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Lower Derivatives of Functions of Finite Variation and Generalized BCH Sets

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1. INTRODUCTION

Let n be a positive integer. Let $\mathbb{R}^{n+1}_+ = \{(x, t) \in \mathbb{R}^{n+1} : t > 0\}$ denote the upper half-plane in $\mathbb{R}^{n+1}$. For $E \subset \mathbb{R}^n$, let E^c denote its complement $\mathbb{R}^n \setminus E$. If E is measurable, let $|E|$ denote its Lebesgue measure. We will let d denote a continuous, translation invariant pseudo-distance on $\mathbb{R}^n$ for which there is an $r > 0$ such that the d-balls satisfy $|B(0, t)| \approx t^r$ (see Sect. 2). By *modulus of continuity* we mean a real-valued, monotone nondecreasing, subadditive function ω on $\mathbb{R}$ such that $\omega(0) = 0$. When convenient we will assume ω is constant outside the interval $[0, 1]$. For $\alpha \geqslant 1$, we say that ω is *α-allowed* if $\omega'(0) = \infty$ and $d^{-\alpha r}(x, 0)\, \omega(d^{\alpha r}(x, 0))$ is locally integrable.

Let $\alpha \geqslant 1$, $c > \varepsilon > 0$, and u a real-valued function on $\mathbb{R}^{n+1}_+$. Let ω be a modulus of continuity. Define

$$\mathbf{D}_{\omega, \alpha} u(x) = \liminf_{t \to 0} \frac{|u(x + y, t)|}{\omega(|B(0, t)|)}, \tag{1.1}$$

where the lim inf is taken over those (y, t) satisfying

$$(c - \varepsilon)\, t < d^{\alpha}(y, 0) < (c + \varepsilon)\, t. \tag{1.2}$$

In the sequel, c and ε shall be fixed once and for all, so we suppress them in this notation.

In this paper we characterize the set of points of $\mathbb{R}^n$ where the generalized lower derivative defined by (1.1) and (1.2) is positive in case u is of finite *γ-variation*, $\gamma \geqslant 1$. This class of functions, which we denote by γBV, was first defined in [4]. See Section 2 for the definition. To obtain this characterization, we define a set $E \subset \mathbb{R}^n$ to be an *(ω, α)-set* (the

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generalized BCH set of the title) if E is a closed set of measure 0 such that, for each bounded set B containing E,

$$\int_B \frac{\omega(d^{\alpha r}(x, E))}{d^{\alpha r}(x, E)}\, dx < \infty. \tag{1.3}$$

Our principal result is

THEOREM 1. *Let* $\alpha, \gamma \geqslant 1$, $c > \varepsilon > 0$, *and* $u \in \gamma BV$, $u \geqslant 0$. *Let* ω *be a modulus of continuity such that* ω^γ *is an* α*-allowed modulus of continuity. If* $\alpha > 1$, *then*

$$\{x : \mathbf{D}_{\omega, \alpha} u(x) > 0\} \tag{1.4}$$

is a countable union of (ω^γ, α)*-sets. Conversely, given a set* $E = \bigcup E_j$ *which is a countable union of* (ω^γ, α)*-sets contained in a fixed bounded open set* B, *there exists a function* $u(x, t) \in \gamma BV$ *such that, for every* $x \in E$,

$$\mathbf{D}_{\omega, \alpha} u(x) = \infty.$$

Examples of functions of finite γ-variation are $u(x, t) = [\mu(B(x, t))]^{1/\gamma}$, where μ is a finite measure on $\mathbb{R}^n$. Thus, in this case, the theorem characterizes the set of points x such that μ has eventual mass concentration at least of the order of $\omega^\gamma[|B(x+y, t)|]$ along the family of balls approaching x as prescribed by (1.2).

