Potential Analysis **18:** 79–96, 2003.

Tangential Limits of Potentials on Homogeneous Trees

KOHUR GOWRISANKARAN[1] and DAVID SINGMAN[2]
[1]*Department of Mathematics & Statistics, McGill University, Montreal, Quebec, Canada H3A 2K6*
(*e-mail: gowri@.math.mcgill.ca*)
[2]*Department of Mathematics, George Mason University, Fairfax, VA 22030, USA*
(*e-mail: dsingman@gmu.edu*)

(Received: 9 June 2000; accepted: 14 December 2001)

Abstract. Let **T** be a homogeneous tree of homogeneity $q+1$. Let Δ denote the boundary of **T**, consisting of all infinite geodesics $b = [b_0, b_1, b_2, \ldots]$ beginning at the root, 0. For each $b \in \Delta$, $\tau \geqslant 1$, and $a \geqslant 0$ we define the approach region $\Omega_{\tau,a}(b)$ to be the set of all vertices t such that, for some j, t is a descendant of b_j and the geodesic distance of t to b_j is at most $(\tau - 1)j + a$. If $\tau > 1$, we view these as tangential approach regions to b with degree of tangency τ. We consider potentials Gf on **T** for which the Riesz mass f satisfies the growth condition $\sum_{\mathbf{T}} f^p(t)q^{-\gamma|t|} < \infty$, where $p > 1$ and $0 < \gamma < 1$, or $p = 1$ and $0 < \gamma \leqslant 1$. For $1 \leqslant \tau \leqslant 1/\gamma$, we show that $Gf(s)$ has limit zero as s approaches a boundary point b within $\Omega_{\tau,a}(b)$ except for a subset E of Δ of $\tau\gamma$-dimensional Hausdorff measure 0, where $H_{\tau\gamma}(E) = \sup_{\delta>0} \inf\{\sum_i q^{-\tau\gamma|t_i|} :$ E a subset of the boundary points passing through t_i for some i, $|t_i| > \log_q(1/\delta)\}$.

Mathematics Subject Classifications (2000): 31B25, 05C05.

Key words: homogeneous tree, potential, tangential region, nontangential region, Hausdorff measure, growth condition.

1. Introduction

Let us consider the half space $\mathbb{R}^n_+ = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. Let $G(x, y)$ be the Green function corresponding to the Laplace operator and let μ be a positive Radon measure on $\mathbb{R}^n_+$ such that the Green potential $G\mu$ is not identically ∞. This is precisely the condition that μ satisfies $\int \frac{y_n}{(1+|y|)^n}\, \mathrm{d}\mu(y) < \infty$. There are a number of well known results that describe the behaviour of $G\mu(y)$ as y tends to the boundary, viz. $\mathbb{R}^{n-1}$. For a generic potential on the unit disc in the complex plane, the most classical result is the one due to Littlewood [11]. It was generalized to higher dimensions as follows by Privalov [16].

THEOREM LP. *The limit of $G\mu(x', x_n)$ equals zero as $x_n \to 0$ for Lebesgue almost every $x' \in \mathbb{R}^{n-1}$.*

By adding a growth condition, Carleson [3] improved the nature of the exceptional set. More precisely, THEOREM C. *Let further* μ *verify the condition* $\int_{K\cap\mathbb{R}^n_+} y_n^\gamma \,\mathrm{d}\mu < \infty$ *for every compact subset* K *of* $\mathbb{R}^n$*, where* $0 < \gamma \leqslant 1$*. Then* $G\mu(x', x_n) \to 0$ *as* $x_n \to 0$ *except for* x' *in a set* E *of* $(n-2+\gamma)$*-dimensional Hausdorff measure zero.*

For generalisations of these results see [7–9].

It is interesting to analyze the behaviour of potentials along more general approach sets such as nontangential and even tangential regions. One cannot expect results of this type for generic potentials. There are, however, results known for potentials represented by absolutely continuous measures whose Radon–Nikodym derivatives (relative to the Lebesgue measure) satisfy certain growth conditions. In this direction we refer the reader to the articles by Arsove and Huber [1], Wu [19], Mizuta [14, 15], and Berman and Cohn [2]. In this regard we recall the following result of interest to us.

