Potential theory on trees and multiplication operators

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Abstract. The article surveys a number of potential theory results in the discrete setting of trees and in an application to complex analysis. On trees for which the associated random walk is recurrent, we discuss Riesz decomposition, flux, a type of potential called H-potential, and present a new result dealing with the boundary behaviour of H-potentials on a specific recurrent homogeneous tree. On general trees we discuss Brelot structures and their classification. On transient homogeneous trees we discuss clamped and simply-supported biharmonic Green functions.

We also describe an application of potential theory, namely a certain minimum principle for multiply superharmonic functions, that is used to prove a result concerning the norm of a class of multiplication operators from $H^\infty(D)$ to the Bloch space on D, where D is a bounded symmetric domain. The proof of the minimum principle involves the use of the Cartan-Brelot topology.

1. Introduction

In this article we survey some of our potential theory results in discrete and classical settings. Most of the paper considers potential theory in the discrete setting of trees. In one section we consider it in a Brelot space, and in the final section we consider an application of potential theory to a problem involving complex and functional analysis.

On trees, functions are viewed as real-valued, with domain the set of vertices. Once transition probabilities are prescribed on each of the directed edges, harmonic functions are defined by the condition that the value at each vertex equals the average of the values at the neighbouring vertices. Much of classical potential theory can then be developed in this setting. The type of theorems considered depends on whether or not the associated random walk is transient or recurrent. From the probabilistic point of view, the walk is transient if and only if, for each vertex v, the probability that the random walk visits v infinitely often is 0. Otherwise it is recurrent. From the analytic point of view, transience is equivalent to the existence of positive potentials.

The transient theory was largely developed in [**24**] where Cartier considers basic properties of harmonic and superharmonic functions, integral representation of positive harmonic functions, and limit theorems of positive superharmonic functions at

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the boundary of the tree along random paths. Limit theorems along deterministic paths are considered in [**26**], [**34**] and [**35**].

After a preliminary section on the basic facts concerning trees, we describe in §3 the results of [**28**], which deal with general recurrent trees. The potential theory in this setting is analogous to the classical potential theory in the complex plane. We were motivated by the work of Anandam in [**7**] and subsequent articles, where he developed the theory of Brelot spaces without positive potentials. In the recurrent setting, there are no positive nonconstant superharmonic functions, so the focus is on admissible superharmonic functions, that is, superharmonic functions having a harmonic minorant outside a compact set. Anandam also introduced several notions of "potential". A key idea is to develop a theory of flux for admissible superharmonic functions, and further, for any function superharmonic outside a compact set. In the Brelot space setting, Anandam first defined flux for functions harmonic outside of a compact set, then for globally defined superharmonic functions, and finally for functions superharmonic outside a compact set. It is true that if one linearly extends along the edges functions that are harmonic on a tree, then one gets a Brelot space, and so the results of Anandam translate directly to results on the tree. However, our approach is different from his. We have a condensed definition of flux, where we directly define it for functions superharmonic outside finitely many vertices. We also introduce a new notion of potential, called H-potential, which seems quite natural in that it is defined by means of a "greatest harmonic minorant zero" condition, and it can be represented in terms of a kernel, called the H-Green function, which itself is represented by an explicit formula. We use the H-potentials to give a Riesz decomposition theorem for the set of admissible superharmonic functions.

In §4, we develop the idea of producing Brelot spaces by starting with a discrete harmonic structure and extending in ways more general than linearly on the edges. After giving a brief background on Brelot spaces, we describe the work in [**19**], where we first characterize, up to Brelot space isomorphism, all the Brelot space structures on intervals, and then discuss the problem of putting these structures together to produce a Brelot space structure on the tree viewed as a one-dimensional simplicial complex.

In §5, we describe the work in [**18**] where we show that our theories of flux and H-potential can be formulated in a general Brelot space without positive potentials, and so we obtain a Riesz decomposition theorem for admissible superharmonic functions in this general setting. We show that the theories described here and in §3 agree in the special case that we extend linearly along the edges of the tree.

In §6, we look at a particular example of the trees in §3, namely the $(1/4, 1/4, 1/2)$-tree. This is the tree in which all vertices have exactly three neighbours, and aside from the root of the tree in which the outward transition probabilities are $1/3$, all forward probabilities are $1/4$, and all backward probabilities are $1/2$. On this tree, we write down explicit formulas for the various quantities introduced in §3, and in particular for the H-potentials. We then depart from our survey in order to prove a new result concerning the boundary behaviour of the H-potentials in the spirit of the boundary results of [**34**] for ordinary potentials on the same tree with a transient structure.

In §7, we describe the results in [**30**] concerning clamped and simply-supported biharmonic Green functions. These functions arise classically on domains (say on $\mathbb{D}$,

the unit disk in the complex plane) in connection with various physical problems, such as the deflection of a thin plate. For applications in complex analysis, see [**33**]. The biharmonic Green functions can be used to construct functions on $\mathbb{D}$ having a prescribed bilaplacian, zero boundary values, and either vanishing normal derivative at the boundary (for the clamped one) or vanishing Laplacian at the boundary (for the simply-supported one). We consider the problem of defining and calculating these functions on a homogeneous tree.

In §8, we leave the discrete topics and consider instead a problem of complex and functional analysis. The results are given in [**31**]. Let D be a bounded homogeneous domain in $\mathbb{C}^N$, and consider the Banach spaces $X = H^\infty(D)$ of bounded holomorphic functions on D and $Y = \mathcal{B}$ of Bloch functions on D. Let $\psi : D \to \mathbb{C}$ have the property that for all $f \in X$, ψf is an element of Y. Then the mapping $M_\psi : X \to Y$, $f \mapsto \psi f$ is called a multiplication operator. In this section we describe a theorem which gives the norm of this operator. Our focus is on the following minimum principle concerning multiply superharmonic functions v defined on a product $\omega_1 \times \omega_2$ of relatively compact domains in Euclidean space. The minimum principle says that if v is lower bounded and has a $\liminf$ greater or equal to 0 on the distinguished boundary $\partial\omega_1 \times \partial\omega_2$, then $v \geq 0$ on $\omega_1 \times \omega_2$. Though the result seems natural, the proof doesn't seem to be completely trivial. The proof we have of it makes use of a topology on the set of differences of positive superharmonic functions, called the Cartan-Brelot topology, which we describe in §8.

For our other collaborations which deal with potential theory on trees but which we do not survey here, see [**16**], [**20**], [**26**], [**27**], [**35**].

2. Preliminaries on trees

A **tree** is a connected graph with no loops. We fix one vertex **e** and refer to it as the **root** of the tree. We write $v \sim w$ if there is an edge joining vertices v and w, and say that v and w are **neighbours**. The number of neighbours of a vertex is referred to as its **degree**. In this paper we assume that T is a tree with infinitely many vertices, each vertex has finite degree, and there are no **terminal vertices**, that is, no vertices with only one neighbour. For each vertex v, let $[e, v] = [e, v_1, v_2, \ldots, v_n = v]$ be the unique geodesic path from e to v. Denote by $|v| = n$ the number of edges in $[e, v]$ and call it the **modulus** of v. More generally, for two vertices v and w, $|v - w|$ is the number of edges in the geodesic path from v to w. We refer to this as the **hyperbolic distance** from v to w, since by analogy, the hyperbolic distance between pairs of points on the unit disk in the complex plane is unbounded. By a **function** on T we mean a real-valued function on the vertices of T.

If v, w are vertices, we write $v \leq w$ if v is a vertex of $[e, w]$. We say that v is an **ancestor** of w and w is a **descendant** of v. The **children** of a vertex v are the vertices w such that $v \sim w$ and $|w| = |v| + 1$. If v is a vertex, v_k will always refer to the vertex of modulus k in the geodesic $[e, v]$. The **wedge product** $v \wedge w$ is the unique ancestor of both v and w having largest modulus. The **sector** $S(v)$ is the set consisting of v and the descendants of v. For a fixed vertex $w \neq e$ we define the half-sectors $\overline{S(w_k)} := S(w_k) \setminus S(w_{k+1})$ for $k = 0, \ldots, |w| - 1$. Note that the sets $\overline{S(w_k)}$, $k = 0, \ldots, |w| - 1$ together with $S(w)$ give a partition of T. Denote

this partition by $\mathcal{P}(w)$:

$$\mathcal{P}(w) := \left\{\overline{S(w_0)}, \dots, \overline{S(w_{|w|-1})}, S(w)\right\}.$$

We say that a function f on T is of **finite type** if it is constant on each of the elements of $\mathcal{P}(w)$ for some vertex w.

If $a > 1$ and $u \neq e$, then a simple calculation shows that, for a homogeneous tree of degree 3,

$$\sum_{v \in S(u)} 2^{-a|v|} = \frac{2^{-a|u|}}{1 - 2^{1-a}} \tag{2.1}$$

The boundary ∂T of T is the set of infinite geodesic paths beginning at e. If $\omega = [e, \omega_1, \omega_2, \dots)$ is a boundary point of T and v a vertex, we write $v \leq \omega$ if $v = \omega_k$ for some k. The **wedge product** $v \wedge \omega$ is the vertex of largest modulus common to $[e, v]$ and ω. The **interval** $\mathbf{I(v)}$ is the set of boundary points ω such that $v \leq \omega$, or equivalently, $v \wedge \omega = v$. If $v, w \in T \cup \partial T$, we define the **Euclidean distance** from v to w to be 0 if $v = w$ and otherwise $2^{-|v \wedge w|}$. This is by way of analogy with the Euclidean distance between points on the unit disk in the complex plane, since it is bounded. This makes $T \cup \partial T$ a metric space which is a compactification of T, and the sets $\{I(\omega_j), j = 1, \dots\}$ form a base of neighbourhoods for the boundary point $\omega = [e, \omega_1, \dots)$.

In order to define harmonic and superharmonic functions, we introduce **transition probabilities** on the directed edges: $p : T \times T \to [0, 1]$ such that (i) $p(u, v) > 0$ if and only if $u \sim v$, and, fixing $u \in T$, (ii) $\sum_{v \sim u} p(u, v) = 1$. The **Laplacian** Δf of a function f is given by $\Delta f(u) = \sum_{v \sim u} p(u, v) f(v) - f(u)$. We say that f is **harmonic** (respectively, **superharmonic**) at u if $\Delta f(u) = 0$ (respectively, ≤ 0). A function f is **subharmonic** at u if $-f$ is superharmonic at u.

