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Thin Sets and Boundary Behavior of Solutions of the Helmholtz Equation

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Abstract. The Martin boundary for positive solutions of the Helmholtz equation in n-dimensional Euclidean space may be identified with the unit sphere. Let v denote the solution that is represented by Lebesgue surface measure on the sphere. We define a notion of thin set at the boundary and prove that for each positive solution of the Helmholtz equation, u, there is a thin set such that u/v has a limit at Lebesgue almost every point of the sphere if boundary points are approached with respect to the Martin topology outside this thin set. We deduce a limit result for u/v in the spirit of Nagel–Stein (1984).

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1. Introduction

Let n be an integer greater than 1. Let $S = S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit sphere in $\mathbb{R}^n$. For $b \in S$, $t > 0$ let $B_{b,t} = \{x \in S : |x - b| < t\}$ be the intersection with S of an open ball in $\mathbb{R}^n$ centered at b of radius t. We let σ denote unit Lebesgue surface measure on S. For μ a Borel measure on S, define

$$K\mu(x) = \int_S \mathrm{e}^{\lambda\langle x,b\rangle}\,\mathrm{d}\mu(b),$$

where $\langle x, b\rangle = x_1b_1 + \cdots + x_nb_n$ is the usual inner product on $\mathbb{R}^n$, and λ is the positive constant that appears in the Helmholtz equation, $\Delta u = \sqrt{\lambda}\cdot u$. All positive solutions of the Helmholtz equation are of the form $K\mu$ for some positive Borel measure μ on S ([6], Corollary to Theorem 4). For each $b \in S$, let Ω_b denote an unbounded subset of $\mathbb{R}^n$ that converges to b at ∞ in the sense that if $\{x_k\}$ is a sequence in Ω_b such that $|x_k| \to \infty$, then $x_k/|x_k| \to b$. We denote this by '$x_k \to b$ at ∞'.

DEFINITION 1. $\{\Omega_b : b \in S\}$ is called a *collection of approach regions* if for each $b \in S$, Ω_b converges to b at ∞ and the convergence is uniform in the sense that for all $\varepsilon > 0$ there exists T such that for all $b \in S$ and $|x| > T$, we have $|\frac{x}{|x|} - b| < \varepsilon$ if $x \in \Omega_b$.

We define the concept of *thinness* as follows:

DEFINITION 2. Let $E \subset \mathbb{R}^n$, $\alpha > 0$. Let

$$\Lambda(E) = \inf \left\{ \sum_{i=1}^{\infty} \sigma(B_{b_i,t_i}) \colon E \subset \bigcup_{i=1}^{\infty} G_\alpha(b_i, t_i) \right\},$$

where

$$G_\alpha(b,t) = \left\{ x \in \mathbb{R}^n \colon \frac{x}{|x|} \in B_{b,t},\, |x| \geqslant \alpha^2 t^{-2} \right\}.$$

If $\alpha = 1$, we write $G(b,t)$ instead of $G_1(b,t)$. We say that E is *thin at the boundary with respect to measure* if

$$\lim_{t\to\infty} \Lambda(E \cap \{x \in \mathbb{R}^n \colon |x| > t\}) = 0.$$

It is easy to see the definition is independent of α.

We show in Theorems 1 and 2 that for any positive solution $K\mu$ of the Helmholtz equation, and any collection $\{\Omega_b\}$ of approach regions, there exists a set E thin at the boundary with respect to measure such that for σ-a.e. b in S, $(K\mu/K\sigma)(x)$ converges to $(\mathrm{d}\mu/\mathrm{d}\sigma)(b)$ as $x \to b$ *at* ∞, $x \in \Omega_b - E$.

In Theorem 4 we deduce from the above result an almost everywhere pointwise boundary behavior result for an ‘admissible collection’ of approach regions $\{\Omega_b \colon b \in S\}$ (‘admissible collection’ is defined in Definition 5). The basic theorem of this type that is known is the following:

THEOREM (Korányi–Taylor) [9]: *Let μ be a positive, Borel measure on S. Let α be a given positive real number. Then*

$$\lim_{x\to b \text{ at } \infty, x\in P_\alpha(b)} \frac{K\mu}{K\sigma}(x) = \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(b)$$

for $\sigma - a.e. b \in S$, where

$$P_\alpha(b) = \{x \in \mathbb{R}^n \colon |x - |x|b| \leqslant \alpha\sqrt{|x|}\} = \left\{ x \in \mathbb{R}^n \colon \left| \frac{x}{|x|} - b \right| \leqslant \frac{\alpha}{\sqrt{|x|}} \right\}$$

is the paraboloid of aperture α.

We will define what it means for a family of approach regions $\{\Omega_b \colon b \in S\}$ to be *admissible* (see Definition 5) and show that for such a family and a given set

E thin with respect to measure at the boundary, σ-a.e. $b \in S$ has the property that $E \cap \Omega_b \cap \{|x| > T\} = \emptyset$ for all T large enough (Theorem 3). Theorem 4 then follows immediately. The family of paraboloids in the Korányi–Taylor theorem forms an admissible collection, in fact the only admissible collection such that Ω_b is invariant under all unitary transformations of $\mathbb{R}^n$ which fix b. However, in Section 5 we show how to generate examples of admissible collections Ω_b which cannot be contained in $P_\alpha(b)$ for any fixed aperture, α.

The result of Theorem 4 is analogous to the main result in [14] where the Fatou theorem for positive harmonic functions on a half-space in $\mathbb{R}^{n+1}$ is improved by replacing cones with a more general family of 'non-nontangential' approach regions. For subsequent developments, see [3], [5], [8], [10–13], [15–16].