THEOREM WM ([19, 14, 15]). *Let* $p > n/2$, $2p - n < \gamma \leqslant 2p - 1$*. Let further* τ *be a real number satisfying* $1 \leqslant \tau \leqslant \frac{n-1}{n-2p+\gamma}$*. For* $x' \in \mathbb{R}^{n-1}$ *and* $a > 0$*, let* $T_{\tau,a}(x') = \{y \in \mathbb{R}^n_+ : |x' - y|^\tau \leqslant a y_n\}$ *be the approach region at* x' *of aperture* a *and tangency* τ*. Let* $f \geqslant 0$ *be measurable such that* Gf *(the potential corresponding to* $f\,\mathrm{d}x$*) is not identically* ∞*. Suppose* f *satisfies the growth condition* $\int_{K\cap\mathbb{R}^n_+} f^p(y) y_n^\gamma \,\mathrm{d}y < \infty$ *for all compact sets* K *of* $\mathbb{R}^n$*. Then the limit of* $Gf(y)$ *as* y *tends to* x' *within* $T_{\tau,a}(x')$ *is zero except for a set of* $x' \in \mathbb{R}^{n-1}$ *of* $\tau(n - 2p + \gamma)$*-dimensional Hausdorff measure zero.*

In this paper we shall prove results analogous to the above for the case of potentials on a homogeneous tree $\mathbf{T}$ instead of $\mathbb{R}^n_+$. Let the tree be homogeneous of degree $q + 1$ and denote the root of the tree by 0. We denote by $|t|$ the length of the geodesic from 0 to t. Let G be the Green function on $\mathbf{T} \times \mathbf{T}$ ([4], p. 264). The Martin boundary of $\mathbf{T}$ consists of the collection Δ of all infinite geodesics starting at 0. If f is any nonnegative function on $\mathbf{T}$, $Gf(s) = \sum_{t\in\mathbf{T}} G(s,t) f(t)$, if finite, is the potential corresponding to f. We shall recall the details of the above concepts in the next section. We shall as well introduce an approach region of 'tangency τ' and 'aperture a' for each $b \in \Delta$, denoted by $\Omega_{\tau,a}(b)$. We shall discuss Hausdorff measures of various dimensions with which we measure the exceptional sets. We now state our main result.

THEOREM 1. *Let* $\mathbf{T}$ *be a homogeneous tree of degree* $q + 1$*. Let* $0 < \gamma < 1$ *for* $p > 1$ *and* $0 < \gamma \leqslant 1$ *for* $p = 1$*. Let further* $1 \leqslant \tau \leqslant 1/\gamma$*. Let* f *be a nonnegative function defined on* $\mathbf{T}$ *such that* Gf *is finite and* $\sum_{t\in\mathbf{T}} f^p(t) q^{-\gamma|t|} < \infty$*. Then the limit of* $Gf(s)$ *as* s *tends to* $b \in \Delta$ *with* s *in the approach region* $\Omega_{\tau,a}(b)$ *of 'tangency* τ*' and 'aperture* a*' is zero for all* $b \in \Delta$ *except possibly for a set* $E \subset \Delta$ *such that the* $\tau\gamma$*-dimensional Hausdorff measure of* E *is* 0*.*

We shall show in the last section that these results are sharp in that (i) the exceptional sets cannot be improved and (ii) the degree of tangency cannot be increased beyond $1/\gamma$.

2. Generalities on Trees and Hausdorff Measures

We define a tree $\mathbf{T}$ to be a graph that is infinite, locally finite, connected, and simply connected. A basic reference for potential theory on trees is [4]. In this paper $\mathbf{T}$ is assumed to be *homogeneous* of degree $q+1$, that is each vertex has exactly $q+1$ nearest neighbours, where q is a fixed integer $\geqslant 2$ and the edge transition probabilities are all $1/(q+1)$. We fix one vertex 0 which we call the *root* of $\mathbf{T}$. For any two vertices s and t, a *path* joining s and t is a finite sequence of vertices $[s_0, s_1, \ldots, s_n]$ such that $s_0 = s$, $s_n = t$, and for each j from 0 to $n-1$, s_j is adjacent to s_{j+1}. We refer to n as the *length* of this path. There is a unique path joining s and t of minimum length which we call the *geodesic* path from s to t. The length of the geodesic is denoted by $d(s,t)$. We let $|t|$ denote $d(0,t)$ and call it the *modulus* of t.

