Norm of the multiplication operators from H^∞ to the Bloch Space of a bounded symmetric domain

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Abstract

Let D be a bounded symmetric domain in $\mathbb{C}^N$ and let ψ be a complex-valued holomorphic function on D. In this work, we determine the operator norm of the bounded multiplication operator with symbol ψ from the space of bounded holomorphic functions on D to the Bloch space of D when ψ fixes the origin. If no restriction is imposed on the symbol ψ, we have a formula for the operator norm when D is the unit ball or has the unit disk as a factor. The proof of this result for the latter case makes use of a minimum principle for multiply superharmonic functions, which we prove in this work. We also show that there are no isometries among the multiplication operators when the domain does not have exceptional factors or the symbol fixes the origin.

Key words: Multiplication operators, Bloch space, bounded symmetric domains, multiply superharmonic, Cartan-Brelot topology
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1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces of holomorphic functions on a domain Ω in $\mathbb{C}^N$ $(N \in \mathbb{N})$ and let ψ be a complex-valued holomorphic function on Ω such that $\psi f \in \mathcal{Y}$ whenever $f \in \mathcal{X}$. The **multiplication operator with symbol** ψ from $\mathcal{X}$ to $\mathcal{Y}$ is the operator M_ψ defined by

$$M_\psi f = \psi f, \text{ for } f \in \mathcal{X}.$$

Multiplication operators for the case in which $\mathcal{X}$ and $\mathcal{Y}$ are both equal to the Bloch space of the open unit disk $\mathbb{D}$ have been studied in [8], [10], [3], and [1]. For the case of the Bloch space of a bounded homogeneous domain in $\mathbb{C}^N$, see [4]. However, since multiplication operators are degenerate weighted composition operators, many operator theoretic results on multiplication operators, such as boundedness and compactness, are subsumed in [23] and [26].

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The weighted composition operators between the Bloch space of $\mathbb{D}$ and the Hardy space H^∞ of bounded analytic functions on $\mathbb{D}$ were investigated in [22]. Characterizations of the boundedness and the compactness of the weighted composition operators from the Bloch space to H^∞ were given in [16] in the case of the unit disk, and in [21] for the case of the ball. The study of the weighted composition operators from the Bloch space, as well as related spaces known as α-Bloch spaces, to the Hardy space H^∞ was carried out in [20] for the polydisk case. The operator norm of the weighted composition operators from the Bloch space to the weighted Hardy space H^∞_μ (where μ is a weight) was determined in [25] for the case of the ball. In [5], the operator norm of the weighted composition operators from the Bloch space to H^∞ was determined in the case of a general bounded homogeneous domain.

The study of the weighted composition operators from the Hardy space H^∞ to the α-Bloch spaces was carried out in [19] for the polydisk case, and [21] and [30] for the case of the ball. In [6] the bounded weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain were characterized and operator norm estimated were derived.

In this paper, we obtain sharper estimates on the operator norm of the multiplication operators from H^∞ to the Bloch space on a general bounded symmetric domain and determine such norm precisely in the case when the symbol of the operator fixes the origin as well as when the domain is the ball or a bounded symmetric domain that has the unit disk as a factor, up to a biholomorphic transformation, and the symbol is not subjected to any restriction. We use this norm to show that for a large class of bounded symmetric domains D there are no isometries among these multiplication operators, a result that was shown in [6] (Theorem 6.2) only when D is the unit disk.

In Section 2, we present an overview of the Bloch space on a bounded homogeneous domain in $\mathbb{C}^N$, the Cartan classification of bounded symmetric domains, and background results which we shall need in this work.

In Section 3, we prove a minimum principle for multiply superharmonic functions.

In Section 4, we establish the main results of the paper. Specifically, in Theorem 4.2, we obtain new estimates on the norm of a bounded multiplication operator M_ψ from H^∞ to the Bloch space on a bounded symmetric domain, which allow us to determine exactly this norm in the special case when the symbol of the operator fixes the origin. From these estimates, in Theorem 4.3 we also obtain a formula for the operator norm without the above restriction on the symbol when the domain is the unit ball or has the unit disk as a factor. Theorem 4.3 makes use of the minimum principle proved in the previous section.

Finally, in Section 5, we use Theorem 4.2 to prove that if the symbol ψ, defined on a bounded symmetric domain D, fixes the origin, or if ψ is unrestricted but D does not have an exceptional factor, then the operator M_ψ cannot be an isometry.

2. Preliminaries

2.1. Background on the Bloch space

A **homogeneous domain** in $\mathbb{C}^N$ ($N \in \mathbb{N}$) is a domain D such that the group of biholomorphic transformations $\mathrm{Aut}(D)$ mapping D onto itself acts transitively on D, that is, for any pair of points $z, w \in D$ there exists $T \in \mathrm{Aut}(D)$ such that $T(z) = w$. We call the elements of $\mathrm{Aut}(D)$ **automorphisms** of D.

Let f be a complex-valued holomorphic function on a bounded homogeneous domain D in $\mathbb{C}^N$. For $u, v \in \mathbb{C}^N$, let $\langle u, v\rangle = \sum_{k=1}^{N} u_k\overline{v_k}$, and for $z \in D$, let $(\nabla f)(z)u = \langle(\nabla f)(z), \overline{u}\rangle$, where $(\nabla f)(z)$ is the gradient of f at z. For $z \in D$, let H_z be the Bergman metric on D at z. Thus, it is a positive definite Hermitian form which is invariant under automorphisms of D. This means that for $S \in\mathrm{Aut}(D)$ and $u \in \mathbb{C}^N$

$$H_{S(z)}(JS(z)u, \overline{JS(z)u}) = H_z(u, \overline{u}), \tag{1}$$

where $JS(z)$ is the Jacobian matrix of S at z and $JS(z)u$ is the usual matrix product where u is viewed as a column vector.