The idea of the definition of thin set as well as the idea of proving Theorem 4 using a notion of thin set is inspired by H. Aikawa. In [1] and [2] he defines a notion of thin set on a half-space in $\mathbb{R}^{n+1}$ and deduces the result of Nagel–Stein [14] for positive harmonic functions on this half-space. In this paper we adapt his proof to our setting. Significant difficulties arise due to a very different definition of 'admissible collection' and the lack of a group structure on S^{n-1}.

The idea of a set of approach regions $\{\Omega_b : b \in S^{n-1}\}$ being an 'admissible collection' is inspired by the analogous definition in [15]. However, our definition is less restrictive. We merely require a control over the σ-measure of the Ω-projection (see Definition 3) of the tents over those balls which are bounded away from a fixed closed set F in S of σ-measure 0, rather than over all balls. As we show in Proposition 4, this is due to the fact that for the natural examples of admissible collections (other than the ones consisting of paraboloids as in the Korányi–Taylor Theorem), there is necessarily a set F for which there are balls with centers in F such that it is impossible to control the σ-measure of the Ω-projection of tents over these balls.

A result analogous to Theorem 4 was considered in Theorem A of [4]. There, 'admissible' was given a much less restrictive definition. However, there is a gap in the proof of that result. Specifically, the use of Theorem 1.1 of [15] was not justified. With the more restrictive definition of admissible one might be able to apply the technique of [4] to deduce our Theorem 4. However, in this article we prove a stronger result (Theorem 3) and show that Theorem 4 follows as a corollary.

We shall make use of the following covering lemma. It is stated slightly differently from the covering lemma in [7]. However, the proof is the same and so we omit it. Recall S denotes the unit sphere S^{n-1}.

LEMMA 1. *Let E be a subset of S. For each $x \in E$, let $0 < r(x) \leqslant 2$ be chosen. Let $\omega = \cup\{B_{x,r(x)} : x \in E\}$. Pick $k > 4$. Then there exists a sequence $\{x_i\} \subset E$ such that the balls $\{B_{x_i, r_{x_i}}\}$ are pairwise disjoint, $E \subset \cup B_{x_i, k\cdot r_{x_i}}$, $\omega \subset \cup B_{x_i,(2+k)r_{x_i}}$, and for every $x \in E$, there exists i such that $x \in B_{x_i, k\cdot r_{x_i}}$, and $r_{x_i} > r(x)/2$.*

In what follows, c denotes a real value that may vary from line to line but does not depend in an important way on the parameters of interest.

2. Weaktype Inequality

PROPOSITION 1. *There exists a constant c such that for every Borel measure μ on S and each $\varepsilon > 0$,*

$$\Lambda\left\{x \in \mathbb{R}^n : \frac{K\mu}{K\sigma}(x) > \varepsilon\right\} \leqslant c \cdot \frac{\|\mu\|}{\varepsilon}.$$

Proof. Define the maximal function $M\mu$ on $\{x \in \mathbb{R}^n : |x| > 1\}$ by

$$M\mu(x) = \sup\left\{\frac{\mu B_{x/|x|,t}}{\sigma B_{x/|x|,t}} : 2 > t \geqslant 1/\sqrt{|x|}\right\}.$$

We first show that

$$\frac{K\mu}{K\sigma}(x) \leqslant c \cdot M\mu(x), \tag{2.1}$$

where c is independent of x, μ. Let $|x| > 1$. Since $K\sigma(x) \approx \mathrm{e}^{\lambda|x|}|x|^{(1-n)/2}$ as $|x| \to \infty$ ([4], Lemma 4.1), we have

$$\begin{aligned}
\frac{K\mu}{K\sigma}(x) &\leqslant c \cdot |x|^{(n-1)/2} \cdot \mathrm{e}^{-\lambda|x|} \int \mathrm{e}^{\lambda|x|\langle \frac{x}{|x|}, b\rangle}\, \mathrm{d}\mu(b) \\
&= c \cdot |x|^{(n-1)/2} \int \mathrm{e}^{-\lambda(|x|/2)|\frac{x}{|x|}-b|^2}\, \mathrm{d}\mu(b) \\
&\leqslant c \cdot |x|^{(n-1)/2} \sum_{k=0}^{[\sqrt{|x|}]} \int_{\frac{k}{\sqrt{|x|}} \leqslant \left|\frac{x}{|x|}-b\right| \leqslant \frac{k+1}{\sqrt{|x|}}} \mathrm{e}^{-\lambda(|x|/2)\left|\frac{x}{|x|}-b\right|^2}\, \mathrm{d}\mu(b) \\
&\quad + \int_{1 \leqslant \left|\frac{x}{|x|}-b\right| \leqslant 4} c \cdot |x|^{(n-1)/2}\, \mathrm{e}^{-\lambda(|x|/2)\left|\frac{x}{|x|}-b\right|^2}\, \mathrm{d}\mu(b) \\
&\leqslant \sum_{k=0}^{[\sqrt{|x|}]} \mathrm{e}^{-\lambda\frac{k^2}{2}} \cdot (k+1)^{n-1} \cdot \frac{\mu B_{x/|x|,(k+1)/\sqrt{|x|}}}{((k+1)/\sqrt{|x|}))^{n-1}} + c \cdot \frac{\mu B_{x/|x|,4}}{\sigma B_{x/|x|,4}} \\
&\leqslant \left[c \cdot \left(\sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k^2/2}(k+1)^{n-1}\right) + d\right] \cdot M\mu(x) \\
&= c \cdot M\mu(x),
\end{aligned}$$

proving Equation (2.1).