We write that $s \leqslant t$ if s lies on the geodesic from 0 to t. We define $\Pi(s)$ to be the set of all vertices $t \in \mathbf{T}$ such that $s \leqslant t$. For any two vertices s and t, consider all the vertices w such that $w \leqslant s$ and $w \leqslant t$. We let $s \wedge t$ denote the unique one of largest modulus.

Let Δ denote the set of all infinite geodesics beginning at the root. Thus $b \in \Delta$ if $b = [b_0, b_1, \ldots]$, where $b_0 = 0$, and for every positive integer n, $[b_0, b_1, \ldots, b_n]$ is a geodesic. We refer to $b_0, b_1, \ldots$ as the *vertices* of b. If $b = [b_0, b_1, \ldots]$ and $s \in \mathbf{T}$, then we define $b \wedge s$ to be the vertex of b of largest modulus that is in the geodesic from 0 to s. If $b = [b_0, b_1, \ldots]$ and $d = [d_0, d_1, \ldots]$ are in Δ, we define $b \wedge d$ to be the the vertex of largest modulus that is common to b and d. We can then define the distance from b to d to be $q^{-|b\wedge d|}$ if $b \neq d$ and 0 if $b = d$ with a similar definition for the distance between two vertices of $\mathbf{T}$ or between a vertex and a point of Δ. With this distance, $\mathbf{T} \cup \Delta$ becomes a metric space which is a compactification of $\mathbf{T}$.

For each $s \in \mathbf{T}$, define $E(s)$ to be the set of all $b \in \Delta$ such that $b \wedge s = s$. Notice this is a ball in Δ of radius $q^{-|s|}$ and that every ball in Δ is $E(s)$ for some $s \in \mathbf{T}$. Notice also that any two such balls are either disjoint or one contains the other. For each $b \in \Delta$, $\{E(b_j)\}_{j=1}^{\infty}$ forms a base for the neighbourhoods of b. A sequence $\{s_n\}$ of vertices converges to $b \in \Delta$ if and only if $|b \wedge s_n| \to \infty$ as $n \to \infty$.

Let f be a real-valued function defined on the vertices of $\mathbf{T}$. We let $\|f\|_\infty = \sup_{t\in\mathbf{T}} |f(t)|$ and $\|f\|_1 = \sum_{t\in\mathbf{T}} |f(t)|$. Let Ω be a subset of $\mathbf{T}$ having b as a limit point. We write $\lim_{\Omega\ni s\to b} f(s) = L$ provided for every $\epsilon > 0$ there exists an integer j such that for every $s \in \Pi(b_j) \cap \Omega$, $|f(s) - L| < \epsilon$. If Ω equals $\Pi(b_j)$ for some j, we simply write $\lim_{s\to b} f(s) = L$. If Ω consists only of the vertices of b, we say f has a *radial* limit at b.

DEFINITION 1. Let $b \in \Delta$, $\tau \geqslant 1$ and $a \geqslant 0$. The region $\Omega_{\tau,a}(b)$ is given by

$$\Omega_{\tau,a}(b) = \{s \in \mathbf{T} : \exists j \in \mathbb{Z} \text{ such that } b \wedge s = b_j, \\ \text{and } d(s, b_j) \leqslant (\tau - 1)j + a\}.$$

If $a = 0$, we simply write $\Omega_\tau(b)$. If $\tau = 1$, we call the region the *nontangential* approach region to b of aperture a. If $\tau > 1$, we call it the *tangential* approach region to b of aperture a and tangency τ.