A **Bloch function** on D is a holomorphic function f on D such that

$$Q_f = \sup_{z \in D} Q_f(z)$$

is finite, where

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|(\nabla f)(z)u|}{H_z(u, \overline{u})^{1/2}}.$$

Denote by $\mathcal{B}(D)$ the space of Bloch functions on D. The map $f \mapsto Q_f$ is a semi-norm on $\mathcal{B}(D)$, which by (1) is invariant under right composition of automorphisms. Fixing any point $z_0 \in D$, the set $\mathcal{B}(D)$ is a Banach space, called the **Bloch space**, under the norm

$$\|f\|_{\mathcal{B}} = |f(z_0)| + Q_f.$$

Throughout this paper we shall assume that $0 \in D$ and $z_0 = 0$. The Bloch space contains the space $H^\infty(D)$ of bounded holomorphic functions on D [27].

Useful references on Bloch functions include [7] for the one-dimensional case, and [14], [27] and [28] for the multi-variable case. Bloch functions have been defined on more general classes of bounded domains, such as strongly pseudo-convex domains [18]. These domains, however, are not as suitable for the study of operator theoretic problems due to their sparse, and possibly trivial, automorphism groups.

2.2. Cartan's classification of bounded symmetric domains

A domain D in $\mathbb{C}^N$ is said to be **symmetric** if for each $a \in D$, there exists an involutory automorphism S of D that has a as an isolated fixed point. Symmetric domains are homogeneous (see [15], pp. 170, 301). Examples of symmetric domains are the unit ball

$$\mathbb{B}_N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\| < 1\},$$

where $\|z\|$ denotes the Euclidean norm of z, and the unit polydisk

$$\mathbb{D}^N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : |z_j| < 1,\ j = 1, \ldots, N\}.$$

Cartan [11] proved that any bounded symmetric domain is biholomorphically equivalent to a finite product of irreducible bounded symmetric domains, unique up to rearrangement of the factors. He then classified all the irreducible domains we call **Cartan domains** into four classes $R_I, R_{II}, R_{III}, R_{IV}$, described below with their Bergman metrics, called **classical domains**, and two classes R_V and R_{VI}, each containing a single domain of dimension 16 and 27, respectively, called **exceptional domains**. For a description of the latter domains see [13]. The classical domains are discussed in [17].

For $M, N \in \mathbb{N}$, denote by $\mathcal{M}_{M,N}$ the set of $M \times N$ matrices over $\mathbb{C}$, let $\mathcal{M}_N = \mathcal{M}_{N,N}$ and let the symbol $>$ in connection with matrices denote positive definiteness. Let $I_N \in \mathcal{M}_N$ be the identity matrix and let Z^* be the adjoint of Z. Then

$$\begin{aligned}
R_I &= \{Z \in \mathcal{M}_{M,N} : I_M - ZZ^* > 0\}, \text{ for } N \geq M \geq 1, \\
H_Z(U, \overline{V}) &= \frac{M+N}{2} \mathrm{Trace}[(I_M - ZZ^*)^{-1} U (I_N - Z^*Z)^{-1} V^*], \\
R_{II} &= \{Z \in \mathcal{M}_N : Z = Z^T, I_N - ZZ^* > 0\}, \text{ for } N \geq 2, \\
H_Z(U, \overline{V}) &= \frac{N+1}{2} \mathrm{Trace}[(I_N - ZZ^*)^{-1} U (I_N - Z^*Z)^{-1} V^*], \\
R_{III} &= \{Z \in \mathcal{M}_N : Z = -Z^T, I_N - ZZ^* > 0\}, \text{ for } N \geq 5, \\
H_Z(U, \overline{V}) &= \frac{N-1}{2} \mathrm{Trace}[(I_N - ZZ^*)^{-1} U (I_N - Z^*Z)^{-1} V^*], \\
R_{IV} &= \left\{ z \in \mathbb{C}^N : \left| \sum z_j^2 \right|^2 + 1 - 2\|z\|^2 > 0,\ \left| \sum z_j^2 \right|^2 < 1 \right\}, \text{ for } N \neq 5, \\
H_z(u, \overline{v}) &= NAu[A(I_N - z^T \overline{z}) + (I_N - z^T \overline{z}) Z^* z (I_N - z^T \overline{z})] v^*,
\end{aligned}$$

where z^T is the transpose of z and $A = |\sum_{j=1}^N z_j^2|^2 + 1 - 2\|z\|^2$. The dimensional restrictions imposed above guarantee the membership of a Cartan domain to a unique class. In the special case of the unit ball, the Bergman metric at $z \in \mathbb{B}_N$ is given by

$$H_z(u, \overline{v}) = \frac{N+1}{2} \frac{|(1-\|z\|^2)\langle u, v\rangle + \langle u, z\rangle\langle z, v\rangle|}{(1-\|z\|^2)^2},$$

where $u, v \in \mathbb{C}^N$. Indeed, the ball $\mathbb{B}_N$ is in R_I with $M = 1$, and for $Z = [z_1 \ \cdots \ z_N]$,

