z),\ p(z',z)G_{z'}(z)\bigr\}.$$

Define $\rho(y,z)$ and $\rho(z,y)$ as follows:

$$\rho(y,z) = p(z',z), \qquad \rho(z,y) = \left(1 + \frac{1}{p(z,z')} + \frac{1}{A_{z,z'}}\right)^{-1}.$$

Note that $0 < \rho(z,y) < p(z,z')$. To complete the proof in this case, we have to consider the cases where $x \notin (z,z')$ and $x \in (z,z')$ separately.

First suppose that $x = (1-t)v + tw$, $0 < t < 1$, where v, w are vertices with $\{v,w\} \neq \{z,z'\}$. Then (7) implies

$$\begin{aligned} G_v(y) &= (1-s)G_v(z) + sG_v(z') \geqslant (1-s)G_v(z) + sp(z',z)G_v(z) \\ &\geqslant p(z',z)G_v(z) \end{aligned} \tag{8}$$

and

$$G_v(y) \leqslant (1-s)G_v(z) + s\frac{G_v(z)}{p(z,z')} \leqslant \frac{G_v(z)}{p(z,z')},$$

so

$$G_v(z) \geqslant p(z,z')G_v(y). \tag{9}$$

We deduce from (8) that

$$\begin{aligned} G_x(y) &= \gamma_t\bigl[p(v,w)(1-t)G_v(y) + p(w,v)tG_w(y)\bigr] \\ &\geqslant \gamma_t\bigl[p(v,w)(1-t)p(z',z)G_v(z) + p(w,v)tp(z',z)G_w(z)\bigr] \\ &= p(z',z)G_x(z) = \rho(y,z)G_x(z) \end{aligned}$$

and a similar argument using (9) yields

$$G_x(z) \geqslant p(z,z')G_x(y) \geqslant \rho(z,y)G_x(y).$$

This completes the case $x \notin (z,z')$.

Suppose now that $x = (1-t)z + tz'$ and $0 < s < t < 1$. Since z, z' are neighboring vertices, we obtain from the first argument of the proof that

$$G_x(z') \geqslant p(z',z)G_x(z) \quad \text{and} \quad G_x(z) \geqslant p(z,z')G_x(z'). \tag{10}$$

Now

$$\begin{aligned} G_x(y) &= (1-s)G_x(z) + sG_x(z') + s(1-t)\gamma_t \geqslant (1-s)G_x(z) + sG_x(z') \\ &\geqslant (1-s)G_x(z) + sp(z',z)G_x(z) \geqslant p(z',z)\big[(1-s)G_x(z) + sG_x(z)\big] \\ &= p(z',z)G_x(z). \end{aligned}$$

Also

$$\begin{aligned} G_x(z) &= \gamma_t\big[p(z,z')(1-t)G_z(z) + p(z',z)tG_{z'}(z)\big] \\ &\geqslant \gamma_t \min\big\{p(z,z')G_z(z),\, p(z',z)G_{z'}(z)\big\} \\ &= \gamma_t A_{z,z'}, \end{aligned}$$

so

$$\gamma_t \leqslant \frac{G_x(z)}{A_{z,z'}}.$$

Thus, by (10) we have

$$\begin{aligned} G_x(y) &= (1-s)G_x(z) + sG_x(z') + s(1-t)\gamma_t \leqslant G_x(z) + \frac{G_x(z)}{p(z,z')} + \frac{G_x(z)}{A_{z,z'}} \\ &= \frac{G_x(z)}{\rho(z,y)}, \end{aligned} \tag{11}$$

so

$$G_x(z) \geqslant \rho(z,y)G_x(y).$$

If $0 < t < s < 1$ the argument is similar. This completes the proof in case $y \in \widetilde{T} - T$ and z is a vertex one of whose edges contains y.

Now suppose that y, z are in $\widetilde{T} - T$ and lie on distinct edges. Thus there exist unique vertices v, w such that y lies on an edge of v, z lies on an edge of w and $d(v,w)$ is as small as possible. Since $\rho(y,v)$, $\rho(v,w)$, and $\rho(w,z)$ have already been defined, we may define

$$\rho(y,z) = \rho(y,v)\rho(v,w)\rho(w,z).$$

Then for any $x \in \widetilde{T}$,

$$\begin{aligned} G_x(y) &\geqslant \rho(y,v)G_x(v) \geqslant \rho(y,v)\rho(v,w)G_x(w) \geqslant \rho(y,v)\rho(v,w)\rho(w,z)G_x(z) \\ &= \rho(y,z)G_x(z). \end{aligned}$$

Suppose $y \in \widetilde{T} - T$ and z is a vertex none of whose edges contains y. Let v be the unique vertex closest to z whose edge contains y. An argument similar to that of the last paragraph shows that the result holds if we define $\rho(y,z) = \rho(y,v)\rho(v,z)$ and $\rho(z,y) = \rho(z,v)\rho(v,y)$.

Finally, suppose $y, z \in \widetilde{T} - T$ and both lie on the same edge (v, w). Define

$$\rho(y,z) = \min\bigl\{\rho(y,v)\rho(v,z),\, \rho(y,w)\rho(w,z)\bigr\}.$$

Then for any $x \in \widetilde{T}$,

$$G_x(y) \geqslant \rho(y,v)G_x(v) \geqslant \rho(y,v)\rho(v,z)G_x(z) \geqslant \rho(y,z)G_x(z).$$

This completes the proof. $\square$

Observe that by Theorem 3.5, if μ is a measure on $\widetilde{T}$, then by integrating G_y and G_z against μ we get

$$G\mu(y) \geqslant \rho(y,z)G\mu(z).$$

Thus we obtain another proof of the fact that either $G\mu$ is identically infinity or is finite everywhere.

4. Trees as BS spaces

In this section, we study those trees whose corresponding nearest-neighbor transition probability is recurrent.

Example 4.1. Let T be a homogeneous tree of degree $q+1$ ($q \geqslant 2$) rooted at e whose corresponding nearest-neighbor transition probability is not isotropic but is defined radially as follows. Let $p(e,v) = 1/(q+1)$ for $|v| = 1$, $p(v^-, v) = 1/(2q)$ for $|v| \geqslant 2$, $p' = p(v, v^-) = \frac{1}{2}$ for $|v| \geqslant 1$. The completion $\widetilde{T}$ of T is a BS space. In order to see this, let us assume that the Green function G_e is finite. By symmetry with respect to e, G_e must be radial, and thus, in order to be harmonic off e it must be of the form $G_e(v) = A + B|v|$ for some constants A, B. Since $\Delta G_e(e) < 0$, it follows that $B < 0$. Clearly, there is no constant A such that $A + B|v| > 0$ for all $v \in T$. Thus the Green function must be infinite.