Thus to finish the proof, it is enough to show that

$$\Lambda\{x \in \mathbb{R}^n : M\mu(x) > \varepsilon\} \leqslant c \cdot \frac{\|\mu\|}{\varepsilon}. \tag{2.2}$$

Let E denote the set in inequality (2.2). For each $x \in E$, there exists t_x such that $2 \geqslant t_x \geqslant 1/\sqrt{|x|}$ and

$$\mu B_{x/|x|,t_x} > \varepsilon \cdot \sigma B_{x/|x|,t_x}.$$

Thus $\{B_{x/|x|,t_x}: x \in E\}$ is a collection of balls on S, so by Lemma 1, there is a countable, pairwise disjoint subset of them $\{B_{x_i/|x_i|,t_i}\}$ such that $\{B_{x_i/|x_i|,c\cdot t_i}\}$ covers the original collection, where c depends only on n. Again, by Lemma 1 and Definition 2, it follows

$$E \subset \bigcup_{x\in E} G\left(\frac{x}{|x|}, t_x\right) \subset \bigcup_i G\left(\frac{x_i}{|x_i|}, c \cdot t_{x_i}\right),$$

and so

$$\Lambda(E) \leqslant \sum \sigma B_{\frac{x_i}{|x_i|},c\cdot t_{x_i}} \leqslant c \cdot \sum \sigma B_{\frac{x_i}{|x_i|},t_{x_i}} \leqslant \frac{c}{\varepsilon} \cdot \sum \mu B_{\frac{x_i}{|x_i|},t_{x_i}} \leqslant \frac{c}{\varepsilon} \cdot \|\mu\|,$$

which is (2.2). □

3. Relative Limits of Solutions of the Helmholtz Equation

We study the boundary behavior of functions of the form $K\mu/K\sigma$ by looking separately at the case where μ is singular and absolutely continuous.

THEOREM 1. *Let μ be a measure on S that is singular with respect to σ. Then there exists a set E thin at the boundary with respect to measure such that $K\mu/K\sigma$ $(x) \to 0$ as $|x| \to \infty$, $x \notin E$.*

Proof. Let $\varepsilon > 0$. We claim that $\{x: \frac{K\mu}{K\sigma}(x) > \varepsilon\}$ is thin at the boundary with respect to measure.

In order to show this, we start by choosing $\eta > 0$. Since μ is singular, there exists a sequence of balls $B_i = B_{b_i,r_i}$ such that $\sum \sigma(B_i) < \eta$ and μ has support in $\cup B_i$. For each positive integer j, let $\mu = \mu_j + \nu_j$, where μ_j is the restriction of μ to $\cup\{B_i: 1 \leqslant i \leqslant j\}$. Since $\|\nu_j\| \to 0$ as $j \to \infty$, we deduce from Proposition 1 that there exists j such that

$$\Lambda\left\{x: \frac{K\nu_j}{K\sigma}(x) > \varepsilon/2\right\} \leqslant \frac{c}{\varepsilon} \cdot \|\nu_j\| < \eta.$$

Fix this j value. Let $r^2 = \min\{r_1^2, \ldots, r_j^2\}$. Choose t such that $t > r^{-2}$ and for all $|x| > t$, $c'|x|^{(n-1)/2}\,\mathrm{e}^{-(\lambda/2)|x|r^2}\|\mu\| < \varepsilon/2$, where c' is a constant such that

$K\sigma(x) \leqslant c' \mathrm{e}^{\lambda|x|}|x|^{(1-n)/2}$ for all x. If $|x| > t$ and $x/|x| \notin \cup\{B_{b_i,2r_i}: 1 \leqslant i \leqslant j\}$, then (recall the first string of inequalities in the proof of Proposition 1)

$$\begin{aligned}\frac{K\mu_j}{K\sigma}(x) &\leqslant c' \cdot |x|^{(n-1)/2} \int_{\cup\{B_i:\, 1\leqslant i\leqslant j\}} \mathrm{e}^{-(\lambda|x|/2)\left|\frac{x}{|x|}-b\right|^2} \,\mathrm{d}\mu_j(b)\\ &\leqslant c' \cdot |x|^{(n-1)/2} \mathrm{e}^{-(\lambda|x|/2)r^2}\|\mu\|\\ &< \varepsilon/2,\end{aligned}$$

so

$$\begin{aligned}&\Lambda\left\{\frac{K\mu}{K\sigma} > \varepsilon, |x| > t\right\}\\ &\quad\leqslant \Lambda\left\{\frac{K\nu_j}{K\sigma} > \varepsilon/2, |x| > t\right\} + \Lambda\left\{\frac{K\mu_j}{K\sigma} > \varepsilon/2, |x| > t\right\}\\ &\quad\leqslant \Lambda\left\{\frac{K\nu_j}{K\sigma} > \varepsilon/2\right\} + \Lambda\left\{\frac{K\mu_j}{K\sigma} > \varepsilon/2, |x| > t, x/|x| \in \textstyle\bigcup_1^j B_{b_i,2r_i}\right\}\\ &\quad\leqslant \eta + \Lambda\left\{x: |x| > r^{-2}, x/|x| \in \textstyle\bigcup_1^j B_{b_i,2r_i}\right\}\\ &\quad\leqslant \eta + c \cdot \sum \sigma B_{b_i,2r_i}\\ &\quad< c \cdot \eta.\end{aligned}$$

This proves the claim.

Let ε_j, η_j be sequences of real numbers decreasing to 0 such that $\sum \eta_j < \infty$. Let

$$E_j = \left\{\frac{K\mu}{K\sigma} > \varepsilon_j\right\}.$$

By what we have just proved, there exists $t_j \nearrow \infty$ such that

$$\Lambda(E_j \cap \{|x| \geqslant t_j\}) < \eta_j.$$

Let

$$E = \bigcup\{E_j \cap \{t_j \leqslant |x| \leqslant t_{j+1}\}\}.$$

Since Λ is countably subadditive,

$$\Lambda(E \cap \{|x| \geqslant t\}) \leqslant \sum_{t_{j+1}\geqslant t} \Lambda(E_j \cap \{|x| \geqslant t_j\}) \leqslant \sum_{t_{j+1}\geqslant t} \eta_j,$$

and the latter goes to 0 as $t \to \infty$, proving that E is thin at the boundary with respect to measure. Finally, if $|x| \geqslant t_k$ and $x \notin E$ then $(K\mu/K\sigma)(x) \leqslant \max\{\varepsilon_l : l \geqslant k\} \to 0$ as $k \to \infty$. □

In order to study the absolutely continuous case, we need the following extension of Lusin's theorem.