$$(I_N - Z^*Z)^{-1}_{j,k} = \frac{\overline{z_j} z_k + \delta_{j,k}(1-\|Z\|^2)}{1-\|Z\|^2},$$

where $\delta_{j,k}$ is the Kronecker delta. In particular, in the case of the unit disk, for $z \in \mathbb{D}$ and $u, v \in \mathbb{C}$, we have

$$H_z(u, \overline{v}) = \frac{u\overline{v}}{(1-|z|^2)^2}.$$

Note that the description of R_I in [17] does not include the restriction $N \geq M$. However, if $W \in R_I$ as defined in [17] has more rows than columns, then $Z = W^*$ is in R_I as defined by us. This follows from the fact that for any $M \times N$ matrix Z, $I_M - ZZ^* > 0$ if and only if $I_N - Z^*Z > 0$.

A bounded symmetric domain D is said to be in **standard form** if it has the form $D = D_1 \times \cdots \times D_k$, where each D_j is a Cartan domain.

Throughout the remainder of the paper, D shall denote a bounded symmetric domain in standard form.

Define the **Bloch constant** of D as

$$c_D = \sup\{Q_f(z) : f \in H^\infty(D), \|f\|_\infty \leq 1, z \in D\}.$$

By Theorem 2 of [12] and Theorem 3 of [29], if D is a Cartan domain, then

$$c_D = \begin{cases} \sqrt{2/(M+N)} & \text{if } D \in R_I, \\ \sqrt{2/(N+1)} & \text{if } D \in R_{II}, \\ \sqrt{1/(N-1)} & \text{if } D \in R_{III}, \\ \sqrt{2/N} & \text{if } D \in R_{IV}, \\ 1/\sqrt{6} & \text{if } D = R_V, \\ 1/3 & \text{if } D = R_{VI}. \end{cases} \tag{2}$$

In particular, if D is the unit ball $\mathbb{B}_N$, then $c_D = \sqrt{2/(N+1)}$.

Furthermore, by Theorem 3 of [12] extended to include the exceptional domains, if $D = D_1 \times \cdots \times D_k$ is in standard form, then

$$c_D = \max_{1 \leq j \leq k} c_{D_j}, \tag{3}$$

so that $c_D < 1$ except when D has the unit disk as a factor, in which case $c_D = 1$.

Remark 2.1. If f is a holomorphic function mapping a bounded symmetric domain D into $\mathbb{D}$ and $Q_f = c_D$, then $\|f\|_\infty = 1$. Indeed, if f is a nonconstant holomorphic function such that $\|f\|_\infty < 1$, then the function $g = f/\|f\|_\infty$ maps D into $\mathbb{D}$ so that $Q_g \leq c_D$, whence $Q_f \leq c_D\|f\|_\infty < c_D$.

The following results will be used in Section 5 to show that there are no isometries among the multiplication operators from $H^\infty(D)$ to $\mathcal{B}(D)$ whose symbol fixes the origin.

Theorem 2.1. (Theorem 7 of [12]) *Let $D = D_1 \times \cdots \times D_k$, with $D_1, \ldots, D_k$ irreducible and let $f \in H^\infty(D)$ with $\|f\|_\infty = 1$ such that $Q_f = c_D$. Then, for each $w_0 \in D$, there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ of automorphisms of D such that $\{f \circ T_n\}$ converges locally uniformly to a holomorphic function $F \in H^\infty(D)$ such that $\|F\|_\infty = 1$ and $Q_F(w_0) = c_D$.*

Theorem 2.2. (Theorem 6 of [12] and Theorem 4 of [29]) *Let $D = D_1 \times \cdots \times D_k$ be a bounded symmetric domain, with $D_1, \ldots, D_k$ irreducible and let $f \in H^\infty(D)$ such that $\|f\|_\infty = 1$ and $Q_f(w) = c_D$ for some $w \in D$. Then $c_D = c_{D_m}$ for some $m \in \{1, \ldots, k\}$, and there exist $x_m \in \partial D_m$ and an automorphism S of D_m such that*

$$f(z_1, \ldots, z_{m-1}, S(zx_m), z_{m+1}, \ldots, z_k) = z$$

for all $z \in \mathbb{D}$ and $z_j \in D_j$, for $j \neq m$.

Using Theorem 2.1 with $w_0 = 0$ and Theorem 2.2, we deduce the following result.

Corollary 2.1. Let $D = D_1 \times \cdots \times D_k$ with $D_1, \ldots, D_k$ irreducible, and let $f \in H^\infty(D)$ such that $\|f\|_\infty = 1$ and $Q_f = c_D$. Then, $c_D = c_{D_m}$ for some $m \in \{1, \ldots, k\}$, and there exist a sequence $\{T_n\}_{n \in \mathbb{N}}$ of automorphisms of D, $x_m \in \partial D_m$ and an automorphism S of D_m such that

$$\lim_{n \to \infty} f \circ T_n(z_1, \ldots, z_{m-1}, S(zx_m), z_{m+1}, \ldots, z_k) = z$$

for all $z \in \mathbb{D}$ and $z_j \in D_j$, for $j \neq m$, where the convergence is uniform on compact subsets of D.