We recall (see Theorem 3.2 and the paragraph preceding it) that in a BP space Ω, as well as in a tree whose underlying random walk is transient, if g is a function defined on the space and harmonic on the complement of a compact set K, then $g = h + b$, where h is harmonic on Ω and b is bounded. This is not true on BS spaces, but we shall describe an obstruction, called the *flux at infinity of* g (which we shall usually refer to just as *the flux*), so that when the flux of g is zero, then $g = h + b$ as above. First we shall describe the situation in a general BS space.

Definition 4.1. Let Ω be a BS space. A function H harmonic off some compact set is called a *standard for* Ω if the following is true: given any function g which is harmonic off an arbitrary compact set, there exists a harmonic function h on the whole space and a unique real number α such that $g - \alpha H - h$ is bounded off a compact set. The constant α is then called the *flux* (*at infinity*) of g with respect to H.

Observation 4.1. If g is harmonic on the whole space, then the flux of g is 0 (take $h = g$). If g is bounded harmonic outside a compact set K, then the flux of g is also zero (take $h = 0$). Furthermore, the flux is linear and unchanged by addition of a function harmonic on the whole space or a function which is bounded harmonic outside a compact set.

Observation 4.2. Let Ω be a BS space, H a standard for Ω. If K_1 and K_2 are compact sets in Ω and f is harmonic on $\Omega\backslash(K_1 \cap K_2)$, then the flux of f is independent of the choice of the set K_1 or K_2. For, if α_1, α_2 are constants and h_1, h_2 are harmonic on Ω such that $f - h_j - \alpha_j H$ is bounded off some compact set K containing K_j $(j = 1, 2)$, then $(\alpha_1 - \alpha_2)H - (h_1 - h_2)$ is bounded outside K, hence its flux $\alpha_1 - \alpha_2$ is 0.

We shall see in Theorem 4.1 that standards for a BS space always exist.

Observation 4.3. The number α depends on the choice of H: Any nonzero multiple of a standard for Ω is again a standard, but the value of the flux will change. Furthermore, if $\widetilde{H}$ is any function harmonic off some compact set with flux $\tilde{\alpha} \neq 0$ with respect to H, then $\widetilde{H}$ is itself a standard of Ω: If g is harmonic off some compact set and α is its flux with respect to H, then since $\widetilde{H} - \tilde{\alpha}H$ and $g - \alpha H$ are both the sum of a harmonic function on Ω and a bounded function, so is

$$g - \frac{\alpha}{\tilde{\alpha}}\widetilde{H} = (g - \alpha H) - \frac{\alpha}{\tilde{a}}\bigl(\widetilde{H} - \tilde{a}H\bigr).$$

We now show that the uniqueness of α and of $\tilde{\alpha}$ leads to the uniqueness of $\alpha/\tilde{\alpha}$. If α' is a constant such that $g - \alpha'\widetilde{H}$ is the sum of a harmonic function on Ω and a bounded function off a compact set, since $\alpha'(\widetilde{H} - \tilde{\alpha}H)$ is the sum of a global harmonic function and bounded function off a compact set, then so is

$$g - \alpha'\tilde{\alpha}H = \bigl(g - \alpha'\widetilde{H}\bigr) + \alpha'\bigl(\widetilde{H} - \tilde{\alpha}H\bigr).$$

By the uniqueness of the flux with respect to H, $\alpha'\tilde{\alpha} = \alpha$. Thus $\alpha' = \alpha/\tilde{\alpha}$.

The following theorem is Theorem 1.17 of [2], together with the note following Lemma 2 of [5].

Theorem 4.1. *Let Ω be a BS space, $K \subset \Omega$ compact, outer regular, and not locally polar. Then there exists a harmonic function $H \geqslant 0$, not identically* 0, *which is unbounded off K and tending to* 0 *on ∂K. Furthermore H is a standard for Ω.*

Theorem 4.2. *Let Ω be a BS space, K a nonempty compact subset. Then any positive harmonic function on $\Omega\backslash K$ which tends to zero on ∂K is a standard for Ω.*

Proof. Let H be a positive harmonic function on $\Omega\backslash K$ tending to zero on ∂K. By Theorem 4.1, there exists a standard for Ω. By Observation 4.3, we need only show that the flux of H with respect to that standard is nonzero. Assume to the contrary that the flux of H is zero. Then there exists a harmonic function h on Ω such that $H - h$ is bounded off a compact set. Extend H to all of Ω by letting it be 0 on K. This extension is continuous, since we assumed that H tends to 0 on ∂K. Let $x_0 \in \partial K$. Then every relatively compact regular neighborhood U of x_0 has some boundary point outside K (where H is positive). Thus

$$\int_{\partial U} H \, \mathrm{d}\rho_{x_0}^U > 0 = H(x_0),$$

so H is subharmonic but not harmonic. Let $M > 0$ be a constant which is an upper bound for $H - h$ on all of Ω. Then $M - (H - h)$ is a positive superharmonic function on Ω, hence it is a constant by Proposition 2.1 and Theorem 1.2. Then $H - h$ is constant so H is harmonic everywhere, a contradiction. □

Proposition 4.1. *Let T be a BS tree rooted at e. If h is nonnegative bounded harmonic off e and $h(e) = 0$, then h must be identically 0.*

Proof. Let M be an upper bound of h. Since h is subharmonic on T, $M - h$ is positive superharmonic on T, hence constant by Theorem 1.2. Thus $h = 0$. □

Proposition 4.2. *Let T be a BS tree rooted at e. There exists a function H on T positive and harmonic off e, unbounded, constant on siblings (i.e., if $u^- = w^-$, then $H(u) = H(w)$), such that $H(e) = 0$ and $\Delta H(e) = 1$. In particular, the linear extension of H to $\widetilde{T}$ is a standard.*

Proof. Define $H(v)$ by induction on $|v|$. Set $H(e) = 0$, and $H(v) = 1$ for $|v| = 1$. Let $|v| = n > 1$ and label the vertices on the geodesic path from e to v as $v_0, v_1, \ldots, v_n$. Assume H has already been defined at v_{n-1} and v_{n-2}. Define

$$H(v_n) = \frac{1}{1 - p(v_{n-1}, v_{n-2})} H(v_{n-1}) - \frac{p(v_{n-1}, v_{n-2})}{1 - p(v_{n-1}, v_{n-2})} H(v_{n-2}).$$

This definition corresponds to the harmonicity condition at v_{n-1}:

$$H(v_{n-1}) = p(v_{n-1}, v_{n-2}) H(v_{n-2}) + \bigl(1 - p(v_{n-1}, v_{n-2})\bigr) H(v_n).$$

Thus $H(v_n)$ is the solution to the second-order linear recurrence relation

$$x_{n+1} = \frac{1}{1 - r_n} x_n - \frac{r_n}{1 - r_n} x_{n-1},$$

where $r_n = p(v_n, v_{n-1})$, with initial conditions $x_0 = 0$ and $x_1 = 1$. Observe that x_n is an increasing sequence, since

$$x_{n+1} - x_n = \frac{r_n}{1 - r_n}(x_n - x_{n-1}), \tag{12}$$

and $x_0 < x_1$. Thus, the function H is necessarily unbounded by Proposition 4.1. □