In [2] related results are considered. In that paper, n and α are taken to be 1, d is the usual Euclidean distance on the real line, and measures on the unit interval $[0, 1] \in \mathbb{R}^2$ are considered (that is, $\gamma = 1$ and u as described in the previous paragraph). Upper and lower derivatives are defined with respect to a modulus of continuity with the difference that the balls are taken to be centered at the point rather than moving toward the point as defined by (1.2). This allowed Berman to deduce results about the perpendicular boundary behavior of positive harmonic functions. By approaching the point as in (1.2), Theorem 1 allows us to deduce results concerning the tangential boundary behavior of positive harmonic functions (as well as many other classes of functions given by Poisson type integrals of measures due to the general nature of the pseudo-distance d). The characterizing set that arose in [2] was a generalized BCH set considered first in [3] and defined as a closed subset E of $[0, 1]$ such that $|E| = 0$ and

$$\sum \omega(|I_j|) < \infty, \tag{1.5}$$

where $\{I_j\}$ are the arcs complementary to E in $[0, 1]$. In Proposition 2 we will characterize an (ω, α)-set in terms of a Whitney decomposition of its complement, thus relating it more closely to the condition of (1.5). As this seems to be the natural generalization of these sets to $\mathbb{R}^n$, we refer to them as generalized BCH sets. We remark that the lim inf results of [2] described above differ somewhat from ours in that the condition on ω of the local integrability of $d(x,0)^{-\alpha r}\,\omega(d(x,0)^{-\alpha r})$ is not needed. We need this condition in order that there be any nonempty (ω, α)-sets at all and, hence, that there exist nonempty sets defined by (1.4). The difference arises from the fact that the balls in the lower derivative defined in this paper are not centered at the point of interest.

The paper is organized as follows. In Section 2 we outline background material we will need. In Section 3 we prove the result relating the (ω, α)-*sets* to Whitney decompositions. In Section 4 we prove Theorem 1. In Section 5 we show the relationship between generalized BCH sets and Hausdorff measure.

2. Preliminaries

Let d denote a translation invariant pseudo-distance on $\mathbb{R}^n$ [6]. This means $d\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ and there exists $K \geqslant 1$ such that for all $x, y, z \in \mathbb{R}^n$, (1) $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) = d(y, x)$, (3) $d(x, z) \leqslant K[d(x, y) + d(y, z)]$, (4) $d(x+z, y+z) = d(x, y)$, (5) $\{B(x, r)\colon r > 0\}$ forms a base of open sets for the open neighborhoods of x in the Euclidean topology, where $B(x, r) = \{y \in \mathbb{R}^n\colon d(x, y) < r\}$, and (6) for each $\alpha > 0$ there exists $\tau(\alpha) < \infty$ such that $|B(0, \alpha t)| \leqslant \tau(\alpha)|B(0, t)|$ for all $t > 0$.

As a consequence, it is shown in [6] that there exists an integer N such that for each $x \in \mathbb{R}^n$, $t > 0$, and $m \in \mathbb{Z}^+$ the d-ball $B(x, t)$ can contain at most N^m points $\{x_i\}$ such that $d(x_i, x_j) > 2^{-m}t$. The quantities N and K are referred to as the constants of the pseudo-distance d. The following result will be used several times.

Covering Theorem [6, Théorème 1.2, p. 69]. *Let E be a bounded subset of $\mathbb{R}^n$ and let $\{B(x, r(x))\colon x \in E\}$ be a covering of E by d-balls. Then there exists a disjoint sequence $\{B(x_i, r(x_i))\}$ from this covering such that $\{B(x_i, kr(x_i))\}$ covers E, where k depends only on the constants of d.*

In this paper we will assume the two additional conditions that d is continuous and there exist c_1, c_2, and r such that

$$c_1 t^r \leqslant |B(0, t)| \leqslant c_2 t^r \qquad \text{for all} \quad t > 0. \tag{2.1}$$

This holds, for example, if $d(x, y)$ is of the form $\sum |x_i - y_i|^{\alpha_i}$.

Let $\gamma \geqslant 1$. We recall a notion of generalized variation introduced in [4] which generalizes a definition of Wiener [14] on the real line. Let u be a nonnegative, real-valued function on the upper half-plane in $\mathbb{R}^{n+1}$. The *γ-variation* $\|u\|_\gamma$ is defined as

$$\sup\left(\sum u(x_j, t_j)^\gamma\right)^{1/\gamma},$$

where the supremum is taken over all sequences $\{x_j\}$ in $\mathbb{R}^n$ such that $\{B(x_j, t_j)\}$ is a mutually disjoint family of d-balls. We denote the set of functions of finite γ-variation by γBV. Wiener considered this in the special case of $u(x, t) = \operatorname{diam}(v[B(x, t)])$ where $v: \mathbb{R} \to \mathbb{R}$.