PROPOSITION 2. *Let $f \in L_1(S)$. Then for each $\varepsilon > 0$ there is a continuous function g such that $\sigma\{f \neq g\} < \varepsilon$ and $\|f - g\|_1 < \varepsilon$.*

Lusin's theorem gives all but the condition that $\|f - g\|_1 < \varepsilon$, and this follows from a careful use of the triangle inequality. We omit the proof.

We next show that the solution of the Dirichlet problem is uniformly continuous.

PROPOSITION 3. *Let f be a continuous function on S. Extend f to $\mathbb{R}^n$ by $(Kf/K\sigma)(x)$ for $|x| \neq 1$. Let $\varepsilon > 0$. Then there exist $\delta > 0, M < \infty$ such that for all $b \in S$ and all $x \in \mathbb{R}^n$, if $|x| \geqslant M$ and $|x/|x| - b| < \delta$, then $|(Kf/K\sigma)(x) - f(b)| < \varepsilon$.*

Proof. Since f is uniformly continuous, we can choose $\delta > 0$ such that for all $b, b' \in S$, $|b - b'| < 2\delta$ implies $|f(b) - f(b')| < \varepsilon/2$. Fix $b_0 \in S$, and let x satisfy $|x/|x| - b_0| < \delta$. Then

$$|(Kf/K\sigma)(x) - f(b_0)| \leqslant \frac{\int e^{\lambda\langle x,b\rangle} |f(b) - f(b_0)| \, d\sigma}{\int e^{\lambda\langle x,b\rangle} \, d\sigma} = I + II,$$

where in the numerator we integrate over $|b - b_0| < 2\delta$, $|b - b_0| \geqslant 2\delta$ in I and II, respectively. Note that

$$I \leqslant (\varepsilon/2) \cdot \frac{\int e^{\lambda\langle x,b\rangle} d\sigma}{\int e^{\lambda\langle x,b\rangle} \, d\sigma} = \varepsilon/2,$$

and

$$\begin{aligned} II &\leqslant c \cdot |x|^{(n-1)/2} \int_{|b-b_0|>2\delta} e^{-\lambda|x|} e^{\lambda\langle x,b\rangle} |f(b) - f(b_0)| \, d\sigma(b) \\ &\leqslant c \cdot |x|^{(n-1)/2} \cdot \|f\|_\infty \int_{|b-b_0|>2\delta} e^{-(\lambda|x|/2)\left|b - \frac{x}{|x|}\right|^2} \, d\sigma(b) \\ &\leqslant c \cdot |x|^{(n-1)/2} \|f\|_\infty \, e^{-(\lambda|x|/2)\delta^2} \\ &\leqslant \varepsilon/2 \text{ if } |x| > M, \end{aligned}$$

where M is chosen such that $c \cdot M^{(n-1)/2} \cdot e^{-\lambda M\delta^2/2} \|f\|_\infty < \varepsilon/2$. □

THEOREM 2. *Let* $f \in L_1(S)$, *and* $\{\Omega_b\}$ *any set of approach regions. Then there exists* E *thin with respect to measure at the boundary and* $F \subset \mathbb{R}^n$ *with* $\sigma(F) = 0$ *such that for every* $b \in S - F$, $(Kf/K\sigma)(x) \to f(b)$ *as* $|x| \to \infty$, $x \in \Omega_b - E$.

Proof. Let $\varepsilon_k, \eta_k \searrow 0$, $\sum \eta_k < \infty$. By Proposition 2, there exist f_k continuous on S such that $\sigma\{f \neq f_k\} < \eta_k$ and $\|f - f_k\|_1 < \varepsilon_k \eta_k$. Let

$$F = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{b \in S\colon f(b) \neq f_k(b)\}$$

Then $\sigma(F) = 0$, since $\sum \eta_k < \infty$. Proposition 1 implies

$$\Lambda \left\{ \frac{K(f - f_k)}{K\sigma} > \varepsilon_k \right\} \leqslant c \cdot \frac{\|f - f_k\|_1}{\varepsilon_k} < c \cdot \eta_k.$$

By Proposition 3, there exists $r_k \searrow 0$ such that for all $b \in S$, if $|y| \geqslant r_k^{-2}$ and $|y/|y| - b| < r_k$ (that is, if $y \in G(b, r_k)$), then

$$\left| \frac{Kf_k}{K\sigma}(y) - f_k(b) \right| < \varepsilon_k. \tag{3.1}$$

Since $\Omega_b \to b$ uniformly at ∞ (recall Definition 1), we can choose $T_k \nearrow \infty$ such that for all $b \in S$ and for all k, $\Omega_b \cap D_{T_{k-1}} \subset G(b, r_k)$, where $D_{T_k} = \{x \in \mathbb{R}^n\colon |x| \geqslant T_k\}$. It follows

$$\Omega_b \cap D_{T_{k-1}} \subset \bigcup_{j \geqslant k} (G(b, r_j) - D_{T_j}).$$

Let $E_j = \{K|f - f_j|/K\sigma > \varepsilon_j\} - D_{T_j}$, $E = \cup E_j$. Let $T > 1$. Then

$$\Lambda(E \cap D_T) \leqslant \sum_{T_j \geqslant T} \Lambda\{K|f - f_j|/K\sigma \geqslant \varepsilon_j\} \leqslant \sum_{T_j \geqslant T} c \cdot \eta_j \to 0$$

as $T \to \infty$. Thus E is thin at the boundary with respect to measure.