3. A minimum principle for multiply superharmonic functions

In this section we digress from the topic of multiplication operators to establish a potential theoretic result which we need in order to prove one of our main results, Theorem 4.3. The key result of this section, Theorem 3.1, can be easily stated and is very natural, but we have not been able to locate it in the literature, even in the classical potential theory of Euclidean space. While intuitively one would expect to prove it easily, we could not come up with a completely elementary argument. Our proof makes use of a topology we refer to as the *Cartan-Brelot topology*, which is distinct from the well-known Cartan-Brelot fine topology. It was introduced in [9]. We merely quote the basic properties of this topology that will be needed. For details, see [24].

Let U be a bounded open subset of $\mathbb{R}^m$. Let $S^+(U)$ denote the set of nonnegative superharmonic functions on U. We define an equivalence relation $\sim$ on $S^+(U) \times S^+(U)$ as follows: $(u_1, v_1) \sim (u_2, v_2)$ if and only if $u_1 - v_1 = u_2 - v_2$. Denote by $[(u, v)]$ the equivalence class of (u, v) and by S the set of all equivalence classes $S^+(U)/\sim$, endowed with the obvious linear space structure. We identify $S^+(U)$ with the set $\{[(u, 0)] : u \in S^+(U)\}$.

Let ω be an open ball in $\mathbb{R}^m$ and let $x \in \omega$. Denote by ρ_x^ω the harmonic measure on $\partial\omega$ corresponding to x. Thus the sub-mean-value property for a superharmonic function v at x can be formulated as $\int v \, d\rho_x^\omega \leq v(x)$.

Let $\mathcal{O}$ denote the set of all open balls in $\mathbb{R}^m$ with rational radii, and let X be any countable dense subset of $\mathbb{R}^m$. For $\omega \in \mathcal{O}$ and $x \in \omega \cap X$, define the functional $\Pi_{\omega,x}$ on S by

$$\Pi_{\omega,x}[(u, v)] = \left| \int u \, d\rho_x^\omega - \int v \, d\rho_x^\omega \right|.$$

Then $\Pi_{\omega,x}$ is a well-defined seminorm, and the countable family of all such seminorms defines a metrizable, locally convex, topological vector space structure on S. We call this topology the **Cartan-Brelot topology**. In the following result we summarize the main properties of the Cartan-Brelot topology which we require.

Proposition 3.1. (a) *The Cartan-Brelot topology is Hausdorff and $S^+(U)$ is closed.*

(b) *The mapping $f : S^+(U) \times U \to \mathbb{R} \cup \{\infty\}$ defined by $f(v, x) = v(x)$ is lower semicontinuous.*

(c) *Every uniformly locally bounded sequence in $S^+(U)$ has a subsequence converging in the Cartan-Brelot topology.*

Let $\Omega = U \times V$, where U and V are domains in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. An extended real-valued function v on Ω is said to be **2-superharmonic** on Ω if the following four properties hold:

(i) v is not identically ∞;

(ii) $v(x) > -\infty$ for all $x \in \Omega$;
(iii) v is lower semicontinuous;
(iv) for each fixed $x_1 \in U$ and $x_2 \in V$, $v(x_1, \cdot)$ is hyperharmonic on V and $v(\cdot, x_2)$ is hyperharmonic on U (i.e. either superharmonic or identically ∞).

We call v **2-subharmonic** if $-v$ is 2-superharmonic. The set of all (respectively, nonnegative) 2-superharmonic functions on U is denoted by 2-$S(U)$ (respectively, 2-$S^+(U)$). Such functions satisfy the following properties.

Proposition 3.2. (a) *If $v_1, v_2 \in$ 2-$S(\Omega)$ and $\alpha_1, \alpha_2 > 0$, then $\alpha_1 v_1 + \alpha_2 v_2$ and $\min(v_1, v_2)$ are in 2-$S(\Omega)$.*
(b) *(Minimum Principle) Let $v \in$ 2-$S(\Omega)$. If $\liminf_{z \to x} v(z) \geq 0$ for all $x \in \partial\Omega$, then $v \geq 0$ on Ω.*

Our main result of this section is the following theorem, which is an improvement of the above minimum principle.

Theorem 3.1. *Let ω_1 and ω_2 be relatively compact open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let $v \in$ 2-$S(\omega_1 \times \omega_2)$ and bounded below. If*

$$\liminf_{(z,z') \to (x,y)} v(z, z') \geq 0, \tag{4}$$

for all $(x, y) \in \partial\omega_1 \times \partial\omega_2$, then $v \geq 0$ on $\omega_1 \times \omega_2$.

Proof. By the Minimum Principle, it suffices to show that (4) holds for all $(x, y) \in (\partial\omega_1 \times \partial\omega_2) \cup (\partial\omega_1 \times \omega_2) \cup (\omega_1 \times \partial\omega_2)$. Due to the hypothesis, by symmetry, we only need to prove this for a fixed $(x_0, y_0) \in \omega_1 \times \partial\omega_2$.

Let us first make the assumption that v is also bounded above. Arguing by contradiction, suppose (4) fails at (x_0, y_0). Then there exists a positive real number ε and a sequence $\{(z_k, z'_k)\}_{k \in \mathbb{N}}$ in $\omega_1 \times \omega_2$ converging to (x_0, y_0) with

$$v(z_k, z'_k) < -\epsilon \text{ for all } k \in \mathbb{N}. \tag{5}$$

For each $k \in \mathbb{N}$, the mapping $v_k : z \mapsto v(z, z'_k)$ defined on ω_1 yields a positive uniformly bounded sequence in $S(\omega_1)$. Choose $M \in \mathbb{R}$ such that $v_k(z) + M \geq 0$ for all $k \in \mathbb{N}$ and $z \in \omega_1$. By assumption, $\{v_k + M\}$ is uniformly bounded above on ω_1. By parts (a) and (c) of Proposition 3.1, there is a subsequence $\{v_{k_j} + M\}_{j \in \mathbb{N}}$ converging in the Cartan-Brelot topology to a function $w_1 \in S^+(\omega_1)$. It follows that $\{v_{k_j}\}$ converges in the Cartan-Brelot topology to $w = w_1 - M$.