We use the following results concerning a modulus of continuity ω. For this and other facts see [3]. We have

$$\frac{A}{A+1}\,\omega(t) \leqslant \omega(At) \leqslant (A+1)\,\omega(t), \qquad A, t \geqslant 0, \tag{2.2}$$

and

$$s^{-1}\omega(s) < 2t^{-1}\omega(t), \qquad 0 < t < s. \tag{2.3}$$

If ω is in addition concave down, then $t^{-1}\omega(t)$ is nonincreasing and $t\omega'(t) \leqslant \omega(t)$ if $\omega'(t)$ exists. In the sequel we will reserve the letters k, K, N, c_1, c_2, r for the constants defined in this section.

3. (ω, α)-Sets

In case $n = 1$, d is the usual Euclidean distance on $\mathbb{R}$, and ω is smooth and concave down, we show that an *(ω, α)-set* E contained in $[0, 1]$ is an ω-set in the sense of [3]. Indeed, let $\{I_j\}$ be the arcs complementary to E in $[0, 1]$. Let $|I_j|/2 = a_j$. Since $\omega'(t) \leqslant t^{-1}\omega(t)$,

$$\infty > \sum \int_{I_j} \alpha \frac{\omega(d(x, E)^\alpha)}{d(x, E)^\alpha}\,dx \geqslant \sum \int_0^{a_j} \alpha \frac{\omega(t^\alpha)}{t^\alpha}\,dt \geqslant \sum \int_0^{a_j} \alpha\omega'(t^\alpha)\,dt$$

$$= \sum \int_0^{a_j^\alpha} \omega'(t)\, t^{(1/\alpha)-1}\,dt \geqslant \sum \omega(a_j^\alpha)\, a_j^{1-\alpha} \geqslant \sum \omega(a_j) \geqslant \sum 3^{-1}\omega(|I_j|).$$

This proves that E is an ω-set.

We now give a characterization of the *(ω, α)-sets* which relates them more closely to the ω-sets of [3]. For this purpose, we define the *α-Whitney decomposition* of an open set. First, we need the following proposition.

PROPOSITION 1. *Let $\alpha \geqslant 1$. Then for every open subset O of $\mathbb{R}^n$ having d-diameter at most 1, there exists a sequence of d-balls $\{B(x_i, r_i)\}$ such that*

(1) $O = \bigcup B(x_i, r_i)$, (2) $\{B(x_i, r_i/k)\}$ *is a disjoint sequence*, (3) *for each* $x \in B(x_i, r_i)$,

$$k^{\alpha-1} r_i \leqslant d^\alpha(x, O^c) \leqslant K^\alpha k^{\alpha-1}(1+2K)^\alpha r_i,$$

and (4) *any point of* O *can belong to at most* M *of the balls* $\{B(x_i, r_i)\}$, *where* M *depends only on* α *and the constants of the pseudo-distance.*

Proof. The proof is similar to that of Théorème 1.3 on p. 70 of [6]. We apply the covering theorem with $r(x) = (2kK)^{-\alpha} d^\alpha(x, O^c)$. This gives us a disjoint sequence $\{B(x_i, \rho_i)\}$ of d-balls such that O is covered by $\{B(x_i, k\rho_i)\}$, where ρ_i denotes $r(x_i)$. Let $r_i = k\rho_i$. Clearly (1) and (2) hold. Let $x \in B(x_i, r_i)$. Then

$$\begin{aligned} 2kK\rho_i^{1/\alpha} = d(x_i, O^c) &\leqslant K[d(x_i, x) + d(x, O^c)] \leqslant K[k\rho_i + d(x, O^c)] \\ &\leqslant K[k\rho_i^{1/\alpha} + d(x, O^c)], \end{aligned}$$

so

$$d^\alpha(x, O^c) \geqslant k^\alpha \rho_i = k^{\alpha-1} r_i. \tag{3.1}$$

Also

$$\begin{aligned} d(x, O^c) &\leqslant K[d(x, x_i) + d(x_i, O^c)] \\ &\leqslant K[k\rho_i + 2kK\rho_i^{1/\alpha}] \leqslant Kk\rho_i^{1/\alpha}(1+2K), \end{aligned}$$

which says

$$d^\alpha(x, O^c) \leqslant K^\alpha k^\alpha (1+2K)^\alpha \rho_i = K^\alpha k^{\alpha-1}(1+2K)^\alpha r_i. \tag{3.2}$$

Property (3) follows from (3.1) and (3.2).