The transition probabilities determine a random walk on the vertices of T that is either **transient** (for all u, v, the walk starting at u visits v finitely many times a.s.) or **recurrent** (for all u, v, the walk starting at u visits v infinitely many times a.s.). For simplicity, we shall simply refer to the tree as being transient or recurrent. It can be shown that the tree is recurrent if and only if there exists a positive function that is superharmonic but not harmonic on T . A good reference for potential theory on transient trees is [**24**].

It can be shown that a tree is recurrent if and only if there exists a positive function that is superharmonic but not harmonic on T ([**24**], §2.3. or [**28**], p. 3-4).

If $[u, v]$ is a directed edge with $|u| = |v| - 1$, we call $[u, v]$ an **outward** edge and $p(u, v)$ an **outward** probability. We similarly define inward edges and inward probabilities.

3. Riesz decomposition on recurrent trees

In this section we consider a general recurrent tree, and describe the work in [**28**].

A **potential** is a positive superharmonic function whose greatest harmonic minorant is 0. Potentials exist precisely on trees whose associated random walk is transient. In this case there is a **Green function**, $G : T \times T \to (0, \infty)$, where $G(u, v)$ is the expected number of times the random walk which begins at vertex u visits vertex v. The potentials are characterized as the functions of the form $Gf(u) := \sum_{v \in T} G(u, v) f(v)$, for f a nonnegative function on T satisfying

$\sum_{v\in T} G(e,v)f(v) < \infty$. The **Martin kernel** $K_\omega(w) := \dfrac{G(w, w\wedge\omega)}{G(e, w\wedge\omega)}$ generates all positive harmonic functions h on T: for each such function, there exists a Borel measure μ_h on ∂T such that $h(\cdot) = \int K_\omega(\cdot)d\mu_h(\omega)$.

EXAMPLE 3.1. **1/3-tree.** If every vertex of T has degree 3, we say T is **homogeneous** of degree 3. For such a tree, if all the transition probabilities are 1/3, we refer to T as the 1/3-tree. The Green function is given by $G(u,v) = 2\times 2^{-|u-v|}$ ([**24**], pg. 264) from which it follows that the Martin kernel is $K_\omega(w) = 2^{2|w\wedge\omega|-|w|}$. Note the similarity with the formula for the Poisson kernel on the unit disk in the complex plane: $P_{e^{i\theta}}(z) = \dfrac{1-|z|^2}{2\pi|e^{i\theta}-z|^2}$. The latter behaves roughly like the distance of z to the boundary of the disk divided by the square of the distance from z to the given boundary point and the former is exactly the hyperbolic distance of w to the boundary divided by the square of the hyperbolic distance from w to the given boundary point ω.

EXAMPLE 3.2. **(1/4,1/4,1/2)-tree.** Again we take T to be homogeneous of degree 3, but instead we define the transition probabilities by

$$p(u,v) = \begin{cases} 1/3 & u = e, v\sim e \\ 1/2 & [u,v] \text{ a backward edge} \\ 1/4 & [u,v] \text{ a forward edge with } u\neq e \end{cases}$$

This is an example of a recurrent tree.

It follows easily from the definition that on a transient tree, every positive superharmonic function s on T can be written uniquely as $h_T + Gf$, where h_T is harmonic on T; h_T is just the greatest harmonic minorant of s and it is immediate that $s - h^T$ has greatest harmonic minorant 0, so is a potential.

The above result is known as the Riesz Decomposition Theorem. More generally we refer to **a Riesz Decomposition Theorem** as one which gives a unique decomposition of each member of some class of superharmonic functions as the sum of a global harmonic function and a member of some special class of "potentials". "Potentials" should be defined by a sentence that includes all of the words "greatest", "harmonic", "minorant" and "zero", and ideally they can be characterized in terms of a "Green function".

Our aim in this section is to describe a Riesz Decomposition Theorem on recurrent trees. In the recurrent setting there are no nonconstant positive superharmonic functions, and so all superharmonic functions with a global harmonic minorant are themselves harmonic. However, there are lots of nonconstant globally superharmonic functions that have a minorant on T which is harmonic except for at most finitely many vertices. Such superharmonic functions are referred to as **admissible**.

We state the main result of this section in Theorem 3.8. It is proved in [**28**]. Before we state the result, we will define and discuss several terms. However, looking at parts (ii) and (iii) of Theorem 3.8 , it is clear that it includes a Riesz Decomposition Theorem in the sense we have given above.

We begin with the term **standard**. Its definition is motivated by the following theorem.

THEOREM 3.3. *([**14**], Theorem 9.7) Let Ω be an unbounded subset of $\mathbb{R}^2$ and K a compact subset of Ω. If u is harmonic on $\Omega\backslash K$, then u has a unique decomposition*

of the form $u = u_H + w$, where u_H is harmonic on Ω and w is a harmonic function on $\mathbb{R}^2 \setminus K$ satisfying $\lim_{x\to\infty} w(x) - \alpha \log|x| = 0$ for some constant α.

Thus u can be written in the form $u(x) = u_H(x) + b + \alpha \log|x|$, where b is bounded outside a compact set. The number α is called the **flux** of u, and we call the function $\chi_{|x|\geq 1}(x) \ln|x|$ a **standard** for Ω. This motivates the following definition on T.

Definition 3.4. Let T be a recurrent tree. A function H on T is called a **standard** if (i) $H \geq 0$, (ii) H is harmonic outside the root, and (iii) given any function h on T which is harmonic off an arbitrary finite set of vertices, there exists a function h_T harmonic on T, a unique real number α, and a bounded function b such that

$$h = \alpha H + h_T + b.$$

One of the advantages of working in the tree setting is that we can make explicit calculations. In [**28**] we calculated a unique function $H \geq 0$ satisfying the following properties:

(1) $H(e) = 0$,
(2) $\Delta H(u) = \delta_e(u)$,
(3) H is constant on children.

The function H is given explicitly by

$$H(v) = \begin{cases} 0 & \text{if } v = e \\ \sum_{k=0}^{n-1} \epsilon_0(v)\epsilon_1(v)\dots\epsilon_k(v) & \text{if } |v| = n \geq 1 \end{cases}$$

where

$$\epsilon_0(v) = 1, \epsilon_k(v) = \frac{p(v_k.v_{k-1})}{1 - p(v_k, v_{k-1})} \text{ for } 1 \leq k \leq n-1.$$

To show the above H is a standard, we introduce and explicitly calculate a family of functions $\{H_v\colon v \in T\}$ uniquely determined by the following properties:

(i) $H_v(v) = 0$, (ii) $\Delta H_v(u) = \delta_v(u)$, (iii) $H_v \geq 0$, (iv) H_v constant on children.

The family $\{-H_v, v \in T\}$ has some of the properties that a Green function would have if there were a Green function on T. Since H will turn out to be a standard and H_v is harmonic outside a finite set (namely $\{v\}$), we anticipate being able to write H_v as

$$H_v(u) = \alpha_v H(u) + h_T(u) - b^v(u),$$

where α_v is a unique constant, h_T is globally harmonic, and b^v is bounded. Direct calculation gives that $h_T \equiv 0$ and α_v is the ratio of the product of the forward and backward probabilities:

$$\alpha_v = \frac{p[e,v]}{p[v,e]} := \frac{p(e,v_1)p(v_1,v_2)\dots p(v_{n-1},v)}{p(v_1,e)p(v_2,v_1)\dots p(v,v_{n-1})}.$$

The function b^v turns out to be constant on each of the elements of the partition $\mathcal{P}(v)$ (and so is of finite type), the constant values given by $b^v(v) = \alpha_v H(v)$ and

$$b^v(v_k) = b^v(v) - \frac{1}{p(v,v_{n-1})}\left[1 + \sum_{m=k+1}^{n-1}\prod_{j=m}^{n-1}\frac{p(v_j,v_{j+1})}{p(v_j,v_{j-1})}\right].$$

The above calculations allow us to prove that H is indeed a standard. Observe that H is harmonic outside the root, it is subharmonic at the root, and it is

necessarily unbounded. Indeed if H were bounded above by a constant M, then $M - H$ would be nonnegative superharmonic, and so constant, which is clearly not the case.

We now discuss the concept of **flux** in the tree setting. The concept was introduced in [**7**] in a Brelot space having no potentials. There the flux was first defined for functions harmonic outside a compact set, then for admissible superharmonic functions, then finally for functions superharmonic outside a compact set. We give instead the following single definition of flux.

Definition 3.5. Let s be superharmonic except possibly for finitely many vertices. Then

$$flux(s) := \sup\{c \in \mathbb{R} : s - cH \text{ is bounded below by a function harmonic on } T\}.$$

Take for example the case that s is nonconstant and globally superharmonic, but not harmonic. Then the numbers c on the list defining $flux(s)$ are all negative (since positive nonharmonic superharmonic functions cannot have a global harmonic minorant). Since H is unbounded, the idea is that we can hope to add a large enough positive multiple of H to s so that it has a global harmonic minorant and that the negative of the smallest such positive multiple should be the flux of s. The superharmonic functions on T thus have $flux(s) \leq 0$, and the admissible superharmonic functions are precisely the ones having finite flux.

We summarize in the next result some of the elementary properties of flux.

Theorem 3.6. *([**28**], Theorem 2.5) (a) If h is harmonic outside a finite set, then the flux of h is the unique constant α in the definition of standard such that $h = \alpha H + h_T + b$ with h_T harmonic on T and b bounded.*

(b) If h is bounded and harmonic outside a finite set or harmonic everywhere, then the $flux(s) = 0$.

(c) If s is superharmonic on T, then $flux(s) \leq 0$, $flux(s) = 0$ if and only if s is harmonic, and $flux(s) > -\infty$ if and only if s is admissible.

(d) If s is superharmonic outside a finite set K of vertices, then the flux of s is equal to the flux of the greatest harmonic minorant of s outside K.

(e) If s is superharmonic outside a finite set and the set A_s in Definition 3.5 whose supremum defines the flux of s satisfies $A_s \neq \emptyset$, then $A_s = (-\infty, flux(s)]$.

By part (a) of the theorem, knowing the flux of a function harmonic outside a finite set of vertices allows us to associate to it, in a natural way, a global harmonic function.

We now turn to the definitions of H-potentials and the H-Green function.