We claim that $w \geq 0$ on ω_1. Indeed, let x_1 be any point in $\partial\omega_1$ and γ a positive real number. Then, from (4) applied to $(x_1, y_0) \in \partial\omega_1 \times \partial\omega_2$, we deduce there exist relatively compact neighborhoods U of x_1 and V of y_0 such that

$$v(z, z') \geq -\gamma \text{ for all } (z, z') \in (U \times V) \cap (\omega_1 \times \omega_2).$$

Without loss of generality, we may assume $z'_{k_j} \in V$ for all $j \in \mathbb{N}$. Thus

$$v(z, z'_{k_j}) \geq -\gamma \text{ for every } z \in U \cap \omega_1 \text{ and every } j \in \mathbb{N}.$$

Now let $x_2 \in U \cap \omega_1$ and let $\{\delta_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of balls of $\mathcal{O}$ such that for each ℓ, $\overline{\delta}_{\ell+1} \subset \delta_\ell \subset \overline{\delta}_\ell \subset U \cap \omega_1$ and $\bigcap_{\ell \in \mathbb{N}} \delta_\ell = \{x_2\}$. Then for every $\ell \in \mathbb{N}$, we have

$$\begin{aligned} \int w(z)\, d\rho_{x_2}^{\delta_\ell}(z) &= \lim_{j \to \infty} \int v_{k_j}(z)\, d\rho_{x_2}^{\delta_\ell}(z) \\ &= \lim_{j \to \infty} \int v(z, z'_{k_j})\, d\rho_{x_2}^{\delta_\ell}(z) \\ &\geq -\gamma \int d\rho_{x_2}^{\delta_\ell}(z). \end{aligned}$$

Letting $\ell \to \infty$ yields $w(x_2) \geq -\gamma$. Since this holds for all $x_2 \in U \cap \omega_1$, letting $\gamma \to 0$, we get $\liminf_{z \to x_1} w(z) \geq 0$. This holds for all $x_1 \in \partial\omega_1$, and so it follows from the Minimum Principle that w is indeed nonnegative on ω_1.

By Proposition 3.1(b), the mapping $f : S^+(\omega_1) \times \omega_1 \to \overline{\mathbb{R}}$ defined by $f(s,x) = s(x)$ is lower semicontinuous if $S^+(\omega_1)$ is equipped with the Cartan-Brelot topology. Thus

$$\liminf_{j\to\infty} v(z_{k_j}, z'_{k_j}) = \liminf_{j\to\infty} v_{k_j}(z_{k_j}) \geq w(x_0) \geq 0,$$

contradicting (5). Therefore the result holds in case when v is bounded above.

The general case follows from part (a) of Proposition 3.2 by applying the special case just proved to $w_k = \min\{v, k\}$ and letting k go to ∞. □

4. Operator norm of $M_\psi : H^\infty(D) \to \mathcal{B}(D)$

In [6] the following result was shown.

Theorem 4.1. (Theorem 3.1 of [6]) *Let ψ be holomorphic on D. Then $M_\psi : H^\infty(D) \to \mathcal{B}(D)$ is bounded if and only if $\psi \in H^\infty(D)$. If M_ψ is bounded, then*

$$\max\{\|\psi\|_{\mathcal{B}}, c_D\|\psi\|_\infty\} \leq \|M_\psi\| \leq \|\psi\|_{\mathcal{B}} + c_D\|\psi\|_\infty.$$

We improve the above estimates and determine the norm under some restrictions on the symbol or the domain.

In [12] and [29] it was shown that if D a Cartan domain, then there exists a holomorphic function f mapping D into $\mathbb{D}$ such that $f(0) = 0$ and $Q_f(0) = c_D$. On the other hand, if $D = D_1 \times \cdots \times D_k$, with D_j irreducible for all $j \in \{1,\ldots,k\}$, then $c_D = c_{D_m}$ for some $m = 1,\ldots,k$. Thus, the function f_m on D defined by $f_m(z_1,\ldots,z_k) = f(z_m)$ (where $z_j \in D_j$, $j = 1,\ldots,k$), satisfies the properties $f_m(0) = 0$, $f_m(D) \subset \mathbb{D}$, and $Q_{f_m}(0) = c_D$. By Remark 2.1, it follows that $\|f_m\|_\infty = 1$. Therefore, the set

$$\mathcal{F} = \{f \in H^\infty(D) : f(0) = 0, \|f\|_\infty = 1, Q_f(0) = c_D\}$$

is nonempty.