Let $\epsilon_n = r_n/(1-r_n)$. Then (12) becomes $x_{n+1} - x_n = \epsilon_n \epsilon_{n-1} \dots \epsilon_1$, whence using the initial condition $x_1 = 1$ by induction, we obtain $x_{n+1} = 1 + \sum_{k=1}^{n} \epsilon_1 \dots \epsilon_k$. Thus if $v \in T$, $n \leqslant |v|$, letting $\epsilon_n(v) = p(v_n, v_{n-1})/(1 - p(v_n, v_{n-1}))$, we get

$$H(v) = 1 + \sum_{k=1}^{|v|-1} \epsilon_1(v) \cdots \epsilon_k(v). \tag{13}$$

Let us analyze the case when r_n is the constant $r \in [1/2, 1)$. If $r > 1/2$, then

$$x_n = \frac{1}{\epsilon - 1}(\epsilon^n - 1), \quad \text{where } \epsilon = \frac{r}{1-r}.$$

If $r = 1/2$, then $x_n = n$. Thus

$$H(v) = \begin{cases} \dfrac{1}{\epsilon - 1}\bigl(\epsilon^{|v|} - 1\bigr) & \text{if } r > 1/2, \\ |v| & \text{if } r = 1/2. \end{cases}$$

Brelot theory tells us that every harmonic function f defined outside a finite set of vertices in a BS tree T can be written as the sum of a function harmonic on T, a certain multiple of H, and a bounded harmonic function. Our aim is to give explicit formulas for this representation of f. As a first step we have

Theorem 4.3. *Let T be a BS tree rooted at e and let H be as in Proposition* 4.2. *For $v \in T$ let $v_0, \dots, v_n$ be the vertices on the geodesic path from e to v. Let $\alpha_e = 1$ and $\alpha_v = p_{n-1} \cdots p_0/(r_n \cdots r_1)$ for $|v| = n \geqslant 1$, where $p_j = p(v_j, v_{j+1})$ and $r_j = p(v_j, v_{j-1})$. For each $v \in T$ there exists $H_v \geqslant 0$ (unique up to an additive constant) harmonic except at v such that $\Delta H_v(v) = 1$, and $H_v - \alpha_v H$ takes on a finite number of values. Thus*

$$\operatorname{flux}(H_v) = \alpha_v.$$

Proof. For $v = e$, let $H_v = H$. By definition of flux, H_e has the required properties. Now assume $v \in T$, $|v| = n \geqslant 1$. Let $\alpha, b_0, \dots, b_n$ be constants to be determined later and let H_v be defined by

$$H_v(u) = \alpha H(u) - b_k$$

if u is v_k or a descendant of v_k which is not a descendant of v_{k+1} for $k = 0, \dots, n-1$, or u is v or a descendant of v in the case $k = n$. Since H_v is clearly harmonic off the path $[e, v]$, the problem reduces to finding constants $\alpha, b_0, \dots, b_n$ such that H_v is harmonic at each v_k, $k = 0, \dots, n-1$, and $\Delta H_v(v) = 1$. In order for H_v to be harmonic at e, we need $(1-p_0)H_v(u) + p_0 H_v(v_1) - H_v(e) = 0$, where $u \sim e$, $u \neq v_1$. Thus

$$(1-p_0)\bigl(\alpha H(u) - b_0\bigr) + p_0\bigl(\alpha H(v_1) - b_1\bigr) + b_0 = 0.$$

Since $\Delta H(e) = 1$, we obtain

$$p_0 b_0 - p_0 b_1 + \alpha = 0.$$

Because H is harmonic at each v_k, $k = 1, \ldots, n-1$, harmonicity of H_v at v_k yields the condition $r_k b_{k-1} + p_k b_{k+1} + (1 - r_k - p_k) b_k = b_k$, whence

$$r_k b_{k-1} - (r_k + p_k) b_k + p_k b_{k+1} = 0, \quad \text{for } 1 \leqslant k \leqslant n-1.$$

Furthermore, the condition $\Delta H_v(v) = 1$ yields

$$r_n\big(\alpha H(v_{n-1}) - b_{n-1}\big) + (1 - r_n)\big(\alpha H(u) - b_n\big) - \big(\alpha H(v) - b_n\big) = 1, \tag{14}$$

where u is any descendant of v. Since H is harmonic at v, (14) reduces to

$$r_n b_{n-1} - r_n b_n = -1.$$

Thus we obtain a system consisting of $n+1$ equations in the $n+2$ unknowns $b_0, \ldots, b_n, \alpha$ whose augmented matrix is given by

$$\begin{bmatrix} p_0 & -p_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ r_1 & -(r_1 + p_1) & p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2 & -(r_2 + p_2) & p_2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & r_{n-1} & -(r_{n-1} + p_{n-1}) & p_{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & r_n & -r_n & 0 & -1 \end{bmatrix}.$$

By an elementary row reduction we see that the system is consistent and that b_n is a free variable. The general solution is given by

$$b_k = \begin{cases} b_n - \dfrac{1}{r_n}\left(1 + \displaystyle\sum_{m=k+1}^{n-1} \prod_{j=m}^{n-1} \frac{p_j}{r_j}\right) & \text{for } 0 \leqslant k \leqslant n-2, \\ b_n - \dfrac{1}{r_n} & \text{for } k = n-1, \end{cases}$$

$$\alpha = \frac{p_{n-1} \cdots p_1 p_0}{r_n \cdots r_2 r_1} = \alpha_v.$$

In particular, if we choose $b_n \leqslant 0$, then H_v is positive. Notice that $H_v - \alpha_v H$ is bounded and so the flux of H_v equals α_v. $\square$

We can now calculate the flux at infinity of a function f which is harmonic outside a complete finite set K.

Theorem 4.4. *Under the hypotheses of Theorem* 4.3*, let f be harmonic outside a finite complete subtree K. Then*

$$\operatorname{flux}(f) = \sum_{v\in\partial K}\left[\alpha_v \sum_{u\sim v,\, u\notin K} p(v,u)f(u) + \big(\alpha_{\tilde v}\, p(\tilde v, v) - \alpha_v\big) f(v)\right],$$

where $\tilde v$ is the unique vertex in K which is a neighbor of v. If we extend f to be 0 *inside K, we obtain*

$$\operatorname{flux}(f) = \sum_{v\in\partial K\cup\widetilde K} \Delta f(v)\alpha_v, \tag{15}$$

where $\widetilde K = \{\tilde v\colon\ v \in \partial K\}$.