Definition 3.7. For f superharmonic outside the root, let $D(f)$ be the **greatest harmonic minorant** of f outside the root. Let in particular $f = s - flux(s)H$, where s is admissible. We say that s is an **H-potential** if $D(s - flux(s)H) = 0$.

The H-Green functions G^H is defined by the formula

$$G^H(w, v) := H_v(e) - H_v(w).$$

The following theorem gives the main results of this section and of [**28**].

Theorem 3.8. [**28**] *Let T be a recurrent tree, H a standard on T and G^H the H-Green function.*

(i) *For s superharmonic except possibly for finitely many vertices, the flux of s is given by the formulas*

$$flux(s) = \sum_{v \in T} \Delta s(v)\alpha_v = \lim_{n\to\infty} \sum_{|v|=n} \big(s(v) - s(v^-)\big)\, \alpha_v p(v, v^-).$$

(ii) *(Riesz decomposition) Let s be an admissible superharmonic function on T. Then s can be written uniquely as the sum of a global harmonic function and an H-potential.*

(iii) *The H-potentials are precisely the functions of the form*

$$G^H f(u) := \sum_{v\sim u} G^H(u, v) f(v),$$

where $f \geq 0$ and $f \in L^1_\alpha(T)$, i.e. $\sum_{v\in T} \alpha_v f(v) < \infty$.

A few comments concerning the function $v \mapsto \alpha_v = \frac{p[e,v]}{p[v,e]}$ are in order. We introduced the functions H_v in order to be able to construct functions with prescribed Laplacian on a given finite set, and to in particular be able to prove that H is a standard. It does, however, appear in the literature in other ways. A simple calculation shows that if $v \sim w$, then $\alpha_v p(v, w) = \alpha_w p(w, v)$. This allows one to view the tree as an electrical network, where the edge function $c(\{v, w\}) := \alpha_v p(v, w)$ is known as the **conductance** of the edge. It also allows us to view the tree as a reversible Markov chain. Up to a constant multiple, the function α_v appears in the text [**40**] in the chapter on recurrent Markov chains. If we view the tree as a Markov chain with transition matrix P and states given by the vertices, the authors define a measure α on the state space having the property $\alpha P = \alpha$ and refer to it as a **regular measure**. Its existence is guaranteed by the ergodic theorem. If $\sum_v \alpha_v < \infty$, the chain is called **ergodic**, and if $\sum_v \alpha_v = \infty$, the chain is called **null**. In the ergodic case, α can be normalized to give the equilibrium distribution of the various states, that is the fraction of time a random walk spends in each of the states.

The second formula of part (i) of Theorem 3.8 relates flux to the physics notion of flux of a vector field; the sum can be viewed as an integral of the normal derivative of s over a sphere, where the surface measure is given by the conductance. The first formula in (i) gives a criterion for admissibility of a superharmonic function, namely $\Delta s \in L^1(\alpha)$, where we view α_v as a measure on the tree.

4. Classification of Brelot structures on trees

In this section we describe the work in [**18**] where trees are viewed as Brelot spaces, and the task is to classify the possible Brelot structures one can have.

Brelot spaces were developed to provide an axiomatization of the properties satisfied by solutions of various elliptic partial differential equations on open subsets of $\mathbb{R}^n$ ([**21**]). Specifically, a **Brelot space** is a connected, locally connected, locally compact but not compact Hausdorff space Ω such that on each nonempty open set there is a vector space of continuous real-valued functions, called harmonic functions, satisfying Axioms 1,2 and 3, given as follows.

Axiom 1 (Sheaf Property): A function harmonic on an open set $U \subset \Omega$ is harmonic on any open subset of U; If a function on U is harmonic on a neighbourhood of each of the points of U, then it is harmonic on U.

A relatively compact domain $U \subset \Omega$ is called **regular** if every function f continuous real-valued on the boundary of U has a unique continuous harmonic extension H_f^U to U, and that extension is nonnegative if f is nonnegative.
Axiom 2 (**Regularity property**): The regular domains form a base for the topology on Ω.

Axiom 3 (Harnack property) Any sequence of harmonic function that increases pointwise on a connected open subset of Ω has a pointwise limit that is either harmonic or identically infinity.

By Axiom 2, if U is regular, then for each $x \in U$, there exists a measure ρ_x^U on ∂U such that $H_f^U(x) = \int f d\rho_x^U$ for each f continuous on ∂U. A function $g : \Omega \to (-\infty, \infty]$ is called **superharmonic** if it is lower semicontinuous, not identically ∞, and for every regular domain U and $x \in U$, $\int g d\rho_x^U \leq g(x)$. A nonnegative superharmonic function is called a **potential** if it doesn't majorize any positive harmonic function.

Two Brelot spaces Ω, Ω' are called **Brelot isomorphic** if there is an isomorphism $f : \Omega \to \Omega'$ which carries one harmonic structure onto the other.

Of course the set of harmonic functions on an open set in $\mathbb{R}^n$ is an example of a Brelot space. Another class of examples consists of the set of harmonic functions on a tree in which they are extended linearly along the edges. This is considered in [**15**].

Actually one can create other Brelot structures on trees by extending along the edges in various ways. This point of view was adopted in [**19**] where the problem of considering what, up to Brelot isomorphism, are the Brelot structures on trees, and what is their relation to discrete harmonic structures on the vertices. A classification of Brelot structures on one-dimensional manifolds was considered in [**42**], based on an earlier work on one-dimensional harmonic spaces [**43**]. The classification described here is different from those in that it has more equivalence classes of Brelot spaces. We describe next the main results in [**19**].

We first describe the possible Brelot structures on an interval I. A Brelot structure on I is called **quasi-linear** if constants are harmonic, **quasi-hyperbolic** if it is not quasi-linear but there does exist a positive harmonic function on I, and **quasi-trigonometric** if there are no positive harmonic functions on I. On each interval I the set of harmonic functions is two-dimensional, and we refer to a basis $\{f, g\}$ as a **Brelot basis**. The next theorem characterizes the quasi-linear and quasi-hyperbolic Brelot structures on $I = (0, 1)$.

THEOREM 4.1. *(a) Every continuous 1-to-1 function f on $[0, 1]$ gives rise to a quasi-linear Brelot structure on $(0, 1)$ such that $\{1, f\}$ is a Brelot basis and, conversely, every quasi-linear structure arises in this way. Consequently a quasi-linear Brelot structure on an interval is Brelot isomorphic to linear functions on some interval, and hence there are three distinct classes of quasi-linear structures on intervals represented by the linear structures on $(-\infty, \infty)$, $(0, \infty)$ and $(0, 1)$.*
(b) Let $\mathcal{H}$ be a quasi-hyperbolic structure on $(0, 1)$ and let k be a positive harmonic function on $(0, 1)$. Then there exists a harmonic function h on $(0, 1)$ such that $\{h, k\}$ is a Brelot basis. For any such k, h/k is 1-to-1. Conversely, let k be any

positive continuous function on $(0,1)$ *and let* ϕ *be any homeomorphism from* $(0,1)$ *to some interval. Then the pair* $\{k, k\phi\}$ *is a Brelot basis for a Brelot structure which is either quasi-linear or quasi-hyperbolic. It is quasi-linear if and only if* k *is constant or* ϕ *is a linear combination of* $1/k$ *and* 1.

We explain next why a quasi-trigonometric structure is closely related to a structure generated by $\sin x$ and $\cos x$ on some interval. Let $-\infty \leq a < b \leq \infty$ with $b - a > \pi$. A **trigonometric Brelot structure** on (a, b) is a structure generated by two continuous functions $S(x)$ and $C(x)$ such that for all $x \in (a, b)$ the sign of $S(x)$ equals the sign of $\sin x$, the sign of $C(x)$ equals the sign of $\cos x$, and $S(x)/C(x) = \tan x$. Then the quasi-trigonometric Brelot structures are described by the following theorem.

THEOREM 4.2. *(a) Any quasi-trigonometric Brelot structure on* $(0,1)$ *is Brelot isomorphic to a trigonometric structure on some interval* (a, b).
(b) Let f, g *be continuous functions on* $(0,1)$ *whose corresponding zero sets are disjoint, discrete, alternating, and assume that* f *has at least two zeros. Suppose that the restriction of* f/g *(respectively,* g/f*) to any interval containing no zero of* g *(respectively, no zero of* f*) is 1-to-1. Then* $\{f, g\}$ *is a Brelot basis for a quasi-trigonometric Brelot structure on* $(0,1)$. *Conversely, every quasi-trigonometric Brelot structure has a Brelot basis of this form.*

Having described the Brelot structures on intervals, the next step is to understand them on trees. A discrete harmonic structure on a tree T is determined by a function $p : T \times T \to [0, \infty)$ such that $p(u, v) > 0$ if and only if $u \sim v$. In this section we don't assume that $\sum_{u \sim v} p(u, v) = 1$ for each u, since we don't wish to assume that constant functions are harmonic. We refer to the discrete harmonic structure as $\mathcal{H}_p$.

Let $\tilde{T}$ be the tree viewed as a one-dimensional simplicial complex, that is, for all $u \sim v$, consider the set $[u, v] = \{(1-t)u + tv : -1 \leq t \leq 1\}$. Then $\tilde{T} = \bigcup_{u \sim v} [u, v]$. Before we give the main results, we give several definitions.

DEFINITION 4.3. (a) A Brelot structure on $(0,1)$ is called **extendible** if it is the restriction of some Brelot structure on $(-1, 1)$. This is equivalent to having every harmonic function f on $(0,1)$ satisfy $f(0) := \lim_{t \to 0^+} f(t)$ exist and be nonzero.

(b) A relatively compact domain U is called a **Dirichlet domain** if, for any boundary function f, there exists a unique solution h_f^U to the corresponding Dirichlet problem.

(c) A Dirichlet domain is called a **positive Dirichlet domain** if the solution to the Dirichlet problem with nonnegative boundary values is nonnegative in some neighbourhood of the boundary.

(d) A Dirichlet domain U is called **weakly regular with respect to** $\mathbf{x} \in \mathbf{U}$ if for any function f defined on the boundary of U which is nonnegative and not identically zero, $h_f^U(x) > 0$.

(e) The **weak ball regularity axiom** says that the unit ball centered at any vertex is weakly regular with respect to its center.