The potential theory on the tree is determined by the kernel $P : \mathbf{T} \times \mathbf{T} \to \mathbb{R}$, $P(s,t) = 1/(q+1)$ if s and t are adjacent, and 0 otherwise. Powers of P are defined inductively by $P^n(s,t) = \sum_{u\in\mathbf{T}} P(s,u) \cdot P^{n-1}(u,t)$. A real valued function f on $\mathbf{T}$ is called *harmonic* if for all $s \in \mathbf{T}$, $Pf(s) = \sum_{t\in\mathbf{T}} P(s,t)f(t) = f(s)$. For interesting boundary behaviour results concerning harmonic functions, see [18]. The Green function $G(s,t)$ is defined on $\mathbf{T} \times \mathbf{T}$ by $G(s,t) = I(s,t) + \sum_{j\in\mathbb{Z}^+} P^j(s,t)$, where $I(s,t) = 1$ if $s = t$ and 0 otherwise. For the homogeneous tree $\mathbf{T}$, $G(s,t)$ is given by $\frac{q}{q-1} q^{-d(s,t)}$. The *potentials* for the potential theory on $\mathbf{T}$ are given by $Gf(s) = \sum_{t\in\mathbf{T}} G(s,t)f(t)$, where f is a non-negative function on $\mathbf{T}$ for which the above sum is not identically infinite. It can be shown that $Gf(s)$ is necessarily finite for every s and this happens if and only if f satisfies the condition that $\sum_{t\in\mathbf{T}} q^{-|t|} f(t) < \infty$.

The technique developed by Martin [13] can be applied to deduce that Δ is the Martin boundary for this harmonic space. Details are given in [4]. Let $s, t \in \mathbf{T}$ and let $[s_0, s_1, \ldots, s_{|s|}]$ be the geodesic from 0 to s. It is easy to check that the quotient $G(s,t)/G(0,t)$ takes the constant value $q^{2j-|s|}$ on the set of all t such that $t \wedge s = s_j$. The Martin kernel is thus $P(s,b) = q^{2|b\wedge s|-|s|}$, and for every positive harmonic function f on $\mathbf{T}$ there is a unique measure μ_f on Δ such that for all $s \in \mathbf{T}$, $f(s) = \int P(s,b)\,\mathrm{d}\mu_f(b)$. The measure μ_1 representing the harmonic function $f \equiv 1$ satisfies $\mu_1(E(t)) = \frac{q}{q+1} q^{-|t|}$, if $t \neq 0$, and $\mu_1(E(0)) = 1$.

We now define the *Hausdorff measures* which we shall use to measure the exceptional sets that arise in Theorem 1.

DEFINITION 2. Let E be a subset of Δ. Let $0 < \beta \leqslant 1$. We define

$$H_\beta(E) = \sup_{\delta>0} \inf\left\{\sum_i q^{-\beta|t_i|} : E \subset \bigcup_i E(t_i),\ |t_i| > \log_q(1/\delta)\right\}$$

and

$$C_\beta(E) = \inf\left\{\sum_i q^{-\beta|t_i|} : E \subset \bigcup_i E(t_i)\right\}.$$

We call $H_\beta(E)$ the β-dimensional *Hausdorff measure* of E and $C_\beta(E)$ the β-dimensional *content* of E.

It is easy to see that $H_\beta(E) = 0$ if and only if $C_\beta(E) = 0$.

REMARK 1. H_β is a measure constructed by "Method II" as described in [17], p. 27. It follows by Theorems 44, 47, and 48 in [17] that H_β is inner regular in the sense that a Borel set has positive H_β measure if and only if there exists a compact subset which has positive H_β measure.

The following is adapted from a classical result of Frostman [6]. The proof can easily be modeled on the proof of Frostman's theorem given on page 223 of [10]. We leave it to the reader.

LEMMA 1. *Let K be a compact subset of Δ having positive H_β measure for* $0 < \beta \leqslant 1$. *Then there exists a measure ν with support in K such that* $0 < \nu(K) < \infty$ *and for every $t \in \mathbf{T}$, $\nu(E(t)) \leqslant q^{-\beta|t|}$.*

In what follows, the letter c will be used to represent a constant value which, though possibly different with each occurrence, does not depend in an important way on the parameters of interest.