Proof. The extended function f is harmonic except on $\partial K \cup \widetilde K$. Thus the function $f - \sum_{v\in\partial K\cup\widetilde K} \Delta f(v) H_v$ is harmonic everywhere and so by Observation 4.1 and Theorem 4.3, we get

$$\operatorname{flux}(f) = \sum_{v\in\partial K\cup\widetilde K} \Delta f(v)\operatorname{flux}(H_v) = \sum_{v\in\partial K\cup\widetilde K} \Delta f(v)\alpha_v.$$

Now, for $v \in \partial K$,

$$\Delta f(v) = \sum_{u\sim v,\, u\notin K} p(v,u)f(u) - f(v), \tag{16}$$

and for $w \in \widetilde K$,

$$\Delta f(w) = \sum_{v\in\partial K,\, \tilde v = w} p(w,v)f(v). \tag{17}$$

The result follows from (16) and (17). □

Using Theorem 4.1, we can now get explicit constructions of all the parameters in the definition of flux. Observing that by Theorem 4.3, $H_v - \alpha_v H$ takes on finitely many values we get

Corollary 4.1. *Let f be harmonic in the complement of a complete finite set of vertices K and extend f to be* 0 *on K. Set*

$$\alpha = \sum_{v\in\partial K\cup\widetilde K} \Delta f(v)\alpha_v, \qquad h = f - \sum_{v\in\partial K\cup\widetilde K} \Delta f(v) H_v, \quad \textit{and}$$
$$b = \sum_{v\in\partial K\cup\widetilde K} \Delta f(v)(H_v - \alpha_v H).$$

Then h is harmonic on T, b is bounded, and $f - h - \alpha H = b$.

Proof. Extend f inside K by defining $f|\mathring{K} = 0$. In the proof of Corollary 4.4, we saw that the function

$$h = f - \sum_{v \in \partial K \cup \widetilde{K}} \Delta f(v) H_v$$

is harmonic on T. Set $\alpha_v = \mathrm{flux}(H_v)$ and $\alpha = \mathrm{flux}(f)$. Since $H_v - \alpha_v H$ is bounded, by (15) we see that

$$f - h - \alpha H = \sum_{v \in \partial K \cup \widetilde{K}} \Delta f(v)(H_v - \alpha_v H),$$

a bounded function. □

We end the section with an interesting result on the growth of positive superharmonic functions on a BS tree.

Theorem 4.5. *Let $\widetilde{T}$ be a BS tree and let K be a finite set of vertices. Let s be a function on T which is positive superharmonic on $T \backslash K$. Then s is increasing along each ray in the complement of K. That is, if $[v_0, v_1, \ldots]$ is a ray not intersecting K, then $s(v_j) \leqslant s(v_{j+1})$ for all $j \geqslant 0$. Furthermore, if $s(v_j) = s(v_{j+1})$ for some j, then s must be constant in the sector determined by $[v_j, v_{j+1}]$.*

Proof. Since the random walk associated with T is recurrent, the function F on $T \times T$ defined before Proposition 1.1 is identically 1. The first part of the result follows immediately from part (a) of Proposition 1.2 applied to a sector in $T \backslash K \cup \partial K$. If $s(v_j) = s(v_{j+1}) < s(w)$ for some j, where $v_{j+1} \in [v_j, w]$, then by the first part the mean value of s at the neighbors of v_{j+1} would be bigger than $s(v_{j+1})$, contradicting superharmonicity. Thus $s(v_j) = s(v_{j+1}) = s(w)$. Hence s must be constant on the sector $S(v_j, v_{j+1})$. □

Potential theory on BS trees has received less attention in the literature than potential theory on BP trees. In a future paper [6], we shall extend and expand the results of this section.

5. Conditions for transience on a tree

Given a tree T rooted at e, let ω be a boundary point of T which is not a terminal vertex. Let us denote by ω_n the vertex of length n in the unique geodesic path from e in the class ω. For each $n \geqslant 1$, set

$$\epsilon_n(\omega) = \frac{p(\omega_n, \omega_{n-1})}{1 - p(\omega_n, \omega_{n-1})}.$$

Define $H^*(\omega) = 1 + \sum_{k=2}^{\infty} \epsilon_1(\omega)\epsilon_2(\omega) \cdots \epsilon_{k-1}(\omega) = \lim_{n \to \infty} H(\omega_n)$, where H is the function in (13).

Theorem 5.1. (a) *If T is a tree whose random walk is transient, then there exists some $\omega \in \partial T$ such that $H^*(\omega) < \infty$.*

(b) *If H^* is bounded, then the random walk on T is transient.*

Proof. The random walk on T is transient if and only if $\widetilde{T}$ is a BP space. As in the proof of Proposition 4.2, H is harmonic except at e and $\Delta H(e) = 1$. Extend H linearly to $\widetilde{T}$.

Assume that $\widetilde{T}$ is a BP space. Then by Theorem 3.2, $H = h + b$ where h is harmonic everywhere and b is bounded. Let $B_n = \{v \in T\colon |v| < n\}$. Then

$$h(e) = \int\limits_{\partial B_n} H \,\mathrm{d}\rho_e^{B_n} - \int\limits_{\partial B_n} b \,\mathrm{d}\rho_e^{B_n}.$$

Since b is bounded and constants are harmonic, $\int_{\partial B_n} H \,\mathrm{d}\rho_e^{B_n}$ is bounded. In particular, if v_n is a minimum point for H on ∂B_n, then $\{H(v_n)\}$ is bounded. By the compactness of $T \cup \partial T$ and the fact that T is discrete, there is an $\omega \in \partial T$ which is the limit of some subsequence of $\{v_n\}$, whence $H^*(\omega) < \infty$, proving (a).

Next, assume that H^* is bounded by some constant M. Then $M - H$ is a positive nonconstant superharmonic function. Thus, $\widetilde{T}$ is a BP space by Theorem 1.2. □

The following theorem is clearly much stronger than (b) of Theorem 5.1. The reason we left Theorem 5.1 as it stands is that it is a purely combinatorial statement, yet its proof is purely Brelot theoretic.