We remark that examples given in [**19**] show that a Dirichlet domain can be weakly regular with respect to some points but not to others, a positive Dirichlet domain need not be regular, Dirichlet domains need not be positive Dirichlet

domains, and the weak ball regularity axiom can hold without the unit ball being regular.

The next theorem describes how to generate examples of Brelot structures on trees.

THEOREM 4.4. *Let T be a tree with an extendible harmonic structure on each edge of $\tilde{T}$. For each directed edge τ, let $\imath(\tau) = u$ be the initial vertex of τ and pick a point $u_\tau \in \tau$ different from u such that (u, u_τ) is regular and let $p(u, u_\tau)$ be an arbitrary positive number. Let $U \subset \tilde{T}$ be a connected open set. For each directed edge τ with $\imath(\tau) = u \in U$, let $u'_\tau \in (u, u_\tau) \cap U$. Let $\mathcal{H}(U)$ be the set of functions f such that for each edge $[u, v]$ intersecting U, the restriction of f to $(u, v) \cap U$ is harmonic and for each vertex $u \in U$ we have $f(u) = \sum_{\imath(\tau)=u} p(u, u'_\tau) f(u'_\tau)$. Then $\mathcal{H}$ yields a Brelot structure on $\tilde{T}$.*

And finally, we give a theorem which describes the relation between discrete harmonic structures on T and Brelot structures on $\tilde{T}$.

THEOREM 4.5. *(a) Let T be a tree with a discrete harmonic structure. Assume that each edge of $\tilde{T}$ has a Brelot structure for which the whole edge is a positive Dirichlet domain. Then there is a unique Brelot structure on $\tilde{T}$ which induces the given discrete structure and whose restriction to each edge is the given Brelot structure. Furthermore, it satisfies the weak ball regularity axiom.*

(b) A Brelot structure on $\tilde{T}$ satisfying the weak ball regularity axiom induces the discrete harmonic structure with transition matrix p as follows: if $u \sim v$, $p(u, v)$ is the value at u of the solution to the Dirichlet problem on the unit ball $B_1(u)$ with boundary values equal to the characteristic function of v.

5. Extension of results of section 3 to Brelot spaces

In this section we describe the work in [**18**]. We let Ω be a Brelot space in which constants are harmonic and there are no potentials. This means that every superharmonic function with a harmonic minorant is necessarily constant. In this setting, admissible superharmonic functions were introduced in [**7**]. A superharmonic function s is called **admissible** if outside some compact set K there is a harmonic function h such that $h(x) \leq s(x)$ for all $x \in \Omega \setminus K$.

Some of the discrete ideas discussed above in section 3 can be formulated in this setting. We describe here some of the results in [**18**] where standard and H-potential are defined, and a Riesz decomposition theorem for admissible superharmonic functions is proved.

DEFINITION 5.1. ([**18**], Definition 2.1) Let $K \subset \Omega$ be a nonempty compact set that is not polar (that is, there does not exist a superharmonic function on Ω which is identically ∞ on K). A function H harmonic off K is called a **standard** for Ω associated with K if the following is true: given any function h which is harmonic off a compact set, there exist a unique function h_Ω harmonic on Ω and a unique real number α such that $b = h - \alpha H - h_\Omega$ is bounded off a compact set and $\liminf_{x\to\infty} b(x) = 0$, where the lim inf is taken with respect to the Alexandrov one-point compactification of Ω.

That a standard exists was proved in [**8**]. We next define flux. As we mentioned earlier, Anandam defined flux in this setting. However, the condensed definition we gave in section 3 works well here.

Definition 5.2. ([**18**], Definition 2.3) Let s be a function on Ω superharmonic outside a compact set. Define the flux of s by $flux(s) = \sup A_s$, where $A_s = \{\alpha \in \mathbb{R} : \text{there exists } h_\Omega \text{ harmonic on } \Omega \text{ such that } s - \alpha H \geq h_\Omega\}$.

The H-potentials were introduced in [**28**] in order to give a Riesz decomposition theorem for admissible superharmonic functions on recurrent trees. They were later introduced in [**18**] in the Brelot space setting for a similar purpose. Before we give the definition, we describe other potential type functions given in [**9**] and [**10**].

Definition 5.3. Let s be superharmonic on $\Omega \setminus K_0$, where K_0 is a compact outer regular set. Let $\mathcal{E} = \{U_n\}$ be an increasing exhaustion of Ω consisting of relatively compact regular sets containing K_0. Let $h_n = h_s^{U_n}$, the solution of the Dirichlet problem on U_n with boundary values s on the boundary of U_n. Define $D_{\mathcal{E}} s(x) = \lim\limits_{n\to\infty} h_n(x)$ if this limit exists locally uniformly.

Then the potentials introduced in [**9**] and [**10**] are defined as follows.

Definition 5.4. An admissible superharmonic function s is said to be in the class $\mathcal{P}$ if there exists an exhaustion $\mathcal{E}$ such that $D_{\mathcal{E}}(s-\alpha H)$ exists and is constant, where α is the flux of s. If, furthermore, that constant is 0 for some exhaustion $\mathcal{E}$, then s is called a **BS potential**. Define the class $\mathcal{Q}$ as the collection of all admissible superharmonic functions s satisfying the property: there exists $s' \in \mathcal{P}$ such that the difference of the greatest harmonic minorants of s and s' outside a compact set is bounded.

The following partial Riesz decomposition holds.

Theorem 5.5. [**11**] *Any admissible superharmonic function s on a BS space is a sum of a function in the class $\mathcal{Q}$ and a harmonic function. This decomposition is unique up to an additive constant. If s is harmonic outside a compact set, then the element of $\mathcal{Q}$ can be chosen uniquely to be a BS potential.*

Besides the fact that this theorem doesn't give a unique representation of admissible superharmonic functions, it might prove difficult to show whether or not a given function is in class $\mathcal{P}$, class $\mathcal{Q}$ or is a BS potential. For this reason the H-potentials were introduced in [**18**]. We now define them.

Definition 5.6. Let K_0 be an outer regular compact set and let s be an admissible superharmonic function with flux α. Define $D(s-\alpha H)$ to be the greatest harmonic minorant of $s - \alpha H$ on $\Omega \setminus K_0$. Then s is called an H-potential if

$$\liminf_{x\to\infty} D\left(s - \alpha H\right)(x) = 0,$$

where the lim inf is taken with respect to the Alexandrov one-point compactification of Ω.

Note that in the above definition, $D(s-\alpha H)$ is calculated by taking any regular exhaustion $\{U_n\}$ of ω, solving the Dirichlet problem on $U_n \setminus K_0$ with $s - \alpha H$ on the boundary, and taking the pointwise limit as n goes to ∞.

With these definitions in place, we state the following Riesz decomposition theorem.

THEOREM 5.7. *([**18**], Theorem 3.1) Every admissible superharmonic function can be written uniquely as the sum of an H-potential and a harmonic function.*

We now make a few comments concerning the relation between the discrete results of section 3 and the Brelot space results of this section. As we noted in the previous section, a recurrent tree can be viewed as a Brelot space without potentials. Thus one could use the work of Anandam to obtain some of the discrete results discussed here. One example is the proof of the existence of a standard H in the discrete setting. However, one doesn't need such sophisticated ideas to understand standards on a tree, and it turned out be very useful to have an explicit formula, something which is not available in a Brelot space. Another example is our definition of flux, which we think simplifies its study.

Finally we note the apparent difference in the definition of H-potential in the discrete and the Brelot space setting: in the discrete setting we require that the greatest harmonic minorant outside the root of $s - \alpha H$ is 0, whereas in the Brelot space we require the apparently weaker condition that the liminf at the point at ∞ of the greatest harmonic minorant outside the compact set K_0 of $s - \alpha H$ is 0.

Consider the tree setting again. Let s be admissible with flux α, and let $h = D(s - \alpha H)$ be the greatest harmonic minorant outside the root of $s - \alpha H$. Assume that $\liminf_{|v|\to\infty} h(v) = 0$. We show that s is an H-potential. What makes things simpler here than in a Brelot space is that the only place where h may fail to be harmonic is at the root. Since the Laplacian $\Delta h(e)$ is either positive, zero, or negative, it follows that h is either globally subharmonic, harmonic, or superharmonic. But from Theorem 3.6(d) it follows that $flux(h) = flux(s - \alpha H) = flux(s) - \alpha flux(H) = \alpha - \alpha = 0$, so from Theorem 3.6(c), h must be harmonic on T. The liminf condition then implies that h is lower bounded, and so since the tree is recurrent, h must be constant. Since the liminf is zero at ∞, h must be zero, proving that s is an H-potential.

6. Boundary behaviour of H-potentials in the $(1/4, 1/4, 1/2)-$tree

In this section we return to the theory discussed in section 3 and apply it in particular to prove a new result on the $(1/4, 1/4, 1/2)-$tree We write explicitly what are the H-potentials, and study their boundary behaviour. This is perhaps the simplest nontrivial example of a recurrent tree, and studying the behaviour of H-potentials is analogous to the classical problem of studying the behaviour at ∞ of logarithmic potentials on $\mathbb{R}^2$.

Let T be the homogeneous tree of degree 3, that is, each vertex has exactly three neighbours. If the transition probabilities are all taken to be 1/3, we obtain the 1/3-tree. The 1/3-tree is transient, and the potentials are all of the form $Gf(w) = \sum_{v\in T} G(w, v)f(v)$, where the Green function $G(w, v)$ is given by $G(w, v) = 2 \times 2^{-|w-v|}$, and f is any nonnegative function on T such that $\sum_{v\in T} 2^{-|v|} f(v) < \infty$ (see [**24**], section 4.5).

In [**34**] we considered the boundary behaviour of these potentials. In this section we prove some results concerning the boundary behaviour of H-potentials on the same homogeneous tree but with the transition probabilities of the (1/4,1/4,1/2)-tree.

We first describe the results in [**34**] beginning with the exceptional sets that arise. For $E \subset \partial T$ and $0 < \beta \leq 1$ we define

$$H_\beta(E) = \sup_{\delta>0} \inf \left\{ \sum_i 2^{-\beta|v_i|} : E \subset \cup_i I(v_i), |v_i| > \log_i (1/\delta) \right\}$$

and

$$C_\beta(E) = \inf \left\{ \sum_i 2^{-\beta|v_i|} : E \subset \cup_i I(v_i) \right\}.$$

We call $H_\beta(E)$ the $\beta-$**dimensional Hausdorff measure** of E and $C_\beta(E)$ the $\beta-$**dimensional content** of E. The null sets are the same for H_β and C_β.