For $a \in D$, define

$$M(a) = \sup\{|f(a)| : f \in \mathcal{F}\}.$$

Theorem 4.2. *Let ψ be a bounded holomorphic function on a bounded symmetric domain D. Then*

$$\sup_{a\in D} (|\psi(0)|M(a) + c_D|\psi(a)|) \leq \|M_\psi\| \leq |\psi(0)| + c_D\|\psi\|_\infty.$$

In particular, if $\psi(0) = 0$, then

$$\|M_\psi\| = c_D\|\psi\|_\infty.$$

Proof. To prove the upper estimate, let $f \in H^\infty(D)$ with $\|f\|_\infty = 1$. Then $\psi f \in H^\infty(D)$ and $\|\psi f\|_\infty \leq \|\psi\|_\infty$. Since by definition of c_D, $Q_g(z) \leq c_D\|g\|_\infty$ for each $g \in H^\infty(D)$ and each $z \in D$, we obtain

$$\|\psi f\|_{\mathcal{B}} \leq |\psi(0)||f(0)| + c_D\|\psi f\|_\infty.$$

Taking the supremum over all such functions f, we obtain

$$\|M_\psi\| \leq |\psi(0)| + c_D\|\psi\|_\infty.$$

To prove the lower estimate, fix $a \in D$ and let S_a be an involutory automorphism of D mapping 0 to a, which exists by the results in [15], pp. 170, 301, and 311. Let $f \in \mathcal{F}$. By the invariance of the Bergman metric under biholomorphic maps and recalling that $(f\circ S_a)(a)=f(0)=0$, we have

$$\begin{aligned} Q_{\psi(f\circ S_a)} &\geq Q_{\psi(f\circ S_a)}(a)\\ &= \sup_{u\neq 0}\frac{|(f\circ S_a)(a)\nabla\psi(a)u+\psi(a)(\nabla(f\circ S_a))(a)u|}{H_a(u,\overline{u})^{1/2}}\\ &= |\psi(a)|\sup_{u\neq 0}\frac{|(\nabla(f\circ S_a))(a)u|}{H_a(u,\overline{u})^{1/2}}\\ &= |\psi(a)|\sup_{u\neq 0}\frac{|(\nabla f)(0)JS_a(a)u|}{H_0(JS_a(a)u,\overline{JS_a(a)u})^{1/2}}\\ &= |\psi(a)|\sup_{v\neq 0}\frac{|(\nabla f)(0)v|}{H_0(v,\overline{v})^{1/2}}\\ &= |\psi(a)|Q_f(0)=c_D|\psi(a)|.\end{aligned}$$

Therefore, $\|\psi(f\circ S_a)\|_{\mathcal{B}}\geq|\psi(0)||f(a)|+c_D|\psi(a)|$. Taking the supremum over all $f\in\mathcal{F}$, we get

$$\|M_\psi\|\geq|\psi(0)|M(a)+c_D|\psi(a)|.$$

Finally, taking the supremum over all $a\in D$, we obtain the lower estimate. □

Our next objective is to obtain a formula for $\|M_\psi\|$ when ψ does not fix the origin. We will be able to accomplish this under some restrictions on the domain D. We shall need the following two results. Lemma 4.1 makes use of Theorem 3.1.

Lemma 4.1. *Let u be a function defined on the Cartesian product of two bounded domains D_1 and D_2 in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Assume that u is bounded above and 2-subharmonic. Let $M=\sup_{z\in D_1\times D_2}u(z)$. Then*

$$M=\max_{\lambda\in\partial D_1\times\partial D_2}\limsup_{z\to\lambda}u(z). \tag{6}$$

Proof. Define $f(\lambda)=\limsup_{z\to\lambda}u(z)$ for $\lambda\in\partial D_1\times\partial D_2$. Then f is upper semicontinuous on a compact set, so it achieves its maximum value. Thus, the right side of (6) exists. Denote it by M'. By Theorem 3.1 applied to $M'-u$, we deduce that $u(z)\leq M'$ for all $z\in D_1\times D_2$. Thus $M\leq M'$. As the reverse inequality is obvious, the proof is complete. □

Lemma 4.2. (a) *If $D=\mathbb{B}_N$, then $M(a)\geq\|a\|$ for all $a\in D$.*

(b) *If $D=D_1\times\cdots\times D_k$, where $D_1,\cdots,D_k$ are irreducible and $D_m=\mathbb{D}$ for some $m\in\{1,\dots,k\}$, then $M(a)\geq|a_m|$ for all $a\in D$, $a=(a_1,\dots,a_k)$, with $a_j\in D_j$, $j=1,\dots,k$.*

Proof. To prove (a), fix $\lambda=(\lambda_1,\dots,\lambda_N)\in\partial\mathbb{B}_N$ and for $z\in\mathbb{B}_N$, define $p_\lambda(z)=\sum_{j=1}^N\lambda_jz_j$. By the Cauchy-Schwarz inequality, we see that $|p_\lambda(z)|\leq\|z\|<1$, so p_λ is a polynomial mapping $\mathbb{B}_N$ into $\mathbb{D}$. Moreover, $p_\lambda(0)=0$ and $(\nabla p_\lambda)(0)=\lambda$, so that by (2),

$$\frac{|(\nabla p_\lambda)(0)\overline{\lambda}|}{H_0(\overline{\lambda},\lambda)^{1/2}}=\sqrt{\frac{2}{N+1}}=c_{\mathbb{B}_N}.$$

Hence $Q_{p_\lambda}(0)=c_{\mathbb{B}_N}$. By Remark 2.1, it follows that $\|p_\lambda\|_\infty=1$, so $p_\lambda\in\mathcal{F}$. Thus, $M(a)\geq|p_\lambda(a)|$ for each $a\in\mathbb{B}_N$. Taking the supremum over all $\lambda\in\partial\mathbb{B}_N$, we obtain $M(a)\geq\|a\|$.