To establish (4), let $x \in \bigcap \{B(x_i, r_i) : i \in I\}$, where I is a subset of the positive integers whose size we now estimate. For $i \in I$,

$$d(x, x_i) < r_i < k^{1-\alpha} d^\alpha(x, O^c),$$

so $\{x_i : i \in I\} \subset B(x, k^{1-\alpha} d^\alpha(x, O^c))$ and for $i, j \in I$,

$$d(x_i, x_j) \geqslant \rho_i = \frac{r_i}{k} \geqslant (Kk(1+2K))^{-\alpha} d^\alpha(x, O^c).$$

We deduce that I can have at most

$$N^{(1 + \log_2 K^\alpha k(1+2K)^\alpha)}$$

elements. ∎

DEFINITION 1. Let O be an open subset of $\mathbb{R}^n$ having d-diameter at most 1. We call a sequence of d-balls $\{B(x_i, r_i)\}$ an *α-Whitney decomposition* of O if (1) $O = \bigcup B(x_i, r_i)$, (2) $\{B(x_i, r_i/k)\}$ is a disjoint sequence, (3) there exist $0 < a_1, a_2 < \infty$ such that for each $x \in B(x_i, r_i)$,

$$a_1 r_i \leqslant d^\alpha(x, O^c) \leqslant a_2 r_i,$$

and (4) any point of O can belong to at most M of the balls $\{B(x_i, r_i)\}$, where M depends only on α and the constants of the pseudo-distance.

PROPOSITION 2. *Let E be a closed set of measure 0 whose closure is contained in an open set U of d-diameter at most 1. Then E is an (ω, α)-set if and only if*

$$\sum \omega(|B(x_i, r_i)|) < \infty,$$

where $\{B(x_i, r_i)\}$ is the subcollection of balls of any α-Whitney decomposition of $U \backslash E$ whose d-distance to $(U \backslash E)^c$ is its d-distance to E.

Proof. Let $V = \bigcup B(x_i, r_i)$. Then

$$\begin{aligned}
\int_V \frac{\omega(d^{\alpha r}(x, E))}{d^{\alpha r}(x, E)}\, dx &\leqslant \sum_i \int_{B(x_i, r_i)} \frac{\omega(d^{\alpha r}(x, E))}{d^{\alpha r}(x, E)}\, dx \\
&\leqslant \sum_i \frac{\omega(a_2^r r_i^r)}{a_1^r r_i^r}\, |B(x_i, r_i)| \\
&\leqslant \sum_i a_1^{-r} c_2 (1 + (a_2^r/c_1)\, \omega(|B(x_i, r_i)|, \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\int_V \frac{\omega(d^{\alpha r}(x, E))}{d^{\alpha r}(x, E)}\, dx &\geqslant \sum_i \int_{B(x_i, r_i/k)} \frac{\omega(d^{\alpha r}(x, E))}{d^{\alpha r}(x, E)}\, dx \\
&\geqslant \sum_i \frac{\omega(a_1^r r_i^r)}{a_2^r r_i^r}\, |B(x_i, r_i/k)| \\
&\geqslant c_1 k^{-r} a_2^{-r} \left(\frac{a_1^r c_2^{-1}}{1 + a_1^r c_2^{-1}}\right) \sum_i \omega(|B(x_i, r_i)|). \qquad (3.4)
\end{aligned}$$

The result follows from (3.3) and (3.4). ∎

4. Proof of Theorem 1

Let δ, ε', and $R>0$. Let F be the intersection of

$$\left\{x_0 \in \mathbb{R}^n : \frac{u(x_0+x, t)}{\omega(|B(0, t)|)} \geqslant R \text{ whenever } (c-\varepsilon)\, t < d^\alpha(x, 0) < (c+\varepsilon)\, t,\ t<\varepsilon'\right\}$$

with a d-ball of radius 2δ. Let U be a concentric d-ball of radius δ. To prove the first part of the theorem, it is enough to show that F is a closed set of measure 0 such that

$$\int_U \frac{\omega^\gamma(d^{\alpha r}(x, F))}{d^{\alpha r}(x, F)}\, dx < \infty. \tag{4.1}$$

We first show F is closed. Let $x_0 \in \mathbb{R}^n \setminus F$. Then there exists (x, t) such that

$$(c-\varepsilon)\, t < d^\alpha(x, 0) < (c+\varepsilon)\, t, \qquad t<\varepsilon', \qquad \frac{u(x_0+x, t)}{\omega(|B(0, t)|)} < R.$$