In case $\beta = 1$, $H_1(I_v) = 2^{-|v|}$. On the 1/3-tree, up to a constant factor H_1 is the representing measure on the Martin boundary (which is just ∂T) of the constant harmonic function 1. We thus refer to H_1 as the **Lebesgue measure** on ∂T.

The approach regions to a boundary point $\omega = [e, \omega_1, \omega_2, \dots)$ which we consider are defined as follows: For $\tau \geq 1$, $a \geq 0$, let

$$\Omega_{\tau,a}(\omega) := \bigcup_j \{w \in T : \omega \wedge w = \omega_j, |w - \omega_j| \leq (\tau - 1)j + a\}.$$

We view it as an approach region to ω which is a **radial** approach region if $a = 0$ and $\tau = 1$, a **nontangential** approach region with aperture a if $\tau = 1$ and $a > 0$, and a **tangential** approach region of tangency τ if $\tau > 1$. We say that a function has τ-limit L at a boundary point ω provided that for every $a \geq 0$, $f(w)$ converges to L as $w \to \omega$ within $\Omega_{\tau,a}(\omega)$.

THEOREM 6.1. [**34**] *Let T be homogeneous with degree 3 with the transition probabilities of the 1/3-tree. Let $0 < \gamma < 1$ for $p > 1$ and $0 < \gamma \leq 1$ for $p = 1$. Let further $1 \leq \tau \leq 1/\gamma$. Let f be be a nonnegative function defined on T such that Gf is finite and $\sum_{v \in T} f^p(v) 2^{-\gamma|v|} < \infty$. Then the limit of $Gf(v)$ as v tends to $\omega \in \partial T$ with v in the approach region $\Omega_{\tau,a}(\omega)$ is 0 for all $\omega \in \partial T$ except possibly for a set $E \subset \Omega$ having $\tau\gamma$-dimensional Hausdorff measure 0.*

In [**34**] it was also shown that the approach regions and the exceptional sets in the theorem were in some sense the best possible.

Consider now T with the structure of the $(1/4, 1/4, 1/2)$-tree. Thus, except for the three forward edges starting at the root, all forward probabilities are 1/4 and all backward probabilities are 1/2. Straightforward calculations and results from the previous section give the following formulas:

$$\begin{aligned}
\alpha_v &= \begin{cases} 1 & \text{if } v = e \\ \frac{4}{3} \cdot 2^{-|v|} & \text{if } v \neq e \end{cases} \\
H(v) &= |v| \\
b^v(v_k) &= 4 \left[-1 + \frac{|v|}{3} 2^{-|v|} + 2^{k-|v|} \right], k = 0, \dots, |v| \\
H_v(w) &= 4 + \frac{4}{3} \left[|w| - |v| - 3 \times 2^{|w \wedge v|} \right] 2^{-|v|} \\
G^H(w, v) &= 4 \times 2^{-|v|} \left(2^{|w \wedge v|} - 1 \right) - \alpha_v H(w).
\end{aligned}$$

Define the nonnegative function B_v on T by

$$B_v(w) := 4 \times 2^{-|v|} \left(2^{|w \wedge v|} - 1\right),$$

and for $f \geq 0$ on T

$$Bf(w) := \sum_{v \in T} B_v(w) f(v).$$

By Theorem 3.8(iii), the H-potentials are precisely the functions

$$G^H f(w) = \sum_{v \in T} G^H(w,v) f(v) = \sum_{v \in T} B_v(w) f(v) - H(w) \sum_{v \in T} \alpha_v f(v),$$

where f is a nonnegative function on T for which $\sum_{v \in T} \alpha_v f(v) < \infty$. Since $\Delta G^H f = -f$, it follows from Theorem 3.8(i) that

$$G^H f = Bf + flux(Gf)H. \tag{6.1}$$

What is a natural boundary result to consider for the H-potentials? We motivate this with an informal discussion. Consider the potential theory of the usual Laplacian on the entire real line (which is a recurrent potential theory) and on the right half-line $\{x > 0\}$ (which is a transient potential theory). The Green function on the right half-line is given by $G(x,y) = \begin{cases} x & \text{if } x < y \\ y & \text{if } x \geq y \end{cases}$. Thus $\dfrac{G_y(x)}{G_y(1)} \to \begin{cases} 1 & \text{if } y \to 0 \\ x & \text{if } y \to \infty \end{cases}$ and so the Martin boundary consists of 0 and ∞ with the Martin kernel given by $P_y(x) = \begin{cases} 1 & \text{if } y = 0 \\ x & \text{if } y = \infty \end{cases}$. For the entire real line, the "standard" is given by $H(x) = |x|$, and on the right half-line $|x|$ is the minimal harmonic function corresponding to the Martin boundary point ∞.

Returning to the tree, by analogy we fix a vertex $e' \sim e$, and consider the sector $S(e')$, which, if we make the root an absorbing state, is a transient tree T' having boundary $\{e\} \cup I(e')$. We anticipate from the above example that the standard $H(w) = |w|$ is, when restricted to T', the positive harmonic function with representing measure given by a multiple of the restriction of Lebesgue measure to $I(e')$. Let $G^H f$ be an H-potential on T and let α be the flux of $G^H f$. Then by definition of H-potential, $G^H f - \alpha H$ is a potential on T', so by the Fatou-Naïm-Doob theorem, we would expect to be able to prove that $(G^H f - \alpha H)/H$ has a limit of 0 (in some sense) at Lebesgue a.e. point of $I(e')$. Thus we should consider the limiting behaviour of $G^H f/H = \dfrac{Bf}{H} + \alpha$, and expect to be able to prove it has a radial limit of α at Lebesgue-a.e. point of $I(e')$.

One can view T as a Brelot space if harmonic functions on T are extended linearly along the edges ([**15**]). Thus one can hope to prove the above radial limit theorem as a consequence of the general Fatou-Naïm-Doob theorem on Brelot spaces. However, we intend to prove the following more general limit theorem, and it does not have an analogue in a general Brelot space setting.

THEOREM 6.2. *Let $G^H f$ be an H-potential on the $(1/4, 1/4, 1/2)$-tree, where f satisfies the growth condition $\sum_{v \in T} f^p(v) 2^{-\gamma|v|} < \infty$. Here, $0 < \gamma < 1$ for $p > 1$ and $0 < \gamma \leq 1$ for $p = 1$. Let $1 \leq \tau \leq 1/\gamma$. Then the limit of $G^H f(v)/H(v)$ as*

$v \to \omega \in \partial T$ with $v \in \Omega_{\tau,a}(\omega)$ is the flux of $G^H f$ for all ω except possibly for a set $E \subset \partial T$ of $\tau\gamma-$Hausdorff measure 0.

Note the similarity between Theorem 6.1 and Theorem 6.2. Theorem 6.1 asserts that Gf has $\tau-$limit 0 at $\tau\gamma$-a.e. boundary point, and, by (6.1), Theorem 6.2 asserts that Bf/H has $\tau-$limit 0 at $\tau\gamma$-a.e. boundary point. For the first theorem, this amounts to estimating $w \mapsto \sum_{v\in T} 2^{-|v-w|} f(v)$ and the second theorem to estimating $w \mapsto \sum_{v\in T} 2^{-|v|} \frac{2^{|w\wedge v|-1}}{|w|} f(v)$, and so it is worth asking if the second theorem follows immediately from the first. Estimating the ratio of the kernel defining Bf/H and the kernel defining Gf we get about $\frac{\left(2^{-|v|}2^{|w\wedge v|}/|w|\right)}{2^{-|v-w|}} = \frac{2^{|w-w\wedge v|}}{|w|}$ and this is not upper bounded. For example, if $|v \wedge w| = |w|/2$, the above quantity is $2^{|w|/2}/|w|$, and this is not upper bounded. For another example, if $w \in \Omega_{\tau,a}(\omega)$ and $|v \wedge \omega| > |w \wedge \omega|$, then $w \wedge v = w \wedge \omega$ and so $2^{|w-w\wedge v|} = 2^{|w-w\wedge\omega|}$; this can be as large as $2^{(\tau-1)|w\wedge\omega|+a}$, making $\frac{2^{|w-w\wedge v|}}{|w|}$ unbounded in the interesting case where $\tau > 1$. Thus Theorem 6.2 does not follow immediately from Theorem 6.1.

Nevertheless, the idea of the proof of Theorem 6.1 can be modified to give a proof of Theorem 6.2, and this is what we do here. The main tools needed, as given in [**34**], are as follows. For a nonnegative function h on T, we define, for $w \in T$,

$$h^*(w) := \sum_{v\in S(w)} h(v)$$

and for $\beta > 0, \omega \in \partial T$ we define

$$M_\beta h(\omega) := \sup_{i\geq 1} 2^{i\beta} h^*(\omega_i).$$

It is shown in [**34**] that for every $\lambda > 0$,

$$C_\beta\{\omega \in \partial T : M_\beta h(\omega) > \lambda\} \leq \frac{\|h\|_1}{\lambda}, \tag{6.2}$$

where $\|h\|_1 = \sum_{v\in T} h(v)$. If f is as in the statement of Theorem 6.2, let

$$A := \{\omega \in \partial T : \limsup_{i\to\infty} 2^i \sum_{v\in S(\omega_i)} f(v)2^{-|v|} > 0\} \tag{6.3}$$

and

$$B := \{\omega \in \partial T : \limsup_{i\to\infty} 2^{i\tau\gamma} \sum_{v\in S(\omega_i)} f^p(v)2^{-\gamma|v|} > 0\}. \tag{6.4}$$

In the proof of Theorem 6.1 as done in [**34**], it is shown that $H_{\tau\gamma}(A \cup B) = 0$.

For $\omega \in \partial T$ and $w \in \Omega_{\tau,a}(\omega)$, let the vertices of the geodesic from $w \wedge \omega$ to w be denoted by $w^0, w^1, \ldots, w^{|w|-|w\wedge\omega|}$ and let $n = |w| - |w \wedge \omega| - 1$. In order to estimate $Bf(w)/H(w)$ we write

$$\sum_{v\in T} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v) = \sum_{i=1}^{4} \sum_{v\in T_i} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v),$$

where

$$T_1 := \bigcup_{j=0}^{|w\wedge\omega|-1} \overline{S(\omega_j)}, \quad T_2 := \bigcup_{i=1}^{n} \overline{S(w^j)}, \tag{6.5}$$
$$T_3 := S(w), \quad T_4 := S(w\wedge\omega) \setminus (T_2 \cup T_3)\,.$$

Define

$$\sum_i := \sum_{v\in T_i} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v), \quad i = 1,\dots,4. \tag{6.6}$$

We now write the proof of Theorem 6.2.