3. Proof of Theorem 1 for the Case $p = 1$

Let h be a nonnegative function on the tree $\mathbf{T}$. Let, for a vertex s, $h^*(s) = \sum_{t\in\Pi(s)} h(t)$.

DEFINITION 3. Let $b = [b_0, b_1, b_2, \ldots]$ be in Δ. Let $\beta > 0$ and $h \geqslant 0$ on $\mathbf{T}$. We define the maximal function

$$M_\beta h(b) = \sup_i q^{i\beta} h^*(b_i).$$

We now prove the following result concerning the growth of the maximal function.

LEMMA 2. *Let $h \geqslant 0$ on $\mathbf{T}$ and $\lambda > 0$. Then*

$$C_\beta(\{b \in \Delta : M_\beta h(b) > \lambda\}) \leqslant \frac{\|h\|_1}{\lambda}.$$

Proof. Let $F_\lambda = \{b \in \Delta : M_\beta h(b) > \lambda\}$. By the definition of the maximal function, for each $b \in F_\lambda$ corresponds an integer $i(b)$ such that $h^*(b_{i(b)}) > \lambda q^{-i(b)\beta}$. The collection $\{E(b_{i(b)})\}_{b\in F_\lambda}$ is a covering of F_λ by balls. We recall that any two balls in the family are either disjoint or one is a subset of the other. By using the Well Ordering property of the natural numbers, we can choose a subset $F'_\lambda \subset F_\lambda$ such that $\{E(b_{i(b)})\}_{b\in F'_\lambda}$ is a cover of F_λ and consists of pairwise disjoint balls. Note that $\{E(b_{i(b)})\}_{b\in F'_\lambda}$ pairwise disjoint is equivalent to $\{\Pi(b_{i(b)})\}_{b\in F'_\lambda}$ being pairwise disjoint. Clearly F'_λ is a countable set. Now, by the countable subadditivity of C_β,

$$\begin{aligned} C_\beta(F_\lambda) &\leqslant \sum_{b\in F'_\lambda} q^{-\beta|b_{i(b)}|} \\ &\leqslant \frac{1}{\lambda}\sum_{b\in F'_\lambda} h^*(b_{i(b)}) \\ &\leqslant \frac{1}{\lambda}\sum_{t\in\mathbf{T}} h(t) \\ &= \frac{\|h\|_1}{\lambda}. \end{aligned}$$

□

LEMMA 3. *Let the support of f be in $\{t \in \mathbf{T} : |t| \leqslant M\}$ and let $N > M$. Then, for all $s \in \mathbf{T}$ with $|s| \geqslant N$,*

$$Gf(s) \leqslant \frac{q+1}{q-1}\|f\|_\infty q^{2M-N}.$$

The lemma follows easily by checking the sum $Gf(s) = \sum_{t\in\mathbf{T}} G(s,t)f(t)$. We note that this result gives us the fact that for any function f with finite support as above, $Gf(s)$ tends to zero uniformly as s tends to b without any restriction on the direction of approach.

Proof of the theorem. We recall $p = 1$ and $0 < \gamma \leqslant 1$. Let f be a function on $\mathbf{T}$. We want to show that $Gf(s) \to 0$ as $s \to b$ with $s \in \Omega_{\tau,a}(b)$ except for a set of $H_{\tau\gamma}$-measure zero. To prove the result, it is enough to show that for arbitrary $\epsilon > 0$ and $\delta > 0$

$$C_{\tau\gamma}\left[\left\{b \in \Delta : \limsup_{\Omega_{\tau,a}(b)\ni s\to b} Gf(s) > \delta\right\}\right] < c\epsilon. \tag{1}$$