Consider

$$V = \{z \in \mathbb{R}^n : (c-\varepsilon)\, t < d^\alpha(x_0+x, z) < (c+\varepsilon)\, t\}.$$

Our continuity assumption on d implies this is an open neighborhood of x_0. Thus there exists $\eta>0$ such that $z \in V$ whenever $d(z, x_0)<\eta$. For any such z,

$$(c-\varepsilon)\, t < d^\alpha(x_0+x-z, 0) < (c+\varepsilon)\, t$$

and

$$u(z+(x_0+x-z), t) = u(x_0+x, t) < R\omega(|B(0, t)|),$$

so that $z \in \mathbb{R}^n \setminus F$. This shows that F is closed.

Let $\{t_j\} \searrow 0$. Since $(\omega^\gamma)'(0) = \infty$, we have $|B(0, t_j)|^{-1/\gamma}\, \omega(|B(0, t_j)|) \to \infty$. In the language of [11, 4], there is a sequence $\{x_j\} \to 0$ such that $\{(x_j, t_j)\}$ is 1-admissible and for each j,

$$(c-\varepsilon)\, t_j < d^\alpha(x_j, 0) < (c+\varepsilon)\, t_j.$$

This is shown on p. 709 of [11]. Thus

$$F \subset \left\{x_0 \in \mathbb{R}^n : \limsup_{j \to \infty} \frac{u(x_0+x_j, t_j)}{|B(0, t_j)|^{1/\gamma}} = \infty\right\},$$

and according to Theorem 1 in [4], the latter has vanishing Lebesgue measure. (Note that 1-admissible means the same in [11, 4] even though α-admissible does not for $\alpha > 1$.) Thus $|F| = 0$.

We show now that (4.1) holds. For each $x \in U \setminus F$, let $r(x) = c^{-1} d^{\alpha}(x, F)$. Since $\alpha > 1$, we may assume δ was chosen so small that $r(x)^{1/\alpha} > 2Kkc^{-1/\alpha} r(x)$ and $r(x) < \varepsilon'$. By the covering theorem, there exists a disjoint sequence $\{B(x_i, r_i)\}$ (where r_i denotes $r(x_i)$) such that $\{B(x_i, kr_i)\}$ covers $U \setminus F$. Observe that $u(x_i, r_i) \geqslant Rw(|B(0, r_i)|)$. For each $x \in U \setminus F$ there exists i such that $x \in B(x_i, kr_i)$. Thus

$$\begin{aligned}(r_i c)^{1/\alpha} = d(x_i, F) &\leqslant K[d(x_i, x) + d(x, F)]\\ &\leqslant K[kr_i + d(x, F)]\\ &\leqslant K[(2K)^{-1}(r_i c)^{1/\alpha} + d(x, F)],\end{aligned}$$

so that $d(x, F) \geqslant c^{1/\alpha}(2K)^{-1} r_i^{1/\alpha}$. Applying 2.2 and 2.3 to ω^{γ},

$$\begin{aligned}\int_U \frac{\omega^{\gamma}(d^{\alpha r}(x, F))}{d^{\alpha r}(x, F)}\,dx &\leqslant \sum_i \int_{B(x_i, kr_i) \cap (U \setminus F)} \frac{\omega^{\gamma}(d^{\alpha r}(x, F))}{d^{\alpha r}(x, F)}\,dx\\ &\leqslant \sum_i \frac{2\omega^{\gamma}(c^r (2K)^{-\alpha r} r_i^r)}{c^r (2K)^{-\alpha r} r_i^r}\,|B(0, kr_i)|\\ &\leqslant \sum_i \frac{2\omega^{\gamma}(c^r (2K)^{-\alpha r} c_1^{-1} |B(0, r_i)|)}{c^r (2K)^{-\alpha r} c_2^{-1} |B(0, r_i)|}\,|B(0, kr_i)|\\ &\leqslant A \sum_i \omega^{\gamma}(|B(0, r_i)|) \leqslant A \sum_i R^{-\gamma} u^{\gamma}(x_i, r_i)\\ &\leqslant AR^{-\gamma} \|u\|_{\gamma}^{\gamma} < \infty,\end{aligned}$$

where

$$A = 2\tau(k)\,\frac{((2K)^{-\alpha r} c_1^{-1} c^r + 1)}{(2K)^{-\alpha r} c_2^{-1} c^r}.$$

Inequality (2.3) was used to deduce the second in the string of inequalities above. This completes the proof of the first part of the theorem.