Theorem 5.2. *If H^* is finite on an interval, then the random walk on T is transient.*

Proof. First we show that if H^* is bounded on an interval, then the random walk on T is transient. Assume H^* is bounded on I_v, where v is a vertex of length N. Let T_0 be the tree consisting of the descendants of v. On T_0 define the transition probabilities

$$p_0(w,u) = \begin{cases} p(w,u) & \text{if } w \neq v, \\ \dfrac{p(w,u)}{1 - p(w,w^-)} & \text{if } w = v. \end{cases}$$

Let H_0 be the analogue of the function H on T_0 viewed as a tree rooted at v, let $w \in T_0$ be a descendant of v at distance m. Then

$$\begin{aligned} H(w) &= 1 + \sum_{k=2}^{N+m} \epsilon_1(w) \cdots \epsilon_{k-1}(w) \\ &= 1 + \sum_{k=2}^{N} \epsilon_1(w) \cdots \epsilon_{k-1}(w) + \epsilon_1(w) \cdots \epsilon_N(w) \left(1 + \sum_{k=N+2}^{N+m} \epsilon_{N+1}(w) \cdots \epsilon_{k-1}(w)\right) \\ &= A + BH_0(w), \end{aligned}$$

where A and B are constants. Since H^* is bounded on I_v, H_0^* is bounded on the nonterminal vertices of ∂T_0. By Theorem 5.1, the random walk on T_0 is transient. Thus, letting F and F_0 be the functions corresponding to the trees T and T_0 defined before Proposition 1.1, we obtain that $F_0(v,v) < 1$. But

$$F(v,v) = p(v,v^-)F(v^-,v) + \sum_{w^-=v} p(v,w)F(w,v),$$

by part (d) of Proposition 1.1. Since $F_0(w,v) = F(w,v)$ for $w^- = v$, we have

$$F_0(v,v) = \sum_{w^-=v} p_0(v,w)F(w,v) = \sum_{w^-=v} \frac{p(v,w)}{1-p(v,v^-)}F(w,v),$$

so $\sum_{w^-=v} p(v,w)F(w,v) = (1-p(v,v^-))F_0(v,v)$. Thus

$$F(v,v) = p(v,v^-)F(v^-,v) + \big(1-p(v,v^-)\big)F_0(v,v) < p(v,v^-) + 1 - p(v,v^-) = 1.$$

Therefore, the random walk on T is transient.

Next, assume H^* is finite but unbounded on I_v for some $v \in T$, and that the random walk on T is recurrent. Since H^* is unbounded on I_v, H is unbounded on S_v, the set of descendants of v. Pick $v_1 \in S_v$ such that $H(v_1) \geqslant 1$. Since the random walk on T is recurrent, by the first part of the proof H^* is unbounded on I_{v_1}. Thus H is unbounded on S_{v_1} and so there exists $v_2 \in S_{v_1}$ such that $H(v_2) \geqslant 2$. Inductively, we obtain a sequence $\{v_n\}_{n\in\mathbb{N}}$ such that v_{n+1} is a descendant of v_n for all n and $H(v_n) \geqslant n$. Let ω be the equivalence class of the ray $[v_1, v_2, \ldots]$. Then

$$H^*(\omega) = \lim_{n\to\infty} H(v_n) = \infty,$$

which is a contradiction. Consequently, if H^* is finite on I_v, then the random walk on T must be transient. □

Corollary 5.1. *Let T be a tree rooted at e and let $r \in (0, 1/2)$. Assume there exists $v_0 \in T$ such that for each descendant u of v_0, $p(u,u^-) \leqslant r$. Then T is transient.*

Proof. For every descendant u of v_0, we have

$$\epsilon_n(u) \leqslant \frac{r}{1-r} < 1 \quad \text{for all } n \geqslant |v_0|.$$

Thus H^* is bounded on I_{v_0}. The conclusion follows at once from Theorem 5.2. □

The next example shows that finiteness of the function H^* at a single boundary point does not guarantee transience.

Example 5.1. Let T be a homogeneous tree of degree 3 rooted at e whose transition probabilities are as follows. Fix $p \in (0, 1/3]$ and an infinite ray $\rho = [v_0 = e, v_1, \ldots)$. Define $p(v_n, v_{n-1}) = 1/3$, $p(v_n, v_{n+1}) = p$, $p(v_n, w) = q$ for $w \sim v_n$, $w \notin \rho$, for $n \geqslant 1$, $p(e, v) = 1/3$ for $|v| = 1$, $p(v^-, v) = 1/4$, $p(v, v^-) = 1/2$ for all other values of v. Thus $p + q + 1/3 = 1$. For each $n \in \mathbb{N}$, let w_n be the neighbor of v_n which does not lie on ρ.

Let T_2 be the tree of Example 4.1 with $q = 2$. Observe that except for the probabilities starting along ρ, T_2 and T are exactly the same. Since for all $v \notin \rho$, Γ'_{v,v^-} with respect to T is the same as the corresponding set with respect to T_2, $F(v, v^-)$ is the same for both sets. But the random walk on T_2 is recurrent, so $F(v, v^-) = 1$. In particular, $F(w_n, v_n) = 1$ for all $n \in \mathbb{N}$.

Next observe that the subtree consisting of the descendants of v_n is isomorphic to the subtree of descendants of v_{n+1}. Thus $F(v_{n+1}, v_n) = F(v_{n+2}, v_{n+1})$ for all $n \geqslant 0$. Call this common value β.

By the multiplicative property of F,

$$F(v_{n+1}, v_n) = \frac{1}{3} + pF(v_{n+2}, v_{n+1})F(v_{n+1}, v_n) + qF(w_{n+1}, v_{n+1})F(v_{n+1}, v_n),$$

or $\beta = 1/3 + p\beta^2 + q\beta$. Since $q = 2/3 - p$, we obtain $\beta = 1$ or $\beta = 1/(3p)$. But $\beta \leqslant 1$ and $1/(3p) \geqslant 1$, thus we must have $\beta = 1$. Thus for every neighbor v of v_0, $F(v, v_0) = 1$ and so by part (d) of Proposition 1.1, we get $F(v_0, v_0) = 1$ and thus the random walk on T is recurrent.

Observe that since $(1/3)/(1 - 1/3) = 1/2$, if ω is the equivalence class of ρ, then

$$H^*(\omega) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} < \infty.$$

In a forthcoming note, we shall show that for $1/3 < p < 2/3$, $\beta = 1/(3p) < 1$, so that the random walk on T is transient. On the other hand, the functions H and H^* are independent of p.

6. Other Brelot structures on a tree

We now consider other harmonic structures on a tree. Fix $a > 0$ and consider the functions on T which are the eigenfunctions of the Laplacian with eigenvalue a^2: $\Delta f(v) = a^2 f(v)$ for all $v \in T$. These correspond to the λ-harmonic functions of Definition 1.3 where $\lambda = a^2 + 1$. Defining the operator $L = \Delta - a^2 I$, the *harmonic functions* that we study now are the elements of the kernel of L. If $Lf = 0$, extend f on each edge $[v, u]$ by means of the solution to the Helmholtz equation $y'' = a^2 y$ given by:

$$f\big((1-t)v + tu\big) = \xi(1-t)f(v) + \xi(t)f(u),$$

where for $0 \leqslant t \leqslant 1$,

$$\xi(t) = \frac{e^{at} - e^{-at}}{e^{a} - e^{-a}}.$$

Notice that $\xi''(t) = a^2\xi(t)$. Extending f gives us an L-harmonic function on $\widetilde{T}$.