Accordingly let us fix an $\epsilon > 0$ and a $\delta > 0$. The hypothesis $\sum_{t\in\mathbf{T}} f(t)q^{-\gamma|t|} < \infty$ can be rewritten as $\|h\|_1 < \infty$, where $h(t) = f(t)q^{-\gamma|t|}$. By virtue of Lemma 3, the lim sup in (1) is unchanged if the values of f on a finite set of vertices are changed to 0, and so without loss of generality we may assume $\|h\|_1 < \epsilon\delta$. Now, let us fix a b belonging to the set described on the left side of (1) and an $s \in \Omega_{\tau,a}(b)$. Let $[s_0, \ldots, s_{|s|-|b\wedge s|}]$ be the geodesic from $b \wedge s$ to s. In order to compute $Gf(s)$, we divide the tree $\mathbf{T}$ into four disjoint pieces $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$ defined as:

$$\mathbf{T}_1 = \bigcup_{j=0}^{|b\wedge s|-1} \overline{\Pi(b_j)}, \qquad \mathbf{T}_2 = \bigcup_{j=1}^{n} \overline{\Pi(s_j)}, \quad \text{for } n = |s| - |b\wedge s| - 1,$$
$$\mathbf{T}_3 = \Pi(s) \quad \text{and} \quad \mathbf{T}_4 = \Pi(b\wedge s) - (\mathbf{T}_2 \cup \mathbf{T}_3),$$

where $\overline{\Pi(b_j)} = \Pi(b_j) - \Pi(b_{j+1})$ and $\overline{\Pi(s_j)} = \Pi(s_j) - \Pi(s_{j+1})$. Now

$$Gf(s) = \sum_t G(s,t)f(t) = \sum_{j=1}^{4}\sum_{t\in\mathbf{T}_j} G(s,t)f(t).$$

We shall estimate each sum on the right separately. In each of the estimates we will use the expression

$$G(s,t)f(t) = \frac{G(s,t)}{G(0,t)}f(t)G(0,t) = \frac{q}{q-1}\frac{G(s,t)}{G(0,t)}q^{-(1-\gamma)|t|}h(t). \tag{2}$$

We have

$$\sum_{t\in\mathbf{T}_1}\frac{G(s,t)}{G(0,t)}q^{-(1-\gamma)|t|}h(t) = \sum_{j=0}^{|b\wedge s|-1}\sum_{t\in\overline{\Pi(b_j)}} q^{2j-|s|}q^{-(1-\gamma)|t|}h(t)$$

$$
\begin{aligned}
&\leqslant \sum_{j=0}^{|b\wedge s|-1} \sum_{t\in\overline{\Pi(b_j)}} q^{-\tau\gamma j} q^{2j-|s|} q^{-(1-\gamma)j} h(t) q^{\tau\gamma j} \\
&\leqslant \sum_{j=0}^{|b\wedge s|-1} q^{j[1+\gamma-\gamma\tau]-|s|} h^*(b_j) q^{\tau\gamma j} \\
&\leqslant q^{-|s|} M_{\tau\gamma} h(b) \sum_{j=0}^{|b\wedge s|-1} q^{j[1+\gamma-\gamma\tau]} \\
&\leqslant c M_{\tau\gamma} h(b) q^{-|s\wedge b|} q^{|b\wedge s|(1+\gamma-\gamma\tau)},
\end{aligned}
$$

hence

$$
\sum_{t\in\mathbf{T}_1} G(s,t) f(t) \leqslant c M_{\tau\gamma} h(b). \tag{3}
$$

Next we have

$$
\begin{aligned}
\sum_{t\in\mathbf{T}_2} G(s,t) f(t) &= \frac{q}{q-1} \sum_{j=1}^{n} \sum_{t\in\overline{\Pi(s_j)}} q^{2(|b\wedge s|+j)-|s|} q^{-(1-\gamma)|t|} h(t) \\
&\leqslant c \sum_{j=1}^{n} \sum_{t\in\Pi(s_j)} q^{2(|b\wedge s|+j)-|s|} q^{-(1-\gamma)(|b\wedge s|+j)} h(t) \\
&\leqslant c q^{-|s|} q^{(1+\gamma)|b\wedge s|} \sum_{j=1}^{n} \sum_{t\in\Pi(s_j)} q^{j(1+\gamma)} h(t) \\
&\leqslant c q^{-|s|} h^*(b\wedge s) q^{(1+\gamma)|b\wedge s|} q^{(1+\gamma)(|s|-|b\wedge s|)} \\
&= c h^*(b\wedge s) q^{|s|\gamma} \\
&\leqslant c h^*(b\wedge s) q^{(\tau|b\wedge s|+a)\gamma}
\end{aligned}
$$

and so

$$
\sum_{t\in\mathbf{T}_2} G(s,t) f(t) \leqslant c M_{\gamma\tau} h(b). \tag{4}
$$