Proof. In the proof, c is used to denote a number which may be different with each occurrence, but which does not depend on any of the parameters or functions of interest.

We first prove the theorem in case $p = 1$. Let $\epsilon, \delta > 0$. Let $h(v) = f(v)2^{-\gamma|v|}$. By assumption $\|h\|_1 < \infty$. Note that if f had bounded support, then for some fixed N,

$$0 \le Bf(w) = 4 \sum_{|v|\le N} \left(2^{|v\wedge w|}-1\right) 2^{-|v|} f(v) \le 4(2^N-1) \sum_{|v|\le N} f(v)$$

is bounded and so $Bf(w)/|w| \to 0$ as $|w| \to \infty$. Thus we may assume that h has no support on any given bounded subset of T, and so it is enough to prove the result in case $\|h\|_1 < \epsilon\delta$. It thus suffices to prove that

$$C_{\tau\gamma}\left\{\omega \in \partial T : \limsup_{\Omega_\tau, a(\omega)\ni w\to\omega} \frac{Bf(w)}{|w|} > \delta\right\} < c\epsilon.$$

By (6.2), this will be achieved if, for each i from 1 to 4, $\omega \in \partial T$, and $w \in \Omega_{\tau\gamma}(\omega)$, we can show that

$$\sum_i \le cM_{\tau\gamma}h(\omega),$$

where $\sum_i$ is defined in (6.6). We have

$$\begin{aligned}
\sum_1 &\le \sum_{j=0}^{|w\wedge\omega|-1} \sum_{v\in\overline{S(\omega_j)}} \frac{2^j-1}{|w|} 2^{-(1-\gamma)|v|} h(v) \\
&\le \sum_{j=0}^{|w\wedge\omega|-1} \sum_{v\in\overline{S(\omega_j)}} 2^{-\gamma\tau j} \frac{2^j-1}{|w|} 2^{-(1-\gamma)j} h(v) 2^{\gamma\tau j} \\
&\le \sum_{j=0}^{|w\wedge\omega|-1} \sum_{v\in\overline{S(\omega_j)}} \frac{2^{\gamma j(1-\tau)}}{|w|} h(v) 2^{\gamma\tau j} \le \sum_{j=0}^{|w\wedge\omega|-1} \frac{h^*(\omega_j)}{|w|} 2^{\gamma\tau j} \\
&\le \sum_{j=0}^{|w\wedge\omega|-1} \frac{M_{\tau\gamma}h(\omega)}{|w|} \le M_{\tau\gamma}h(\omega);
\end{aligned}$$

$$\begin{aligned}
\sum_2 &\le \sum_{j=1}^n \sum_{v\in\overline{S(w^j)}} \frac{2^j-1}{|w|} 2^{-(1-\gamma)|v|} h(v) \le \sum_{j=1}^n \sum_{v\in\overline{S(w^j)}} \frac{2^j-1}{|w|} 2^{-(1-\gamma)[|w\wedge\omega|+j]} h(v) \\
&\le \sum_{j=1}^n \frac{2^{\gamma[|w\wedge\omega|+j]}}{|w|} h^*(w\wedge\omega) \le \sum_{j=1}^n \frac{2^{\gamma[\tau|w\wedge\omega|+a]}}{|w|} h^*(w\wedge\omega) \\
&\le 2^{\gamma a} \sum_{j=1}^n \frac{M_{\gamma\tau}h(\omega)}{|w|} \le 2^{\gamma a} M_{\gamma\tau} h(\omega);
\end{aligned}$$

$$\begin{aligned}
\sum_3 &\le \sum_{v\in S(w)} \frac{2^{|w|}-1}{|w|} 2^{-(1-\gamma)|v|} h(v) \le \sum_{v\in S(w)} \frac{2^{|w|}-1}{|w|} 2^{-(1-\gamma)|w|} h(v) \\
&\le \frac{2^{\gamma|w|}}{|w|} h^*(w) \le \frac{2^{\gamma[\tau|w\wedge\omega|+a]}}{|w|} h^*(w\wedge\omega) \le 2^a M_{\gamma\tau} h(\omega);
\end{aligned}$$

$$\begin{aligned}
\sum_4 &\le \sum_{v\in\overline{S(w\wedge\omega)}} \frac{2^{|w\wedge\omega|}-1}{|w|} 2^{-(1-\gamma)|v|} h(v) \\
&\le \sum_{v\in\overline{S(w\wedge\omega)}} \frac{2^{|w\wedge\omega|}-1}{|w|} 2^{-(1-\gamma)|w\wedge\omega|} h(v) \le \frac{2^{\gamma|w\wedge\omega|}}{|w|} h^*(w\wedge\omega) \\
&\le 2^{\gamma\tau|w\wedge\omega|} h^*(w\wedge\omega) \le M_{\gamma\tau} h(\omega),
\end{aligned}$$

completing the proof in case $p = 1$.

We now consider the case of $p > 1$. Thus $0 < \gamma < 1$. Fix $\omega \in \partial T \setminus A \cup B$, defined above in (6.3) and (6.4). Let $\epsilon > 0$. Choose j_0 such that

$$C_{j_0} := \sup_{j\ge j_0} 2^j \sum_{v\in S(\omega_j)} f(v) 2^{-|v|} < \epsilon;$$

we can do this because $\omega \notin A$. Now consider $w \in \Omega_{\tau,a}(\omega)$ with $|w| > j_0$. As before, we consider $\sum_i := \sum_{v\in T_i} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v)$ separately for $i = 1, \dots, 4$. We have

$$\sum_1 = \sum_{j=0}^{j_0-1} \sum_{v\in\overline{S(\omega_j)}} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v) + \sum_{j=j_0}^{|w\wedge\omega|-1} \sum_{v\in\overline{S(\omega_j)}} \frac{2^{|v\wedge w|}-1}{|w|} 2^{-|v|} f(v).$$

The first term is bounded by $\dfrac{j_0 2^{j_0} \sum_T 2^{-|v|} f(v)}{|w|}$, and this goes to 0 as $|w| \to \infty$. The second term is bounded above by

$$c \sum_{j=j_0}^{|w\wedge\omega|-1} \sum_{v\in\overline{S(\omega_j)}} \frac{2^j 2^{-|v|} f(v)}{|w|} \le c C_{j_0} \sum_{j=j_0}^{|w\wedge\omega|-1} 1/|w| < c\epsilon,$$

and so we are done with $\sum_1$.

To deal with $\sum_2, \sum_3$ and $\sum_4$ we will apply Hölder's inequality. Let $p' := p/(p-1)$ be the exponent conjugate to p. We write

$$\frac{2^{|v\wedge w|}}{|w|} 2^{-|v|} f(v) = \left(\frac{2^{|v\wedge w|}}{|w|} 2^{-(1-\frac{\gamma}{p})|v|} \right) \left(2^{-\frac{\gamma}{p}|v|} f(v) \right). \tag{6.7}$$

Before we apply Hölder's inequality we first estimate $\sum_{v\in T_i} 2^{|v\wedge w|p'} 2^{-(1-\frac{\gamma}{p})p'|v|}$ for $i = 2, 3, 4$. We claim that each is bounded above by a multiple of $2^{\frac{p'}{p}\gamma\tau|w\wedge\omega|}$. To prove it, we make use of (2.1) and the fact that $\left(1-\dfrac{\gamma}{p}\right)p'$ is greater than 1. Recalling that $n = |w| - |w\wedge\omega| - 1$, we have

$$\begin{aligned}
\sum_{v\in T_2} 2^{|v\wedge w|p'} 2^{-(1-\frac{\gamma}{p})p'|v|} &\leq c\sum_{j=1}^{n} 2^{(j+|w\wedge\omega|)p'} 2^{-(1-\frac{\gamma}{p})p'(j+|w\wedge\omega|)} \\
&\leq c\sum_{j=1}^{n} 2^{(j+|w\wedge\omega|)p'\frac{\gamma}{p}} \\
&\leq c2^{(n+|w\wedge\omega|)p'\frac{\gamma}{p}} \\
&\leq c2^{|w\wedge\omega|\tau\gamma\frac{p'}{p}},
\end{aligned}$$

since $n + |w\wedge\omega| < |w| \leq \tau|w\wedge\omega| + a$. This proves the claim for the sum over T_2. We omit the proofs for the other two cases (actually they are easier to prove) and the claim is established.

By (6.7), we thus have for $i = 2, 3, 4$, that

$$\begin{aligned}
\sum_i &\leq \left(\sum_{v\in T_i} 2^{|v\wedge w|p'} 2^{-(1-\frac{\gamma}{p})|v|p'}\right)^{1/p'} \left(\sum_{S(w\wedge\omega)} 2^{-\gamma|v|} f^p(v)\right)^{1/p} \\
&\leq c2^{\frac{\gamma\tau}{p}|w\wedge\omega|} \left(\sum_{S(w\wedge\omega)} 2^{-\gamma|v|} f^p(v)\right)^{1/p} \\
&= c\left(2^{\gamma\tau|w\wedge\omega|} \sum_{v\in S(w\wedge\omega)} 2^{-\gamma|v|} f^p(v)\right)^{1/p},
\end{aligned}$$

and, since ω is not in the set B, this goes to 0 as $|w|$ goes to ∞. This completes the proof. □

Note that in the above proof for $p > 1$, the factor of $1/|w|$ was needed in dealing with $\sum_1$, but not in dealing with $\sum_2, \sum_3$ or $\sum_4$.

7. Biharmonic Green functions on homogeneous trees

Our aim in this section is to explain how one might define various biharmonic Green functions on the 1/3−tree and to see what one can say about them. The details appear in [**30**]. One of the interesting features of that paper is the calculations we were able to do, in particular explicit calculations of the Euclidean and hyperbolic simply-supported biharmonic Green functions as well as a function we called the biharmonic Martin kernel. On the surface these calculations appear to be hard, so we made use of a computer algebra system in order to do some of them.