We now give the local definition of L-harmonic. Let U be an open set in $\widetilde{T}$ and f a continuous function on U. Assume first $x = (1 - t_0)v + t_0 u$, with $0 < t_0 < 1$, $v, u \in T$. Define f to be L-harmonic at x if there exist $b, c \in \mathbb{R}$ and $\epsilon > 0$ such that $f((1 - t)v + tu) = \xi(1 - t)b + \xi(t)c$ for all t, $|t - t_0| < \epsilon$. Observe that the function $g(t) = f((1 - t)v + tu)$ satisfies the equation $g''(t) = a^2 g(t)$ (in fact, this is equivalent to L-harmonicity on the edge).

If $x = v \in T$, v not a terminal vertex, we say that f is L-harmonic at v if for some $\epsilon > 0$ and for all $t \in (0, \epsilon)$

$$f(v) = \frac{1}{\alpha(t)} \sum_{u \sim v} p(v, u) f\bigl((1 - t)v + tu\bigr),$$

where α is the function on $[0, 1]$ mapping 0 to 1 and 1 to λ given by

$$\alpha(t) = \xi(1 - t) + \lambda\xi(t).$$

We now say that f is L-harmonic on U if it is L-harmonic at each $x \in U$, x not a terminal vertex of T. The first Brelot space axiom clearly holds.

Consider the base of domains defined by the ϵ-balls as in Section 2. We show that we can solve the Dirichlet problem on each such domain. Fixing a vertex v, for any neighbor u of v, let $u_t = (1 - t)v + tu$. Let us first solve the Dirichlet problem on a neighborhood of $x = u_{t_0}$, with $v, u \in T$, $0 < t_0 < 1$ by taking this neighborhood to be $B_\epsilon(x)$, where $\epsilon \leqslant \min\{t_0, 1 - t_0\}$. If f is defined on $\partial B_\epsilon(x) = \{u_{t_0 \pm \epsilon}\}$, then f can be extended as follows:

$$f(u_t) = \frac{1}{\xi(2\epsilon)} \bigl[\xi(\epsilon + t_0 - t) f(u_{t_0 - \epsilon}) + \xi(\epsilon - t_0 + t) f(u_{t_0 + \epsilon})\bigr]$$

for $|t - t_0| \leqslant \epsilon$. By construction, f is L-harmonic inside $B_\epsilon(x)$.

Next, let $x = v \in T$, where v is a nonterminal vertex. For $0 < \epsilon \leqslant 1$, $\partial B_\epsilon(x) = \{u_\epsilon\colon u \sim v\}$. If f is defined on $\partial B_\epsilon(x)$, then let

$$f(v) = \frac{1}{\alpha(\epsilon)} \sum_{u \sim v} p(v, u) f(u_\epsilon) \tag{18}$$

and

$$f(u_t) = \frac{1}{\xi(\epsilon)} \bigl[\xi(\epsilon - t) f(v) + \xi(t) f(u_\epsilon)\bigr], \quad \text{for } 0 \leqslant t \leqslant \epsilon.$$

Thus the second axiom of Brelot is satisfied.

If all the neighbors of v and the corresponding edges are contained in U, (18) implies that $\lambda f(v) = \sum_{u\sim v} p(v,u)f(u)$.

The proof of the third axiom of Brelot is almost identical to that which was given in the proof of Theorem 2.1. Thus the L-harmonic functions give a Brelot structure on $\widetilde{T}$.

Notice that under this structure, $\widetilde{T}$ is not a BH space since the nonzero constants are not L-harmonic. In the remainder of this section we shall assume that the harmonic structure on $\widetilde{T}$ is induced by the operator $L = \Delta - a^2 I$, for $a > 0$.

Observation 6.1. By Observation 1.2, if s is L-superharmonic on $\widetilde{T}$, $v \sim w \in T$, $x, y \in [v, w]$, and $0 < t < 1$, then

$$s\big((1-t)x + ty\big) \geqslant h^{(x,y)}_{s|\{x,y\}}\big((1-t)x + ty\big) = \xi(1-t)s(x) + \xi(t)s(y).$$

Moreover, for any $v \in T$, and $0 < t < 1$,

$$\begin{aligned} s(v) &\geqslant h^{B_t}_{s|\partial B_t}(v) = \frac{1}{\alpha(t)} \sum_{u\sim v} p(v,u) s\big((1-t)v + tu\big) \\ &\geqslant \frac{1}{\alpha(t)} \sum_{u\sim v} p(v,u)\big(\xi(1-t)s(v) + \xi(t)s(u)\big) \\ &= \frac{1}{\alpha(t)} \left(\xi(1-t)s(v) + \xi(t) \sum_{u\sim v} p(v,u)s(u)\right), \end{aligned}$$

whence $\lambda s(v) \geqslant \sum_{u\sim v} p(v,u)s(u)$. In particular, $s|T$ is λ-superharmonic on T.

Proposition 6.1. *Let s be λ-superharmonic* (*respectively, λ-harmonic*) *on any tree T. Then the extension of s defined on the edges by*

$$s\big((1-t)v + tw\big) = \xi(1-t)s(v) + \xi(t)s(w), \quad \textit{for all } t \in (0,1),\ v, w \in T$$

is L-superharmonic (*respectively, L-harmonic*) *on $\widetilde{T}$.*

Proof. First observe that s restricted to the interior of each edge is L-harmonic, hence L-superharmonic there. Next, let $v \in T$ and $0 < \epsilon < 1$. Then

$$\begin{aligned} \sum_{u\sim v} p(v,u) s\big((1-\epsilon)v + \epsilon u\big) &= \sum_{u\sim v} p(v,u)\xi(1-\epsilon)s(v) + \sum_{u\sim v} p(v,u)\xi(\epsilon)s(u) \\ &= \xi(1-\epsilon)s(v) + \xi(\epsilon)\sum_{u\sim v} p(v,u)s(u) \\ &\leqslant \xi(1-\epsilon)s(v) + \lambda\xi(\epsilon)s(v) = \alpha(\epsilon)s(v), \end{aligned} \tag{19}$$

proving L-superharmonicity at each vertex v. If s is λ-harmonic, the inequality in (19) is an equality, proving L-harmonicity on $\widetilde{T}$. $\square$

Definition 6.1. We shall call the extension of s in Proposition 6.1 the ξ*-extension* of s.

Proposition 6.2. *If s is L-superharmonic on $\widetilde{T}$, then s is finite-valued.*

Proof. First, assume $s(v) = \infty$ for some $v \in T$. Then for each $u \sim v$ and each $t \in (0,1)$,

$$s\big((1-t)v + tu\big) \geqslant \xi(1-t)s(v) + \xi(t)s(u),$$

so s is identically infinity on $[v, u)$. On the other hand, by Observation 6.1,

$$\lambda s(u) \geqslant \sum_{w\sim u} p(u,w)s(w) = \infty,$$

since v is a neighbor of u and $s(v) = \infty$. Thus, s is identically infinity on the whole edge $[v, u]$. By connectedness, $s = \infty$ on $\widetilde{T}$, contradicting L-superharmonicity.