To prove (b), let $p_m:D\to\mathbb{D}$ be the projection map $p_m(z)=z_m$. Then $p_m(0)=0$, $\|p_m\|_\infty=1$, and

$$\begin{aligned} Q_{p_m}(0) &= \sup_{u\neq 0}\frac{|(\nabla p_m)(0)u|}{H_0(u,\overline{u})^{1/2}}=\sup_{u\neq 0}\frac{|u_m|}{\left(\sum_{j=1}^k H_0^{D_j}(u_j,\overline{u_j})\right)^{1/2}}\\ &= \sup_{u_m\neq 0}\frac{|u_m|}{H_0^{\mathbb{D}}(u_m,\overline{u_m})^{1/2}}=1=c_D,\end{aligned}$$

where $H_0^{D_j}$ denotes the Bergman metric at 0 relative to the domain D_j.

Therefore, $p_m \in \mathcal{F}$, so $M(a) \geq |p_m(a)| = |a_m|$ for each $a \in D$. $\square$

Theorem 4.3. *If D is the unit ball in $\mathbb{C}^N$ or a bounded symmetric domain in $\mathbb{C}^N$ that has $\mathbb{D}$ as a factor, then*

$$\|M_\psi\| = |\psi(0)| + c_D\|\psi\|_\infty = \begin{cases} |\psi(0)| + \sqrt{\frac{2}{N+1}}\|\psi\|_\infty & \text{if } D = \mathbb{B}_N, \\ |\psi(0)| + \|\psi\|_\infty & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.2, it suffices to show that

$$\|M_\psi\| \geq |\psi(0)| + c_D\|\psi\|_\infty. \tag{7}$$

Suppose $D = \mathbb{B}_N$. Then by Lemma 4.2 and Theorem 4.2, we have

$$\sup_{a\in\mathbb{B}_N} \left(|\psi(0)|\|a\| + c_{\mathbb{B}_N}|\psi(a)|\right) \leq \|M_\psi\|. \tag{8}$$

If ψ is constant, (7) follows immediately from (8). If ψ is nonconstant, then by the Maximum Modulus Principle, if $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in D such that $\sup_{n\in\mathbb{N}} |\psi(z_n)| = \|\psi\|_\infty$, then $\|z_n\| \to 1$ as $n \to \infty$. Thus, (8) yields (7).

Let us now suppose D is a bounded symmetric domain that has $\mathbb{D}$ as a factor. As we observed in Section 2, $c_D = 1$. Without loss of generality, we may assume $D = \mathbb{D} \times D_2$, for some bounded symmetric domain D_2. By Lemma 4.2 and Theorem 4.2, we see that

$$\|M_\psi\| \geq \sup_{a\in D} \left(|\psi(0)||a_1| + |\psi(a)|\right). \tag{9}$$

On the other hand, applying Lemma 4.1 to the functions $a \mapsto |\psi(0)||a_1| + |\psi(a)|$ and $a \mapsto |\psi(a)|$ gives that the right-hand side of (9) equals $|\psi(0)| + \|\psi\|_\infty$, verifying (7) in this case. This completes the proof. $\square$

Remark 4.1. Using (3) and (2) as well as the proof of Theorem 4.3, it is straightforward to see that $\|M_\psi\| = |\psi(0)| + c_D\|\psi\|_\infty$ also when D has the unit ball $\mathbb{B}_n$ as a factor provided that $\mathbb{B}_n$ and the other irreducible factors D_j of D satisfy the following dimensional restrictions

- $D_j \in R_I$ with $D_j \in \mathcal{M}_{m_j,n_j}$ and $m_j + n_j \geq n + 1$;
- $D_j \in R_{II}$ with $D_j \in \mathcal{M}_{n_j}$ and $n_j \geq n$;
- $D_j \in R_{III}$ with $D_j \in \mathcal{M}_{n_j}$ and $n_j \geq n + 2$;
- $D_j \in D_{IV}$ with $D_j \in \mathbb{C}^{n_j}$ and $n_j \geq n + 1$;
- $D_j = R_V$ with $n \leq 11$;
- $D_j = R_{VI}$ with $n \leq 17$.

Thus, we conclude the section by posing the following

Conjecture. Let D be a bounded symmetric domain. If $\psi \in H^\infty(D)$, then

$$\|M_\psi\| = |\psi(0)| + c_D\|\psi\|_\infty.$$

5. Isometries

In [6] it was shown that there exist no isometries among the multiplication operators from $H^\infty(\mathbb{D})$ to $\mathcal{B}(\mathbb{D})$. We next apply Theorem 4.2, to extend this result to a large class of domains in $\mathbb{C}^N$.