We first describe what is meant by a biharmonic Green function Γ on the unit disk in the complex plane $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If we think of $\mathbb{D}$ as being a thin metal plate, then for points $P, Q \in \mathbb{D}$, $\Gamma(P, Q)$ represents the deflection at P due to a unit load placed at Q. Outside of Q, the function $P \mapsto \Gamma(P, Q)$ is

not harmonic, but rather is biharmonic, that is, it is annihilated by the Euclidean bilaplacian, Δ_E^2. In order to uniquely determine Γ, one must impose boundary conditions. In addition to the condition that the deflection on the boundary be zero, one must prescribe one additional condition, namely a **clamped condition** or a **simply-supported condition**. The clamped condition is that the plate is clamped horizontally at the boundary, which means that the normal derivative is 0 at the boundary; the simply-supported condition is that the boundary rests on a support, which means that the Euclidean Laplacian is 0 at the boundary. We thus get two biharmonic Green functions Γ_C and Γ_S uniquely determined as follows:

Clamped Euclidean biharmonic Green function:

$$\begin{cases} (\Delta_E)_P^2 \Gamma_C(P,Q) = \delta_Q(P) & \text{for } P,Q \in \mathbb{D} \\ \Gamma_C(p,Q) = 0 & \text{for } (p,Q) \in \partial\mathbb{D} \times \mathbb{D} \\ \left(\frac{d}{dn}\right)_P \Gamma_C(p,Q) = 0 & \text{for } (p,Q) \in \partial\mathbb{D} \times \mathbb{D} \end{cases}$$

Simply-supported Euclidean biharmonic Green function:

$$\begin{cases} (\Delta_E)_P^2 \Gamma_S(P,Q) = \delta_Q(P) & \text{for } P,Q \in \mathbb{D} \\ \Gamma_S(p,Q) = 0 & \text{for } (p,Q) \in \partial\mathbb{D} \times \mathbb{D} \\ \lim_{P\to p} (\Delta_E)_P \Gamma_S(P,Q) = 0 & \text{for } (p,Q) \in \partial\mathbb{D} \times \mathbb{D} \end{cases}$$

Biharmonic Green functions can be used to construct functions having prescribed bilaplacian and various prescribed boundary conditions. For a discussion of the clamped Euclidean biharmonic Green function on $\mathbb{D}$, see [**33**]. For a detailed discussion of both biharmonic Green functions on a Riemannian manifold, see [**50**].

Each Riemannian metric on $\mathbb{D}$ defines a distance between points and a Laplace-Beltrami operator. The Euclidean metric defines the usual distance on D, and so is bounded, whereas the hyperbolic metric defines a distance for which the diameter of $\mathbb{D}$ is infinite. The Euclidean metric $ds^2 = dx^2 + dy^2$ associates the Euclidean Laplacian $\Delta_E f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$ and the hyperbolic metric $ds^2 = (1-x^2-y^2)^{-2}(dx^2 + dy^2)$ associates the hyperbolic Laplacian $\Delta_H f = (1-x^2-y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$. A function is called **Euclidean (respectively, hyperbolic-biharmonic)** if it is annihilated by the bilaplace operator Δ_E^2 (respectively, Δ_H^2). The Euclidean and hyperbolic harmonic functions are the same, but the Euclidean and hyperbolic biharmonic functions are quite different due to the extra factor of $(1-x^2-y^2)^2$ in Δ_H.

In section 2 we defined, for vertices v, w, the hyperbolic and Euclidean distances between them, namely $|v-w|$ and $2^{-|v\wedge w|}$, respectively. Since the Laplacian we have so far discussed in this paper is defined by averaging a given function over nearest neighbours, i.e. over vertices a hyperbolic distance away of 1, we refer to this Laplacian as the **hyperbolic Laplacian:** $\Delta_H f(v) = \sum_{w\sim v} p(v,w) f(w) - f(w)$.

Motivated by the fact that on the unit disk $\mathbb{D}$ we have

$$\Delta_E f(x,y) \quad = \quad (1-x^2-y^2)^{-2} \Delta_H f(x,y),$$

and the factor $1-x^2-y^2$ behaves roughly like the Euclidean distance of $x+iy$ to the boundary of the disk, we define the **Euclidean Laplacian** on the $1/3-$tree to

be

$$\Delta_E f(u) = 2^{2|u|}\Delta_H f(u),$$

since the factor $2^{2|u|}$ is the Euclidean distance of u to the boundary of T raised to the power -2. We can thus define the **Euclidean simply-supported biharmonic Green function** Γ_S on T to be a function satisfying

$$\begin{cases} (\Delta_E)_u^2\Gamma_S(u,v) = \delta_v(u) & \text{for all } u,v \in T \\ \lim_{n\to\infty}\Gamma_S(\omega_n, v) = 0 & \text{for all } \omega \in \partial T, v \in T \\ \lim_{n\to\infty}(\Delta_E)_u\Gamma_S(\omega_n, v) = 0 & \text{for all } \omega \in \partial T, v \in T. \end{cases}$$

In order to define what is the clamped Euclidean biharmonic Green function, we must first define what is the normal derivative of a function f at boundary point ω. Assuming that $f(\omega) = \lim_{m\to\infty} f(\omega_m)$ exists, we define it as a limit of difference quotients by

$$\frac{df}{dn}(\omega) = \lim_{m\to\infty}\frac{f(\omega_m) - f(\omega)}{2^{-m}},$$

since the Euclidean distance of ω_m to ω is 2^{-m}. Thus we define the **clamped biharmonic Green function** Γ_C to satisfy

$$\begin{cases} (\Delta_E)_u^2\Gamma_C(u,v) = \delta_v(u) & \text{for all } u,v \in T \\ \lim_{n\to\infty}\Gamma_C(\omega_n, v) = 0 & \text{for all } \omega \in \partial T, v \in T \\ \left(\frac{d}{dn}\right)_\omega \Gamma_C(\omega, v) = 0 & \text{for all } \omega \in \partial T, v \in T. \end{cases}$$

The main result in [**30**] is the following.

THEOREM 7.1. *(a) Γ_S and Γ_C both exist and are unique.*
(b) Γ_S exists and is given by the two formulas

$$\Gamma_S(u,v) = 4\cdot 2^{-2|v|}\sum_{w\in T} 2^{-2|w|}2^{-|u-w|}2^{-|v-w|} = \frac{x_S(u\wedge v, v)}{2^{|u-u\wedge v|}} + \frac{y_S(u\wedge v, v)}{2^{3|u-u\wedge v|}}, \tag{7.1}$$

where

$$x_S(u\wedge v, v) = \frac{2\left(2^{-3|v|-|u\wedge v|}\left[25 + 15|u\wedge v|\right] - 4\cdot 2^{-5|v|+|u\wedge v|}\right)}{7} \tag{7.2}$$

$$y_s(u\wedge v, v) = -\frac{8}{7}\,2^{-3|v|-|u\wedge v|}. \tag{7.3}$$

Let's write a few comments about the the proof. The uniqueness of Γ_S is proved using a simple maximum principle argument. The first formula in (7.1) is proved by just simply checking that the sum has the required properties of Γ_S, and so the existence of Γ_S in (a) is established. In order to understand where the second equality in (7.1) comes from, let's consider what are the radial Euclidean biharmonic functions on $T\setminus\{e\}$, that is the functions f such that $f(u)$ is constant on each n sphere $|u| = n$ for $n \geq 1$. Denote by x_n the value $f(u)$ for $|u| = n$. Then for $|u| = n$, $\Delta_E f(u) = 2^{2n}\left[\frac{2}{3}x_{n+1} + \frac{1}{3}x_{n-1} - x_n\right]$ and

$$\begin{aligned}\Delta_E^2 f(u) &= 2^{2n}\left[\frac{2}{3}2^{2n+2}\left(\frac{2}{3}x_{n+2} + \frac{1}{3}x_n - x_{n+1}\right)\right. \\ &\quad \left. + \frac{1}{3}2^{2n-2}\left(\frac{2}{3}x_n + \frac{1}{3}x_{n-2} - x_{n-1}\right) - 2^{2n}\left(\frac{2}{3}x_{n+1} + \frac{1}{3}x_{n-1} - x_n\right)\right].\end{aligned}$$

Setting this to zero and simplifying gives the fourth order homogeneous difference equation

$$64x_{n+2} - 120x_{n+1} + 70x_n - 15x_{n-1} + x_{n-2} = 0.$$

The corresponding characteristic polynomial is $64r^4 - 120r^3 + 70r^2 - 15r + 1$ and this factors as $(r-1)(2r-1)4r-1)(8r-1)$ and so the roots are $1, 1/2, 1/2^2, 1/2^3$. Thus the radial biharmonic function f can be written in the form

$$f(u) = c_0 + c_1 2^{-|u|} + c_2 2^{-2|u|} + c_3 2^{-3|u|}. \tag{7.4}$$

Now consider each of these four terms with respect to their boundary behaviour and the boundary behaviour of their Euclidean Laplacian. The first term is the only one that does not go to zero at infinity. Outside the root, the Euclidean Laplacian of the second term is 0, the Euclidean Laplacian of the third term is constant, and the Euclidean Laplacian of the fourth term is a multiple of $2^{-|u|}$. Thus of the four terms, only the second and fourth terms both go to 0 at infinity and have Euclidean Laplacian which also goes to 0 at infinity.

Now fix $v \in T$ and consider the corresponding partition $\mathcal{P}(v)$ of T. A typical member of it is a set $\{u \in T : |u \wedge v| = k\}$ for $k = 0, \ldots, |v|$. On each such set, it follows by symmetry that $u \mapsto \Gamma_S(u, v)$ is radial, and so by the result of the previous paragraph, it can be written as a linear combination of $2^{-|u-u\wedge v|}$ and $2^{-3|u-u\wedge v|}$. The coefficients in this linear combination are denoted by $x_S(u \wedge v, v)$ and $y_S(u \wedge v, v)$ as in (7.2) and (7.3) above. It turns out that the infinite sum in (7.1) can be explicitly found in case u is either $u \wedge v$ or a forward neighbour of $u \wedge v$. This then gives two linear equation satisfied by $x_S(u \wedge v, v)$ and $y_S(u \wedge v, v)$, and so can be solved, resulting in the values of x_S and y_S in (7.2) and (7.3).