Let $b \in S - F$, $\varepsilon > 0$. Pick k such that $\varepsilon_k < \varepsilon/2$, and $f_j(b) = f(b)$ for all $j \geqslant k$. Let $y \in D_{T_{k-1}} \cap \Omega_b - E$. Then there exists $j \geqslant k$ such that $y \in G(b, r_j) - D_{T_j}$, so

$$\begin{aligned} |(Kf/K\sigma)(y) - f(b)| &\leqslant |(Kf/K\sigma)(y) - (Kf_j/K\sigma)(y)| \\ &\quad + |(Kf_j/K\sigma)(y) - f_j(b)| + |f_j(b) - f(b)| \\ &< \varepsilon_j + \varepsilon_j + 0 \\ &< \varepsilon \end{aligned}$$

□

4. Admissible Sets and Pointwise Results

In order to define what is an admissible collection of approach regions, we make the following definitions.

DEFINITION 3. Let $\{\Omega_b : b \in S\}$ be a collection of approach regions. Let X be a subset of $\mathbb{R}^n$. We define the Ω-*projection of* X to be

$$\Omega^*(X) = \{b \in S : \Omega_b \cap X \neq \emptyset\}.$$

DEFINITION 4. Let $B = B_{c,r}$ be an open ball on S, and α a fixed positive constant. Let $\{P_\alpha(b)\}$ be the parabolic family of approach regions (recall $P_\alpha(b) = \{x \in \mathbb{R}^n : |x/|x| - b| \leqslant \alpha |x|^{-1/2}\}$). Define the *tent*, $T_\alpha(B)$, over B to be the set whose projection with respect to this parabolic family is B.

The reader can check that

$$T_\alpha(B) = \left\{ x \in \mathbb{R}^n : B_{\frac{x}{|x|}, \frac{\alpha}{\sqrt{|x|}}} \subset B \right\}.$$

We refer to the point $(\alpha^2/r^2)c$ as the *vertex* of the tent.

DEFINITION. 5. Fix $\alpha > 0$. We say the collection of approach regions $\{\Omega_b : b \in S\}$ is an *admissible collection* if (1) each set Ω_b is starlike with respect to the origin, and (2) there exists a closed set F of σ-measure 0 having the following property: for every $\varepsilon > 0$ and $\gamma > 1$ there exists $c = c_{\varepsilon,\gamma}$ such that for every ball $B = B_{x,r}$ for which the distance of $B_{x,\gamma r}$ to F is greater than ε, we have

$$\sigma(\Omega^*(T_\alpha(B))) \leqslant c \cdot \sigma(B).$$

Thus we only ask for control of the Ω-projection of tents over balls that stay away from F.

Using the covering lemma, it is easy to show that the definition is independent of α.

After a few more definitions, we show how to write the definition of *admissible collection* in an equivalent way.

DEFINITION 6. Let Ω be any subset of $\mathbb{R}^n$. Let $t > 0$. The t-*section* of Ω is defined to be $\Omega(t) = \{b \in S : bt \in \Omega\}$.

DEFINITION 7. Let $b \in S$, and r, α given positive numbers. Let $S_\alpha(b, r) = \{c \in S : B_{b,r} \cap \Omega_c(\alpha^2 r^{-2}) \neq \emptyset\}$. If $\alpha = 1$, denote it by $S(b, r)$.

We leave it to the reader to show that for $B_{b,r}$ a ball on S,

$$\Omega^*(T_\alpha(B_{b,r})) \subset S_\alpha(b, r) \subset \Omega^*(T_\alpha(B_{b,2r})).$$

Thus we may replace $\sigma(\Omega^*(T_\alpha(B)))$ by $\sigma(S_\alpha(B))$ in Definition 5.

THEOREM 3. *Let* $\{\Omega_b : b \in S\}$ *be an admissible collection, with* F *its associated closed set of measure 0. Let* E *be thin with respect to measure at the boundary. Then for* σ*-a.e.* $b \in S$, E *lies eventually outside* Ω_b *in the sense that there exists* $T = T(b)$ *such that for all* y *with* $|y| > T$, $y \notin \Omega_b \cap E$.

We deduce immediately from Theorems 1, 2 and 3 the following

THEOREM 4. *Let* μ *be a positive regular Borel measure on* S. *Let* $\{\Omega_b : b \in S\}$ *be an admissible collection. Then there is a subset* H *of* S *having* σ*-measure 0 such that for all* $b \in S - H$, $(K\mu)/(K\sigma)(y)$ *converges to* $(\mathrm{d}\mu/\mathrm{d}\sigma)(b)$ *as* $|y| \to \infty$, $y \in \Omega_b$.

Proof of Theorem 3. For each $\varepsilon > 0$, let

$$B_\varepsilon = \bigcup_{b\in F} B_{b,\varepsilon},\ G_\varepsilon = \bigcup_{b\in F} G(b,\varepsilon),\ E_\varepsilon = E - G_\varepsilon.$$

Now fix $\varepsilon > 0$ once and for all. Let $\{c_k\}$ be any decreasing, summable sequence of positive real numbers. Since E, and hence E_{2e}, are thin with respect to measure at the boundary, there exists $T_1 < T_2 < \cdots$ with $\lim_{k\to\infty} T_k = \infty$ such that for every k there is a sequence of balls $\{B_{b_{k,j},r_{k,j}}\}_j$ with $E_{2\varepsilon} \cap \{|x| > T_k\} \subset \bigcup_{j=1}^{\infty} G(b_{k,j}, r_{k,j})$ and $\sum_{j=1}^{\infty} \sigma B_{b_{k,j},r_{k,j}} < c_k$. We may assume that c_1 is small enough so that for each k and j, $r_{k,j} < \varepsilon/3$ and $E_{2\varepsilon} \cap G_{b_{k,j},r_{k,j}} \neq \emptyset$ (otherwise we may discard $B_{b_{k,j},r_{k,j}}$).