Theorem 5.1. (a) *If D is a bounded symmetric domain in standard form, then there are no isometric multiplication operators from $H^\infty(D)$ to $\mathcal{B}(D)$ whose symbol fixes the origin.*

(b) *If D is a bounded symmetric domain in standard form without exceptional factors, then there exist no isometric multiplication operators from $H^\infty(D)$ to $\mathcal{B}(D)$.*

Proof. (a) Suppose $D = D_1 \times \cdots \times D_k$, with D_j irreducible for all $j = 1, \ldots, k$, and $M_\psi : H^\infty(D) \to \mathcal{B}(D)$ is an isometry with $\psi \in H^\infty(D)$ such that $\psi(0) = 0$. We shall obtain a contradiction. Since $\|M_\psi\| = 1$, then by Theorem 4.2, we have

$$c_D\|\psi\|_\infty = 1. \tag{10}$$

Next, observe that $\|\psi^2\|_{\mathcal{B}} = \|M_\psi \psi\|_{\mathcal{B}} = \|\psi\|_\infty$, so that by (10),

$$1 = \left\| \frac{\psi^2}{\|\psi\|_\infty} \right\|_{\mathcal{B}} = Q_{\psi^2/\|\psi\|_\infty} \leq c_D\|\psi\|_\infty = 1.$$

Hence $Q_{\psi^2/\|\psi\|^2_\infty} = 1/\|\psi\|_\infty = c_D$. By Corollary 2.1 applied to $f = \frac{\psi^2}{\|\psi\|^2_\infty}$, we see that there exist a sequence $\{T_n\}$ of automorphisms of D, $m \in \{1, \ldots, k\}$, $x_m \in \partial D_m$ and S an automorphism of D_m such that $c_D = c_{D_m}$ and

$$\lim_{n\to\infty} \psi^2(T_n(z_1, \ldots, z_{m-1}, S(zx_m), z_{m+1}, \ldots, z_k) = z\|\psi\|^2_\infty \tag{11}$$

for each $z \in \mathbb{D}$, $z_j \in D_j, j \neq m$. Since the set $\{(\psi \circ T_n)/\|\psi\|_\infty : n \geq 0\}$ is a normal family, some subsequence $\{(\psi \circ T_{n_\iota})/\|\psi\|_\infty\}_{\iota\in\mathbb{N}}$ converges locally uniformly to a holomorphic function h. Fixing $z_j \in D_j$, for each $j \neq m$, it follows from (11) that

$$h(z_1, \ldots, z_{m-1}, S(zx_m), z_{m+1}, \ldots, z_k)^2 = z, \tag{12}$$

for all $z \in \mathbb{D}$. It follows that $h(z_1, \ldots, z_{m-1}, S(0), z_{m+1}, \ldots, z_k)^2 = 0$. Differentiating (12) with respect to z and substituting $z = 0$ then gives $0 = 1$. This contradiction shows that M_ψ cannot be an isometry.

(b) Suppose $D = D_1 \times \cdots \times D_k$, with D_j irreducible but not an exceptional domain for all $j = 1, \ldots, k$. If at least one of the domains $D_1, \ldots, D_k$, say D_j, is of type R_I, R_{II} or R_{III}, then letting p_ℓ be the projection map of D given by $p_\ell(z) = z_\ell$, where $\sum_{\iota=1}^{j-1} \dim(D_\iota) \leq \ell \leq \sum_{\iota=1}^{j} \dim(D_\iota)$, we obtain

$$Q_{\psi p_\ell} = \|\psi p_\ell\|_{\mathcal{B}} = \|p_\ell\|_\infty = 1.$$

Yet, since $\|\psi p_\ell\|_\infty \leq \|\psi\|_\infty$, by definition of c_D we have $Q_{\psi p_\ell} \leq c_D\|\psi\|_\infty$. Hence

$$1 \leq c_D\|\psi\|_\infty. \tag{13}$$

On the other hand, since $\|M_\psi\| = 1$, by Theorem 4.2 we obtain

$$c_D\|\psi\|_\infty \leq \sup_{a\in D} (|\psi(0)|M(a) + c_D|\psi(a)|) \leq 1. \tag{14}$$

Therefore, from (13) and (14) we obtain $c_D\|\psi\|_\infty = 1$ and

$$\sup_{a\in D} (|\psi(0)|M(a) + c_D|\psi(a)|) = 1.$$

Let δ be a small positive number such that the ball $\mathbb{B}_\delta$ centered at 0 of radius δ is properly contained in D and let $D_\delta = D\backslash\mathbb{B}_\delta$. The Maximum Modulus Principle implies that $\sup_{a\in D_\delta}|\psi(a)| = \|\psi\|_\infty$. By Proposition 4.1 of [2], $Q_{p_\ell}(0) = c_D$ so that $|a_\ell| \leq M(a)$. Thus

$$|\psi(0)|\delta + 1 = |\psi(0)|\delta + c_D\|\psi\|_\infty \leq \sup_{a\in D_\delta}|\psi(0)||a_\ell| + c_D|\psi(a)| \leq 1.$$

Consequently, $\psi(0) = 0$. By part (a), we obtain a contradiction. Therefore M_ψ cannot be an isometry.

If all factors of D are of type IV, then for r and s distinct in $\{1,\dots,N\}$, let $p^+_{r,s}$ and $p^-_{r,s}$ be the functions from D to $\mathbb{D}$ defined by $p^+_{r,s}(z) = z_r + iz_s$ and $p^-_{r,s}(z) = z_r - iz_s$. Again by Proposition 4.1 of [2], $Q_{p^+_{r,s}}(0) = Q_{p^-_{r,s}}(0) = c_D$ so that $|a_r \pm a_s| \leq M(a)$. Proceeding as above for the case of the projections p_ℓ, we obtain $\psi(0) = 0$, and hence M_ψ cannot be an isometry. □

We end the paper by posing the following conjecture.

Conjecture. There exist no isometries among the bounded multiplication operators from the Hardy space $H^\infty(D)$ to the Bloch space of any bounded symmetric domain D.

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