Next, assume $s(x) = \infty$, for some x in the interior of the edge $[v, u]$. Then, for each $t \in (0, 1)$, $s((1-t)v + tx) \geqslant \xi(1-t)s(v) + \xi(t)s(x) = \infty$. Similarly, $s((1-t)u + tx) \geqslant \xi(1-t)s(u) + \xi(t)s(x) = \infty$. So s is infinity on (v, u). Again by L-superharmonicity, $s(v) \geqslant (1/\alpha(\epsilon)) \sum_{w\sim v} p(v,w)s((1-\epsilon)v + \epsilon w) = \infty$. Thus, $s(v) = \infty$. By the first case, we get a contradiction, completing the proof. □

Proposition 6.3. *Let T be any tree. Then the space $\widetilde{T}$ under the harmonic structure induced by $L = \Delta - a^2 I$ has potentials.*

Proof. As above, let $\lambda = a^2 + 1$. Observe that the positive constants are positive L-superharmonic but not L-harmonic. Thus, by Theorem 1.2, $\widetilde{T}$ has potentials. □

Let T be a homogeneous tree of degree $q + 1$ and, as in Section 1.1, let $\alpha = ((q+1)\lambda - \sqrt{(q+1)^2\lambda^2 - 4q})/(2q)$ which is the smaller positive root of the quadratic equation associated with the recurrence relation (1), and let β be the larger.

Proposition 6.4. *The function $p(v) = \alpha^{|v|}$, $v \in T$, is a potential on T with harmonic support at $\{e\}$ with respect to the structure induced by L.*

Proof. Assume $v \in T$, $|v| = n > 0$. Then

$$\mu_1 p(v) = \frac{q\alpha^{n+1} + \alpha^{n-1}}{q+1} = \frac{q\alpha + \alpha^{-1}}{q+1} p(v) = \lambda p(v).$$

Thus $Lp(v) = 0$ for all $v \neq e$. Furthermore, $\mu_1 p(e) = \alpha$, so $Lp(e) = \alpha - \lambda$, a negative number. Thus p is positive λ-superharmonic and λ-harmonic off e. Assume h is a nonnegative λ-harmonic function such that $h \leqslant p$. Let $\tilde{h}$ be the radialization of h, i.e.,

$$\tilde{h}(v) = \frac{1}{c_{|v|}} \sum_{|w|=|v|} h(w),$$

where c_n is the number of vertices of length n. Since h is nonnegative, $0 \leqslant \tilde{h} \leqslant p$. Furthermore, $\tilde{h}$ is λ-harmonic.

If v_n is any vertex of length n, then

$$\frac{q}{q+1}\tilde{h}(v_{n+1}) + \frac{1}{q+1}\tilde{h}(v_{n-1}) = \lambda\tilde{h}(v_n)$$

(which is the recurrence relation (1)) for $n \geqslant 2$, and $\tilde{h}(v_1) = \lambda\tilde{h}(e)$. Thus $\tilde{h}(v_n) = c_0\beta^n + c_1\alpha^n$ and $0 \leqslant c_0\beta^n + c_1\alpha^n \leqslant \alpha^n$ for all n. Since $\beta > \alpha$, it follows that $c_0 = 0$, so $\tilde{h}$ is a multiple of p. On the other hand, since p is not harmonic at e, $c_1 = 0$. Thus $\tilde{h}$ is identically 0, whence $h = 0$. Therefore p is a potential. □

Observation 6.2. In this homogeneous case, the Green function for the L operator is given by

$$G(u, v) = G_v(u) = \frac{1}{\lambda - \alpha}\alpha^{d(u,v)}.$$

Let T be any tree, $a \geqslant 0$, $\lambda = a^2 + 1$. Define the operator G on the space of functions on T by

$$G = \sum_{j=0}^{\infty} \frac{1}{\lambda^{j+1}}\mu_1^j,$$

where μ_1^j is the j-fold composition of the operator μ_1 with itself. Notice that for $a > 0$, $\lambda > 1$ and so by Observation 1.1, G is a bounded operator of norm $\|G\| \leqslant 1/(\lambda - 1)$. Furthermore

$$I + \mu_1 G = I + G\mu_1 = \lambda G. \tag{20}$$

Proposition 6.5. *Let f be a nonnegative function on a tree T. Then either there is no nonnegative solution s to $Ls = -f$, or Gf is a nonnegative solution and any nonnegative solution s to $Ls = -f$ satisfies the inequality $s \geqslant Gf$.*

Proof. Assume there exists $s : T \to [0, \infty)$ such that $Ls = -f$. Then $\mu_1 s - \lambda s = -f$, so

$$\frac{1}{\lambda}\mu_1 s + \frac{1}{\lambda} f = s.$$

Composing with μ_1 yields

$$\frac{1}{\lambda}\mu_1^2 s + \frac{1}{\lambda}\mu_1 f = \mu_1 s = \lambda s - f,$$

whence

$$s = \frac{1}{\lambda} f + \frac{1}{\lambda^2}\mu_1 f + \frac{1}{\lambda^2}\mu_1^2 s.$$

By induction, it follows that for all $n \in \mathbb{N}$

$$s = \frac{1}{\lambda} f + \frac{1}{\lambda^2}\mu_1 f + \cdots + \frac{1}{\lambda^{n+1}}\mu_1^n f + \frac{1}{\lambda^{n+1}}\mu_1^{n+1} s \geqslant \frac{1}{\lambda} f + \frac{1}{\lambda^2}\mu_1 f + \cdots + \frac{1}{\lambda^{n+1}}\mu_1^n f.$$

Letting $n \to \infty$, we deduce $s \geqslant Gf$. Thus $Gf < \infty$ and by (20), $LGf = -f$. □

We now show that G induces the Green function on $T \times T$.

Proposition 6.6. *Let $v \in T$. Then the function on T defined by $G_v = G\delta_v$ is a potential on T with harmonic support at $\{v\}$. Furthermore, every potential with harmonic support at v is a positive multiple of G_v.*

Proof. Since $LG_v = -\delta_v \leqslant 0$, G_v is positive L-superharmonic on T and L-harmonic on $T\backslash\{v\}$. If h is an L-harmonic minorant of G_v, then $L(G_v - h) = -\delta_v$, so by Proposition 6.5, $G_v - h \geqslant G_v$. It follows that $h \leqslant 0$. Thus G_v is a potential.

Let p be a potential with harmonic support at v. By scaling, it suffices to show that if $Lp(v) = -1$, then $p = G_v$. Thus, assuming $Lp(v) = -1$, $Lp = -\delta_v$. Applying Proposition 6.5 to $f = \delta_v$ and $s = p$, we obtain $p \geqslant G_v$. But $h = p - G_v$ is nonnegative L-harmonic and $h \leqslant p$. Thus $h = 0$, whence $p = G_v$. □

Recall the axiom of proportionality (Definition 1.14).