Let $b \in B_{b_{k,j},2r_{k,j}}$, $c \in F$, $z \in G\ (b_{k,j}, r_{k,j}) \cap E_{2\varepsilon}$. Notice

$$\left| c - \frac{z}{|z|} \right| \geqslant 2\varepsilon,$$

since if not, $|z| \geqslant r_{k,j}^{-2} \geqslant 9\varepsilon^{-2} \geqslant (2\varepsilon)^{-2}$ would imply $z \in G(c, 2\varepsilon) \subset G_{2\varepsilon}$, contrary to our assumption that $z \in E_{2\varepsilon}$. Thus

$$\left| \frac{z}{|z|} - b \right| \leqslant \left| \frac{z}{|z|} - b_{k,j} \right| + |b_{k,j} - b| < r_{k,j} + 2r_{k,j} = 3r_{k,j} < \varepsilon,$$

and so

$$2\varepsilon \leqslant \left| c - \frac{z}{|z|} \right| \leqslant |c - b| + \left| b - \frac{z}{|z|} \right| < |b - c| + \varepsilon,$$

implying $|b - c| > \varepsilon$. It follows $B_{b_{k,j},2r_{k.j}}$ is at least a distance of ε from F. Since the approach regions form an admissible family, there exists c_ε such that for all $k, j, \sigma(S(b_{k,j}, r_{k,j})) \leqslant c_\varepsilon \sigma(B_{b_{k,j},r_{k,j}})$.

Let $S_k = \bigcup_{j=1}^{\infty} S(b_{k,j}, r_{k,j})$. Since

$$\sum_{k=1}^{\infty} \sigma(S_k) \leqslant \sum_{k,j} \sigma(S(b_{k,j}, r_{k,j})) \leqslant c_\varepsilon \sum_{k,j} \sigma(B_{b_{k,j}, r_{k,j}}) \leqslant c_{\varepsilon \sum_{k=1}^{\infty} c_k < \infty},$$

$\sigma(\overline{\lim}\, S_k) = 0$, where $\overline{\lim}\, S_k = \bigcap_{k=1}^{\infty} \bigcup_{l=k}^{\infty} S_l$.

Let $b \in (S^{n-1} - B_{3\varepsilon}) - \overline{\lim}\, S_k$. Then there exists k_0 such that for all $k \geqslant k_0$ and all j, $b \notin S(b_{k,j}, r_{k,j})$, so

$$B_{b_{k,j}, r_{k,j}} \cap \Omega_b(r_{k,j}^{-2}) = \emptyset. \tag{4.1}$$

We claim there does not exist y in $E_{2\varepsilon} \cap \Omega_b$ for $|y| \geqslant T_{k_0}$. For if there were such a y, then $y \in G(b_{k,j}, r_{k,j})$ for some $k \geqslant k_0$, and a positive integer j. By definition, $|y| \geqslant 1/r_{k,j}^2$. Since Ω_b is starlike with respect to the origin,

$$\frac{y}{|y|} \in B_{b_{k,j}, r_{k,j}} \cap \Omega_b(|y|) \subset B_{b_{k,j}, r_{k,j}} \cap \Omega_b(r_{k,j}^{-2}),$$

which contradicts (4.1). It follows that as we approach b at ∞ within Ω_b, we eventually leave E (since for $b \in S^{n-1} - B_{2\varepsilon}$, it is impossible for a sequence contained in $G_{2\varepsilon}$ to approach b at ∞). Since ε is an arbitrarily chosen positive number, the theorem follows. $\square$

5. Examples of Admissible Collections

In this section, we fix a positive aperture α once and for all. We describe a way to generate examples of admissible collections in $\mathbb{R}^3$ other than the paraboloids of the Korányi–Taylor theorem.

For each θ_1, θ_2 with $0 \leqslant \theta_1 \leqslant \pi$, and $0 \leqslant \theta_2 < 2\pi$, define the unitary transformation U_{θ_1,θ_2} whose matrix with respect to the standard bases is given by the product

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus U_{θ_1,θ_2} first rotates by θ_1 in the first two coordinates (fixing the third), then rotates the resulting point by θ_2 in the second and third coordinates (fixing the first).

Consider the mapping

$$U\colon (0, \pi) \times [0, 2\pi) \to S$$

given by

$$U(\theta_1, \theta_2) = U_{\theta_1,\theta_2}(1, 0, 0) = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2).$$

Notice that U is one-to-one and maps onto $S - \{(\pm 1, 0, 0)\}$; $\{(\pm 1, 0, 0)\}$ corresponds to the image of the set where θ_1 is 0 or π. The surface measure σ is given by

$$\mathrm{d}\sigma = \sin\theta_1 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2. \tag{5.1}$$

We identify each point b of $S - \{(\pm 1, 0, 0)\}$ with (θ_1, θ_2), its preimage under U and occasionally we will write U_b instead of U_{θ_1,θ_2}.