The results concerning Γ_C are more demanding, both for existence and uniqueness. We only comment briefly on the existence proof. It makes use of the explicit formulas for Γ_S as well as a new function, $B : T \times \partial T \to \mathbb{R}^+$, which we call the **biharmonic Martin kernel**, used to produce Euclidean biharmonic functions. It is defined by

$$B(u, \omega) = 2 \sum_{w \in T} \frac{2^{2|w\wedge\omega| - |w|}}{2^{|u-w|}\, 2^{2|w|}},$$

chosen this way in order to have the property that for each $\omega \in \partial T$, $\Delta_H B(u, \omega) = -K_H(u, \omega) = -2^{2|u\wedge\omega| - |u|}$, (recall the Martin kernel discussed in example 3.1, which we now call the hyperbolic Martin kernel, K_H). By integrating out ω with respect to a measure μ one obtains a Euclidean biharmonic function $B\mu(u) = \int B(u, \omega) d\mu(\omega)$ such that $\Delta_E(B\mu) = -K_H \mu$. In order to use it to prove the existence theorem, one needs to sum it in closed form. This can be done, and the result is

$$B(u, \omega) = \frac{2^{-|u|}}{7} \left[25 + 15|u \wedge \omega| - 2^{2(1-|u-u\wedge\omega|)} \right].$$

From here one can calculate the normal derivative at ∂T, and thus construct examples of Euclidean biharmonic functions with prescribed normal derivative. Then $\Gamma_C(u, v)$ is obtained from $\Gamma_S(u, v)$ by subtracting off an appropriate function $\Phi(u, v)$ which for each fixed v is of the form $(B\mu)(u)$, with μ chosen so that $(B\mu)(u)$ has the appropriate normal derivative at ∂T.

8. Multiplication operators and a minimum principle

In this section we begin by discussing the results in [**31**]. We will then focus attention on a certain minimum principle which is needed to prove the main result, and which can be formulated in a fairly general Brelot space.

Let X, Y be Banach spaces of holomorphic function on a domain $D \subset \mathbb{C}^N$. Let $\psi : D \to \mathbb{C}$ be holomorphic such that $\psi f \in Y$ whenever $f \in X$. Then the **multiplication operator with symbol** ψ is the function $M_\psi : X \to Y$, $f \mapsto \psi f$. The aim is to characterize the symbol ψ in order that M_ψ is of a certain special type, such as for example bounded, or compact, or an isometry. It is also of interest to calculate the norm of M_ψ.

There are many possible choices for D (for example the unit disk in $\mathbb{C}$, the unit ball in $\mathbb{C}^N$, the unit polydisk in $\mathbb{C}^N$, a bounded homogeneous domain, a bounded symmetric domain) and for the spaces X and Y (Hardy spaces, Bergman spaces, Dirichlet spaces, Bloch spaces, weighted versions of these spaces). There are numerous examples in the literature of this kind of research, for example [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**13**], [**22**], [**39**], [**45**], [**46**], [**47**], [**48**], [**49**], [**52**], [**53**], [**57**]. The one which we focus on in this section, [**31**], takes D to be a bounded symmetric domain, X to be $H^\infty(D)$ and Y to be the Bloch space on U. We explain these terms below.

The **Bloch space** $\mathcal{B}$ on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the set of holomorphic functions on $\mathbb{D}$ such that $\beta_f := \sup_{|z|<1} \left(1 - |z|^2\right) |f'(z)|$ is finite. Then $\|f\|_{\mathcal{B}} := |f(0)| + \beta_f$ is a norm, and it makes $\mathcal{B}$ a Banach space. A geometric condition equivalent to the Bloch condition is that $f \in \mathcal{B}$ if and only if the radii of schlicht disks in $f(\mathbb{D})$ are upper bounded. One has the inclusions that H^∞ and BMOA are properly contained in $\mathcal{B}$, none of the Hardy spaces H^p are contained in $\mathcal{B}$ if $p < \infty$, and $\mathcal{B}$ is not a subset of the Nevanlinna class.

The Bloch space can be defined on bounded homogeneous domains in $\mathbb{C}^N$. The domain D is called a **homogeneous domain** if for all $z, w \in D$ there exists $\phi \in Aut(D)$ (i.e. a holomorphic self map that is a bijection) such that $\phi(z) = w$. In [**36**], [**54**], and [**55**] the definition of Bloch function was extended to homogeneous domains by the condition

$$Q_f := \sup_{z \in D} \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left| \sum_{j=1}^N \frac{\partial f}{\partial z_j}(z) u_j \right|}{\sqrt{H_z(u, \overline{u})}} < \infty,$$

where for each $z \in D$, H_z is the Bergman metric on D at z.

A domain D in $\mathbb{C}^N$ is called **symmetric** if for every $z \in D$ there exists $\phi \in Aut(D)$ such that $\phi \circ \phi = id$, and z is an isolated fixed point of ϕ. Such domains are actually homogeneous. Cartan in [**23**] showed that any bounded symmetric domain is biholomorphically equivalent to a unique finite product of irreducible bounded symmetric domains. He classified the irreducible domains into the four classical domains $R_I, R_{II}, R_{III}, R_{IV}$ (see [**41**]), and two exceptional domains R_V, R_{VI} (see [**33**]).

A bounded symmetric domain D is said to be in **standard form** if it can be written as a finite product of Cartan domains. To each domain D one can associate the **Bloch constant** c_D defined by

$$c_D = \sup\{Q_f : f \in H^\infty(D), \|f\|_\infty \leq 1\}.$$

The values of the Bloch constants are known for each of the Cartan domains (see [**25**] and [**56**]), $c_D \leq 1$ with strict inequality unless $D = \mathbb{D}$, and if $D = D_1 \times \cdots \times D_k$ is a product of Cartan domains, then $c_D = \max_{1 \leq j \leq k} c_{D_j}$ ([**25**]). Thus for any bounded symmetric domain, $c_D \leq 1$ with equality if and only if D has the unit disk as a factor.

We now state the main result of [**31**].

THEOREM 8.1. *Let D be a bounded symmetric domain in $\mathbb{C}^N$ and $M_\psi : H^\infty(D) \to \mathcal{B}(D)$.*

(1) *M_ψ is bounded if and only if $\psi \in H^\infty(D)$.*
(2) *If $\psi(0) = 0$ then $\|M_\psi\| = c_D \|\psi\|_\infty$.*
(3) *Suppose D has $\mathbb{D}$ as a factor. Then*

$$\|M_\psi\| = |\psi(0)| + c_D \|\psi\|_\infty.$$

(4) *If D is a bounded symmetric domain without exceptional factors, then there exist no isometric multiplication operators from $H^\infty(D)$ to $\mathcal{B}(D)$.*

The theorem suggests the following two **conjectures**:

(1) The norm equality in (3) above holds even if $\mathbb{D}$ is not a factor of D;
(2) There exist no isometries among the bounded multiplication operators from $H^\infty(D)$ to the Bloch space of any bounded symmetric domain D.

The proof of Theorem 8.1 makes use of a minimum principle for multiply superharmonic functions.

DEFINITION 8.2. Let $\omega_1 \times \omega_2$ be a product of relatively compact open subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively. We say that $v : \omega_1 \times \omega_2 \to \mathbb{R} \cup \{\infty\}$ is **multiply superharmonic** if

(1) v is not identically ∞,
(2) $v(x) > -\infty$ for all x,
(3) v is lower semicontinuous,
(4) for each fixed $x_1 \in \omega_1$ and $x_2 \in \omega_2$, $v(x_1, \cdot)$ is hyperharmonic (i.e. $\equiv \infty$ or superharmonic) on ω_2 and $v(\cdot, x_2)$ is hyperharmonic on ω_1.

Then the minimum principle which is used to prove Theorem 8.1 is the following.

THEOREM 8.3. ***(Minimum principle)*** *Let v be multiply superharmonic on $\omega_1 \times \omega_2$ and bounded below. If*

$$\liminf_{(z,z') \to (x,y)} v(z, z') \geq 0,$$

for all $(x, y) \in \partial\omega_1 \times \partial\omega_2$, then $v \geq 0$ on $\omega_1 \times \omega_2$.

The proof of this which we have in mind makes use of the **Cartan-Brelot topology**.

DEFINITION 8.4. **(Cartan-Brelot topology)** Let U be a fixed bounded open subset of $\mathbb{R}^m$. Let $S = S^+(U) \times S^+(U)/\sim$, where $\sim$ is the equivalence relation defined by $(u_1, v_1) \sim (u_2, v_2)$ if $u_1 + v_2 = u_2 + v_1$ and $S^+(U)$ denotes the set of positive superharmonic functions on U. Let $\mathcal{O}$ be the set of balls of rational radii

and X any countable dense subset of $\mathbb{R}^m$. For $\omega \in \mathcal{O}$ and $x \in \omega \cap X$, let ρ_x^ω denote the harmonic measure. Define the seminorm $\Pi_{\omega,x}$ on S by

$$\Pi_{\omega,x}\left[(u,v)\right] = \left| \int u d\rho_x^\omega - \int v \rho_x^\omega \right|.$$

The Cartan-Brelot topology is the topology generated by this countable family of seminorms.

A related topology is considered in [**38**]. The following basic properties of the Cartan-Brelot topology hold:

Lemma 8.5. (1) *The Cartan-Brelot topology is Hausdorff and $S^+(U)$ is closed.*
(2) *The mapping $f : S^+(U) \times U \to \mathbb{R} \cup \{\infty\}$ defined by $f(v,x) = v(x)$ is lower semicontinuous.*
(3) *Every uniformly locally bounded sequence in $S^+(U)$ has a subsequence converging in the Cartan-Brelot topology.*

In proving Theorem 8.1 above in [**31**], all the details are given except for showing the proofs of the above basic properties.

Actually one can formulate the Cartan-Brelot topology in any Brelot space having a countable base of regular domains. The lemma and the minimum principle can then be proved in such a Brelot space if in addition it satisfies the following axiom.

Axiom of Domination: *For U any relatively compact set and g a locally bounded positive superharmonic function on Ω that is harmonic on U, any positive superharmonic function which majorizes g on $\Omega \setminus U$ also majorizes it on U.*

The proof of the lemma and the minimum principle in the Brelot space setting can be found in [**51**]. Actually the proof of Theorem 8.3 above was adapted from the one in [**51**].

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