Let $E = \bigcup E_i$, where E_i is an $(\omega^{\gamma}, \alpha)$-*set* whose closure is contained in a fixed ball B. Since, for each i,

$$\int_B \frac{\omega^{\gamma}(d^{\alpha r}(z, E_i))}{d^{\alpha r}(z, E_i)}\,dz < \infty,$$

it follows that there exist sequences $\{e_j\} \searrow 0$ and $\{d_j\} \nearrow \infty$ such that for each i,

$$\sum_j \int_{E_{ij}} d_j \frac{\omega^\gamma(d^{\alpha r}(z, E_i))}{d^{\alpha r}(z, E_i)} dz < 2^{-i},$$

where

$$E_{ij} = \{x \in \mathbb{R}^n : d(x, E_i) < e_j\} \subset B.$$

For each $(x, t) \in \mathbb{R}^{n+1}_+$, define

$$u_i(x, t) = \left(\sum_j \int_{E_{ij} \cap B(x, t)} d_j \frac{\omega^\gamma(d^{\alpha r}(z, E_i))}{d^{\alpha r}(z, E_i)} dz\right)^{1/\gamma}$$

and

$$u(x, t) = \left(\sum_i u_i^\gamma(x, t)\right)^{1/\gamma}.$$

Clearly $u \in \gamma BV$. Let $x_0 \in E_i$ and let $(x, t) \in \mathbb{R}^n \times (0, 1)$ satisfy $(c - \varepsilon) t < d^\alpha(x, 0) < (c + \varepsilon) t$, with t so small that $B(x_0 + x, t) \subset E_{ij}$ for some j. Let $z \in B(x_0 + x, t)$. Since $t < 1$ we have

$$d(z, E_i) \leqslant d(z, x_0) \leqslant K[d(z, x_0 + x) + d(x, 0)] < At^{1/\alpha},$$

where $A = K[1 + (c + \varepsilon)^{1/\alpha}]$, so that

$$\begin{aligned} u^\gamma(x_0 + x, t) \geqslant u_i^\gamma(x_0 + x, t) &\geqslant \int_{B(x_0 + x, t)} d_j \frac{\omega^\gamma(d^{\alpha r}(z, E_i))}{d^{\alpha r}(z, E_i)} dz \\ &\geqslant \int_{B(x_0 + x, t)} (d_j/2) \frac{\omega^\gamma(A^{\alpha r} t^r)}{A^{\alpha r} t^r} dz \\ &\geqslant (d_j/2) \int_{B(x_0 + x, t)} \frac{\omega^\gamma(A^{\alpha r} c_2^{-1} \,|B(0, t)|)}{A^{\alpha r} c_1^{-1} \,|B(0, t)|} dz \\ &\geqslant (d_j/2) \frac{A^{\alpha r} c_2^{-1}}{1 + A^{\alpha r} c_2^{-1}} A^{-\alpha r} c_1 \omega^\gamma(|B(0, t)|). \end{aligned}$$

The third inequality follows by applying (2.3) to ω^γ. Dividing by $\omega^\gamma(|B(0, t)|)$ and letting $j \to \infty$, we get that the required lim inf is ∞. This completes the proof. ∎

5. Hausdorff Measure and (ω, α)-Sets

We first recall the definition of Hausdorff measure. Let E be any subset of $\mathbb{R}^n$ and let v be a continuous, nondecreasing function on $[0, \infty)$ such that $v(0)=0$. Then

$$\mathscr{H}_v(E)=\sup_{s>0} \inf \left\{\sum v(|B_i|): B_i \in \mathscr{C}_s\right\},$$

where $\mathscr{C}_s$ is the set of all coverings of E by d-balls of d-diameter at most s. In this section we deal with the problem of determining whether or not $\mathscr{H}_v(E)=0$ for every (ω, α)-set E. For related results on the real line with $\alpha=1$, see [1, 13, 3]. The definition of the following function is inspired by the *type function* defined on the real line in [1]. See also [3].