DEFINITION 8. Let $x \in \mathbb{R}^n$, $x \neq 0$. Define the 'α-balloon set'

$$C_\alpha(x) = \left\{ y \in \mathbb{R}^n : 0 < |y| \leqslant |x| \text{and} \left| \frac{y}{|y|} - \frac{x}{|x|} \right| \leqslant \alpha \left(\frac{1}{|y|} - \frac{1}{|x|} \right)^{1/2} \right\}.$$

Let $\{x_k\}$ be a sequence in $\mathbb{R}^3$ chosen so that $|x_k|$ is an increasing sequence with limit ∞, x_k converges to $e = (1, 0, 0)$ at ∞, and $\{x_k/|x_k|\}$ lies in the x_1x_2-plane and is of the form $(\cos\theta_k, \sin\theta_k, 0)$, for some $\theta_k \in (0, \pi)$. Suppose in addition we have

$$\sqrt{|x_k|} \left| \frac{x_k}{|x_k|} - e \right| \to \infty \tag{5.2}$$

and

$$\sqrt{|x_k|} \left| \frac{x_{k+1}}{|x_{k+1}|} - e \right| \leqslant M, \tag{5.3}$$

for some constant M. The meaning of (5.2) is that $\{x_k\}$ is not contained in a paraboloid $P_\alpha(e)$ for any fixed aperture α. We can construct such a sequence as follows. Consider the curves $C_1: t \mapsto (t, \sqrt{t}, 0)$, $C_2: t \mapsto (t, t^{3/4}, 0)$, for $t \geqslant 0$. Let x_1 be any point on C_2. Pick a point y_1 on C_1 for which $|y_1| = |x_1|$. Let x_2 be the point on the intersection of C_2 with the line that contains the origin and y_1. If we continue this procedure, we generate the sequence $\{x_k\}$.

Define

$$\Omega = \Omega_e = \bigcup_{k=1}^{\infty} C_\alpha(x_k),$$

and for $b \in S$, define

$$\Omega_b = U_b(\Omega).$$

PROPOSITION 4. *The set* $\{\Omega_b : b \in S\}$ *is an admissible collection of approach regions. The associated closed set of* σ*-measure* 0 *is* $F = \{e\} = \{(1,0,0)\}$. *In addition, there exists a sequence* $\{r_k\}$ *of radii decreasing to* 0 *such that*

$$\sigma(\Omega^*(T_\alpha(B_{e,r_k})))/r_k^2 \to \infty \text{ as } k \to \infty.$$

The last part of the statement shows why we must include the exceptional set F in our definition of admissible collection. The key to the proof of the first part of the proposition is the following lemma which essentially shows that we may replace the tent in Definition 5 by its vertex.

LEMMA 2. *Fix a ball* $B = B_{c,r}$, *and* $\gamma > 1$. *Choose a positive integer* k *such that* $r \geqslant \frac{\alpha}{\sqrt{(1-\gamma^{-1})|x_k|}}$. *Suppose* $|x| = |x_k|$ *and* $C_\alpha(x) \cap T_\alpha(B_{c,r}) \neq \emptyset$. *Then* $\alpha^2 r^{-2} c \in C_{2\sqrt{\gamma}\alpha}(x)$.

Proof of Lemma. Let $y \in C_\alpha(x) \cap T_\alpha(B_{c,r})$. Then

$$\left|\frac{y}{|y|} - \frac{x}{|x|}\right| \leqslant \alpha\sqrt{\frac{1}{|y|} - \frac{1}{|x|}} \text{ and } B_{y/|y|,\alpha/\sqrt{|y|}} \subset B_{c,r}$$

(so $|y| \geqslant \alpha^2 r^{-2}$). Thus

$$\begin{aligned}
\left|c - \frac{x}{|x|}\right| &\leqslant \left|c - \frac{y}{|y|}\right| + \left|\frac{y}{|y|} - \frac{x}{|x|}\right| \\
&\leqslant r + \alpha\sqrt{\frac{1}{|y|} - \frac{1}{|x|}} \\
&\leqslant r + \frac{\alpha}{\sqrt{|y|}} \\
&\leqslant 2r \\
&= 2\sqrt{\gamma}\alpha\sqrt{\frac{r^2}{\alpha^2} - \frac{r^2(1-\gamma^{-1})}{\alpha^2}} \\
&\leqslant 2\sqrt{\gamma}\alpha\sqrt{\frac{r^2}{\alpha^2} - \frac{1}{|x|}},
\end{aligned}$$

and so $\alpha^2 r^{-2} c \in C_{2\sqrt{\gamma}\alpha}(x)$. □

Proof of the Proposition. It is easy to show that the α-balloons are starlike with respect to the origin, hence so is Ω_b for each $b \in S$.

Let $B = B_{c,r}$ be a ball on S, $\bar{\gamma} > 1$, and suppose the distance of $B_{c,\bar{\gamma}r}$ to $e = \{(1,0,0)\}$ is positive. We need to show there is a constant, c, such that

$$\sigma(\Omega^*((T_\alpha(B))) \leqslant c \cdot \sigma(B),$$

where c depends only on α, $\bar{\gamma}$, M (recall 5.3), and the above distance.

Let $\gamma = \frac{\bar{\gamma}^2}{\bar{\gamma}^2 - 1}$. Suppose the α-balloon $C_\alpha(x)$ intersects the tent $T_\alpha(B_{c,r})$. Then $|x| \geqslant \alpha^2 r^{-2} = \alpha^2(\bar{\gamma} r\sqrt{1-\gamma^{-1}})^{-2}$. Also $\emptyset \neq C_\alpha(x) \cap T_\alpha(B_{c,r}) \subset C_\alpha(x) \cap T_\alpha(B_{c,\bar{\gamma}r})$. We deduce (using the lemma with r replaced by $\bar{\gamma}r$) that

$$\Omega^*(T_\alpha(B)) = \{b\colon \Omega_b \cap T_\alpha(B) \neq \emptyset\} \subset \{b\colon \alpha^2(\bar{\gamma}r)^{-2}c \in \Omega'_b\},$$

where

$$\Omega'_b = U_b\left(\bigcup_{k=1}^{\infty} C_{2\sqrt{\gamma}\alpha}(x_k)\right).$$