Theorem 6.1. *Let T be any tree. Then the axiom of proportionality holds for $\widetilde{T}$ under the Brelot structure given by L.*

Proof. If p_1 and p_2 are potentials on $\widetilde{T}$ with harmonic support at $v \in T$, then $p_1|T$ and $p_2|T$ are potentials on T with the same harmonic point support. Thus, by Proposition 6.6, they are multiples of one another. So assume p_1 and p_2 are potentials on $\widetilde{T}$ with harmonic support at $x \in (v, w)$, where $v, w \in T$. Then p_1 and p_2 are potentials on T which are L-harmonic except possibly at v, w. Thus off $[v, w]$, $p_j = \alpha_j G_v + \beta_j G_w$, for some $\alpha_j, \beta_j \geqslant 0$, $j = 1, 2$. Let G_v^ξ be the ξ-extension of G_v. Near v, define $\widetilde{G}_v^\xi$ to be G_v^ξ off (v, w), while for $y = (1-s)v + sw$, $0 \leqslant s \leqslant 1$, define

$$\widetilde{G}_v^\xi(y) = \xi(1-s)G_v(v) + \xi(s)\widetilde{G}_v(w),$$

where $\widetilde{G}_v(w)$ is the quantity defined by

$$\lambda G_v(v) = \sum_{u \neq w} p(v, u)G_v(u) + p(v, w)\widetilde{G}_v(w).$$

Define $\widetilde{G}_w(v)$ by reversing the roles of v and w. Then for $j = 1, 2$,

$$p_j = \begin{cases} \alpha_j \widetilde{G}_v^\xi + \beta_j G_w^\xi & \text{on } [v, x], \\ \alpha_j G_v^\xi + \beta_j \widetilde{G}_w^\xi & \text{on } [x, w]. \end{cases}$$

So for $x = (1-t)v + tw$, $0 \leqslant t < 1$, we have

$$p_j(x) = \alpha_j \widetilde{G}_v^{\xi}(x) + \beta_j G_w^{\xi}(x) = \alpha_j G_v^{\xi}(x) + \beta_j \widetilde{G}_w^{\xi}(x).$$

This fixes α_j/β_j, $j = 1, 2$. Moreover

$$\begin{aligned}
&\alpha_j\big[\xi(1-t)G_v(v) + \xi(t)\widetilde{G}_v(w)\big] + \beta_j\big[\xi(1-t)G_w(v) + \xi(t)G_w(w)\big] \\
&\quad = \alpha_j\big[\xi(1-t)G_v(v) + \xi(t)G_v(w)\big] + \beta_j\big[\xi(1-t)\widetilde{G}_w(v) + \xi(t)G_w(w)\big].
\end{aligned}$$

Thus

$$\alpha_j \xi(t)\big[\widetilde{G}_v(w) - G_v(w)\big] = \beta_j \xi(1-t)\big[\widetilde{G}_w(v) - G_w(v)\big].$$

Consequently,

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\xi(1-t)}{\xi(t)}\left[\frac{\widetilde{G}_w(v) - G_w(v)}{\widetilde{G}_v(w) - G_v(w)}\right],$$

proving proportionality. $\square$

By Theorem 6.1 and Theorem 3.4, we know that there is a Green function on $\widetilde{T}$. By Observation 3.1, we obtain

Corollary 6.1. *If T is any tree, then the Green function G on $\widetilde{T}$ under the structure inherited by the operator L can be chosen so that its restriction to $T \times T$ equals the Green function of Proposition* 6.6.

References

[1] V. Anandam, Espaces harmoniques sans potentiel positif, Ann. Inst. Fourier (Grenoble) 22 (4) (1972) 97–160.

[2] V. Anandam, Harmonic spaces with positive potentials and nonconstant harmonic functions, Rend. Circ. Mat. Palermo (2) 21 (1972) 149–167.

[3] V. Anandam, Admissible subharmonic functions and associated measures, J. London Math. Soc. (2) 19 (1) (1979) 65–78.

[4] V. Anandam, Subharmonic extensions outside a compact set in $\mathbb{R}^n$, Proc. Amer. Math. Soc. 84 (1) (1982) 52–54.

[5] V. Anandam, Superharmonic functions and the flux, Bull. Sci. Math. 106 (1982) 85–92.

[6] I. Bajunaid, J.M. Cohen, F. Colonna, D. Singman, Potential theory on trees without potentials, Preprint.

[7] M. Brelot, Lectures on Potential Theory, in: Tata Inst. Fund. Res. Lectures on Math., Vol. 19, 1967.

[8] M. Brelot, Historical introduction, Potential Theory (CIME, I Ciclo, Stresa, 1969) Edizioni Cremonese, Rome, 1970, pp. 1–21.

[9] M. Brelot, Axiomatique des Fonctions Harmoniques, in: Sém. Math. Sup., Vol. 14, Les Presses de l'Université de Montréal, Montréal, 1969.

[10] C.A. Berenstein, E. Casadio Tarabusi, J.M. Cohen, M.A. Picardello, Integral geometry on trees, Amer. J. Math. 113 (1991) 441–470.

[11] P. Cartier, Fonctions harmoniques sur un arbre, Sympos. Math. 9 (1972) 203–270.

[12] J.M. Cohen, F. Colonna, Spectral analysis on homogeneous trees, Adv. in Appl. Math. 20 (1998) 253–274.
[13] N. Du Plessis, An Introduction to Potential Theory, Oliver & Boyd, Edinburgh, 1970.
[14] A. Figà-Talamanca, M.A. Picardello, Harmonic Analysis on Free Groups, in: Lecture Notes in Pure and Appl. Math., Vol. 87, Dekker, New York, 1983.
[15] K. GowriSankaran, D. Singman, Minimal fine limits on trees, Preprint, 2001.
[16] R.M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, Ann. Inst. Fourier (Grenoble) 12 (1962) 415–571.
[17] A. Korányi, M.A. Picardello, Boundary behaviour of eigenfunctions of the Laplace operator on trees, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 13 (1986) 389–399.
[18] A. Korányi, M.A. Picardello, M. Taibleson, Hardy spaces on nonhomogeneous trees, in: Sympos. Math., Vol. 29, Academic Press, New York, 1987, pp. 205–265.
[19] F.Y. Maeda, Dirichlet Integrals on Harmonic Spaces, in: Lecture Notes in Math., Vol. 803, Springer-Verlag, 1980.
[20] A.M. Mantero, A. Zappa, The Poisson transform and representations of a free group, J. Funct. Anal. 51 (1983) 372–399.
[21] B. Rodin, L. Sario, Principal Functions, Van Nostrand, 1968.
[22] M.H. Taibleson, Hardy spaces of harmonic functions on homogeneous isotropic trees, Math. Nachr. 133 (1987) 273–288.