Definition 2. Let E be a compact subset of $\mathbb{R}^n$ having measure 0. Let U be an open set containing E. We define the *α-type function of E* to be

$$T_{E,\alpha}(t)=|\{x \in U \backslash E: d^{\alpha r}(x, E)=d^{\alpha r}(x, (U \backslash E)^c) \leqslant t\}|.$$

Lemma 1. *Let $\{B(x_i, r_i)\}$ be those d-balls in a given α-Whitney decomposition of $U \backslash E$ such that for each i, $d(x, E)=d(x, (U \backslash E)^c)$ for each $x \in B(x_i, r_i)$. Then for all t sufficiently small,*

$$T_{E,\alpha}(a_1^r t) \leqslant \sum \{|B(x_i, r_i)|: r_i^r \leqslant t\} \leqslant \tau(k)\, T_{E,\alpha}(a_2^r t).$$

Proof. The first inequality follows from the fact that

$$\{x \in U \backslash E: d^{\alpha r}(x, E)=d^{\alpha r}(x, (U \backslash E)^c) \leqslant a_1^r t\} \subset \bigcup \{B(x_i, r_i): r_i^r \leqslant t\}.$$

As for the second,

$$\begin{aligned}\sum \{|B(x_i, r_i)|: r_i^r \leqslant t\} &\leqslant \tau(k) \sum \{|B(x_i, (r_i/k))|: r_i^r \leqslant t\}\\ &=\tau(k)\left|\bigcup \{B(x_i, r_i/k): r_i^r \leqslant t\}\right|\\ &\leqslant \tau(k)\, T_{E,\alpha}(a_2^r t).\end{aligned}$$

This completes the proof. ∎

Lemma 2. *Let ω be a C^2-smooth, concave downward, α-allowed modulus of continuity. In order that E be an (ω, α)-set, it is necessary that*

$$\int_0^1 T_{E,\alpha}(t)\, \omega''(t)\, dt > -\infty.$$

Proof. Choose $\{B_i\} = \{B(x_i, r_i)\}$ as in Lemma 1. Then

$$\begin{aligned}
-\int_0^1 T_{E,\alpha}(t)\,\omega''(t)\,dt &\leqslant -\int_0^1 \sum \{|B_i| : a_1^r r_i^r \leqslant t\}\,\omega''(t)\,dt \\
&\leqslant -\sum \int_{a_1^r r_i^r}^1 |B_i|\,\omega''(t)\,dt = \sum |B_i|(\omega'(a_1^r r_i^r) - \omega'(1)) \\
&\leqslant \sum |B_i|\, a_1^{-r} r_i^{-r}\,\omega(a_1^r r_i^r) \\
&\leqslant c_2 a_1^{-r}(1 + a_1^r c_1^{-1}) \sum \omega(|B_i|) \\
&< \infty.
\end{aligned}$$

The last line follows by Proposition 2 and the fact that $t\omega'(t) \leqslant \omega(t)$. This completes the proof. ∎

THEOREM 2. *Let v, $\omega \not\equiv 0$ be moduli of continuity, where v is continuous and ω is C^2-smooth, concave downward, and α-allowed. Then $\mathscr{H}_v(K) = 0$ for every (ω, α)-set K if*

$$\int_0^1 \frac{t^{1/\alpha}}{v(t^{1/(\alpha r)})}\,\omega''(t)\,dt = -\infty.$$

Proof. Let K be a compact (ω, α)-set contained in a fixed open ball B. Let $\bar{v}$ be a Borel measure on K such that $\bar{v}[B(x, r)] \leqslant v(r)$ for every d-ball $B(x, r)$. Let $t > 0$. Then there is a finite set $\{x_i : 1 \leqslant i \leqslant N\}$ in K such that $\{B(x_i, t) : 1 \leqslant i \leqslant N\}$ is a disjoint family, and $\{B(x_i, kt) : 1 \leqslant i \leqslant N\}$ covers K. If t is sufficiently small,

$$Nc_1 t^r \leqslant \sum_{i=1}^N |B(x_i, t)| = \left|\bigcup_{i=1}^N B(x_i, t)\right| \leqslant T_{K,\alpha}(t^{\alpha r}).$$

Thus

$$\bar{v}(K) \leqslant \sum_{i=1}^N \bar{v}[B(x_i, kt)] \leqslant (1+k)\,Nv(t) \leqslant c_1^{-1}(1+k)\,v(t)\,t^{-r}\,T_{K,\alpha}(t^{\alpha r}),$$

and so for all t sufficiently small we have

$$c_1 \frac{t^{1/\alpha}}{v(t^{1/(\alpha r)})}\,\bar{v}(K) \leqslant (1+k)\,T_{K,\alpha}(t).$$