Let y be chosen in Ω'_e such that $|y| = \alpha^2(\bar{\gamma}r)^{-2}$. Pick a positive integer k_0 such that

$$|x_{k_0-1}| \leqslant |y| = \alpha^2(\bar{\gamma}r)^{-2} < |x_{k_0}|.$$

Then $y \in C_{2\sqrt{\gamma}\alpha}(x_k)$ for some $k \geqslant k_0$. Suppose $k \geqslant k_0+1$. Then

$$\left|\frac{y}{|y|} - \frac{x_k}{|x_k|}\right| \leqslant 2\sqrt{\gamma}\alpha\sqrt{\frac{1}{|y|} - \frac{1}{|x_k|}} < \frac{2\sqrt{\gamma}\alpha}{\sqrt{|y|}},$$

and so

$$\begin{aligned}
\left|\frac{y}{|y|} - e\right| &\leqslant \frac{2\sqrt{\gamma}\alpha}{\sqrt{|y|}} + \left|\frac{x_k}{|x_k|} - e\right| \\
&\leqslant \frac{2\sqrt{\gamma}\alpha}{\sqrt{|y|}} + \frac{M}{\sqrt{|x_{k-1}|}} \\
&\leqslant \frac{2\sqrt{\gamma}\alpha}{\sqrt{|y|}} + \frac{M}{\sqrt{|x_{k_0}|}} \\
&\leqslant \frac{2\sqrt{\gamma}\alpha + M}{\sqrt{|y|}}.
\end{aligned}$$

The second inequality above follows from (5.3). Thus y is in the paraboloid $P_{2\sqrt{\gamma}\alpha+M}(e)$, and so

$$\left\{\frac{y}{|y|}\colon y \in \Omega'_e, |y| = \alpha^2(\bar{\gamma}r)^{-2}\right\} \subset B_1 \cup B_2,$$

where

$$B_1 = B_{e,\frac{(2\sqrt{\gamma}\alpha+M)}{\alpha}\bar{\gamma}r}, \qquad B_2 = B_{x_{k_0}/|x_{k_0}|,2\sqrt{\gamma}\bar{\gamma}r}.$$

It follows

$$\begin{aligned}\Omega^*(T_\alpha(B)) &\subset \left\{b\colon (U_b)^{-1}\left(\frac{\alpha^2}{(\bar{\gamma}r)^2}c\right) \in \Omega'_e\right\}\\ &= \{b\colon (U_b)^{-1}(c) \in \Omega'_e(\alpha^2(\bar{\gamma}r)^{-2})\}\\ &\subset \{b\colon (U_b)^{-1}(c) \in B_1 \cup B_2\}.\end{aligned}$$

In order to move c (via U_b^{-1}) to a point of $B_1 \cup B_2$ (first by a rotation U_{0,θ_2} then by a rotation $U_{\theta_1,0}$) the only control we have over the third coordinate of the inverse image is with θ_2 (since $U_{\theta_1,0}$ does not change the third coordinate). It follows that θ_2 must lie in an interval of length the order of r, and θ_1 lies in the union of two intervals each of length the order of r, and so, using (5.1), we see $\sigma(\Omega^*(T_\alpha(B))$ is bounded by a multiple of r^2. This completes the proof that the collection is admissible.

We now show how to choose the sequence $\{r_k\}$, described in the proposition. Fix a positive integer, k. Choose $r_k = \sqrt{2}\alpha/\sqrt{|x_k|}$. We have

$$\Omega^*(T_\alpha(B_{e,r_k})) = \{b \in S\colon \Omega_b \cap T_\alpha(B_{e,r_k}) \neq \emptyset\} \tag{5.4}$$

$$\supset \{b\colon U_b(C_\alpha(x_k)) \ni (\alpha^2)r_k^{-2}e\}, \tag{5.5}$$

since $\alpha^2 r_k^{-2}e$ is in $T_\alpha(B_{e,r_k})$ (it is the vertex). Thus it suffices to show that r_k^{-2} multiplied by the σ-measure of the last set tends to infinity as $k \to \infty$.

Suppose $y \in C_\alpha(x_k)$, $|y| = \alpha^2 r_k^{-2}$, and $y/|y|$ can be written $(\cos\theta, \sin\theta, 0)$, (that is, it projects onto the equator). A simple computation shows.

$$\left|\frac{y}{|y|} - \frac{x_k}{|x_k|}\right| = 2|\sin(\theta - \theta_k)/2|.$$

Since $y \in C_\alpha(x_k)$ and $|y| = \alpha^2 r_k^{-2}$,

$$2|\sin(\theta - \theta_k)/2)| \leqslant \alpha\sqrt{\frac{r_k^2}{\alpha^2} - \frac{r_k^2}{2\alpha^2}} = \frac{r_k}{\sqrt{2}}.$$

It follows that

$$\begin{aligned}\Omega^*(T_\alpha(B_{e,r_k})) &\supset \{b\colon U_b(C_\alpha(x_k)) \ni \alpha^2 r_k^{-2}e\}\\ &\supset \{b\colon b = (\cos(-\theta), \sin(-\theta), 0), |\sin(\theta_k - \theta)/2| \leqslant r_k/(2\sqrt{2})\}.\end{aligned}$$

Call the latter set B. Note that B is essentially an interval along the equator having linear measure the order of r_k and distance to e about $|x_k/|x_k| - e|$. For any $b \in B$, operate on $U_b(C_\alpha(x_k))$ by U_{0,θ_2} for any θ_2. Let b' be the image of b. Then $U_{b'}(C_\alpha(x_k)) \supset \alpha^2 r_k^{-2}e$, since e is fixed by U_{0,θ_2}. It follows that $\Omega^*(T_\alpha(B_{e,r_k}))$

contains the annular region consisting of the image of B under all unitary transformations fixing e. This set has σ-measure of the order $|x_k/|x_k| - e|r_k$. If we divide by r_k^2 and replace r_k by $\sqrt{2}\alpha|x_k|^{-1/2}$ we see that the σ-measure is of the order of $|x_k/|x_k| - e|\sqrt{|x_k|}$, which by (5.2) tends to ∞ as $k \to \infty$. This completes the proof. □

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