In view of this, the concavity of ω, and Lemma 2, we deduce that $\bar{v}(K) = 0$. It follows from Frostman's theorem (see [9, Théorème III, Chap. II, p. 27]) that $\mathscr{H}_v(K) = 0$. ∎

We do not know if the integral condition of the above theorem is necessary. We show this is true in the case where $n=1$, d is the usual Euclidean distance on $\mathbb{R}$, $t^{1/\alpha}\omega'(t)\to 0$ as $t\to 0$, and $t^{-1}\omega(t)\approx\omega'(t)$, that is, if there exist constants b_1, b_2 such that

$$b_1\,\omega'(t)\leqslant t^{-1}\omega(t)\leqslant b_2\,\omega'(t).$$

Note that in this case, a subset E of $[0,1]$ is an (ω,α)-*set* if and only if

$$\sum\int_0^{|I_j|^\alpha}\omega'(s)\,s^{(1/\alpha)-1}\,ds<\infty,$$

where $\{I_j\}$ are the arcs complementary to E in $[0,1]$. These conditions hold if, for example, $\omega(t)=t^\beta$, where $0<\beta<1$ and $\alpha(1-\beta)<1$ or $\omega(t)=-t\log(t)$. We remark, however, that there do exist moduli of continuity on $\mathbb{R}$ which are not comparable to $t\omega'(t)$, so this is not the general situation.

THEOREM 3. *Let $v,\omega\not\equiv 0$ be moduli of continuity, where v is continuous and ω is α-allowed, C^2-smooth, and concave downward. Under the conditions of the previous paragraph, $\mathscr{H}_v(E)=0$ for every (ω,α)-set E if and only if*

$$\int_0^1\frac{t^{1/\alpha}}{v(t^{1/\alpha})}\,\omega''(t)\,dt=-\infty.$$

LEMMA 3. *Let E be a subset of $[0,1]$. Then E is an (ω,α)-set if and only if*

$$\int_0^1 T_{E,\alpha}(t)\,\omega''(t)\,dt>-\infty.$$

Proof. The necessity of the condition was shown in Lemma 2. Suppose now the condition holds. With the same reasoning as in the proof of Lemma 4.1 of [3],

$$T_{E,\alpha}(t)\approx\sum_{J_1}|I_j|+\sum_{J_2}t^{1/\alpha},$$

where

$$J_1=\{j\in\mathbb{Z}^+:|I_j|^\alpha\leqslant t\}\qquad\text{and}\qquad J_2=\{j\in\mathbb{Z}^+:|I_j|^\alpha>t\}.$$

Thus

$$-\int_0^1\sum_{J_1}|I_j|\,\omega''(t)\,dt<\infty\qquad\text{and}\qquad-\int_0^1\sum_{J_2}t^{1/\alpha}\omega''(t)\,dt<\infty.$$

From the first we get

$$-\sum \int_{|I_j|^\alpha}^{1} |I_j|\, \omega''(t)\, dt \leqslant \sum |I_j|\, \omega'(|I_j|^\alpha) < \infty, \tag{5.1}$$

and from the second an integration by parts gives

$$\begin{aligned} \infty &> -\sum \int_0^{|I_j|^\alpha} t^{1/\alpha} \omega''(t)\, dt \\ &= -\sum |I_j|\, \omega'(|I_j|^\alpha) + \sum \int_0^{|I_j|^\alpha} (1/\alpha)\, \omega'(t)\, t^{(1/\alpha)-1}\, dt. \end{aligned}$$

From (5.1) we deduce

$$\sum \int_0^{|I_j|^\alpha} (1/\alpha)\, \omega'(t)\, t^{(1/\alpha)-1}\, dt < \infty.$$

By our remarks preceding the statement of Theorem 3, the proof is complete. ∎

Proof of Theorem 3. Suppose the integral in the statement of the theorem is finite. For $t \leqslant 1$ we have

$$t^{-1/\alpha} v(t^{1/\alpha}) \leqslant 2t^{-1} v(t),$$

so by the concavity of ω

$$\int_0^1 \frac{t}{v(t)}\, \omega''(t)\, dt > -\infty.$$

A construction in [1] shows there is a set E such that $0 < \mathscr{H}_v(E) < \infty$ and $t/v(t) \approx T_{E,\alpha}(t^\alpha)$. It follows that

$$\int_0^1 T_{E,\alpha}(t)\, \omega''(t)\, dt > -\infty,$$

so by the lemma, E is an (ω, α)-set. ∎

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