For the sum over $\mathbf{T}_3$ we have

$$
\begin{aligned}
\sum_{t\in\mathbf{T}_3} G(s,t) f(t) &= \frac{q}{q-1} \sum_{t\in\Pi(s)} q^{|s|} q^{-(1-\gamma)|t|} h(t) \\
&\leqslant c \sum_{t\in\Pi(s)} q^{|s|} q^{-(1-\gamma)|s|} h(t) \\
&= c q^{\gamma|s|} h^*(s) \\
&\leqslant c q^{|s|\gamma} h^*(b\wedge s),
\end{aligned}
$$

and so, as before,

$$\sum_{t\in\mathbf{T}_3} G(s,t)f(t) \leqslant cM_{\tau\gamma}h(b). \tag{5}$$

Lastly,

$$\begin{aligned}\sum_{t\in\mathbf{T}_4} G(s,t)f(t) &= \frac{q}{q-1}\sum_{t\in\mathbf{T}_4} q^{2|b\wedge s|-|s|}q^{-(1-\gamma)|t|}h(t)\\ &\leqslant c\sum_{t\in\Pi(b\wedge s)} q^{2|b\wedge s|-|b\wedge s|}q^{-(1-\gamma)|b\wedge s|}h(t)\\ &\leqslant c\sum_{t\in\Pi(b\wedge s)} q^{|b\wedge s|\tau\gamma}h(t)\\ &= cq^{\tau\gamma|b\wedge s|}h^*(b\wedge s)\end{aligned}$$

giving

$$\sum_{t\in\mathbf{T}_4} G(s,t)f(t) \leqslant cM_{\tau\gamma}h(b). \tag{6}$$

By combining (3), (4), (5), and (6) we get

$$\left\{b\in\Delta : \limsup_{\Omega_{\tau,a}(b)\ni s\to b} Gf(s) > \delta\right\} \subset \{b\in\Delta : M_{\tau\gamma}h(b) > c\delta\}. \tag{7}$$

It follows from Lemma 2 that

$$C_{\tau\gamma}\left(\left\{b\in\Delta : \limsup_{\Omega_{\tau,a}(b)\ni s\to b} Gf(s) > \delta\right\}\right) \leqslant c\frac{\|h\|_1}{\delta} < c\epsilon.$$

This completes the proof. □

REMARK 2. A generic potential on **T** satisfies $\gamma = p = 1$. The theorem applied in this case says that every potential has nontangential limit zero H_1-a.e. (which is the same as almost everywhere with respect to the representing measure of the constant harmonic functions). This is in marked contrast to the situation in classical potential theory where there exist potentials on a half space in $\mathbb{R}^n_+$ that fail to have nontangential limits at every boundary point.

4. Proof of Theorem 1 for the Case $p > 1$

Recall that the function f on the tree verifies the two conditions $Gf < \infty$ and $\sum_{\mathbf{T}} f^p(t)q^{-\gamma|t|} < \infty$ where $0 < \gamma < 1$. We start with the following simple exercise in summing a geometric series. We do not include the proof.

LEMMA 4. *Let $a > 1$ and $|s| \geqslant 1$. Then* $\sum_{t\in\Pi(s)} q^{-a|t|} = \frac{q^{-a|s|}}{1-q^{(1-a)}}$.

We shall organize the remainder of the proof of Theorem 1 in several steps.

Step 1. Let s be in $\mathbf{T}$ and $b \in \Delta$. We show that for every i, $\sum_{t\in\mathbf{T}-\Pi(b_i)} G(s,t) f(t) \to 0$ as $s \to b$. Note that

$$G(s,t) = \frac{q}{q-1}\frac{G(s,t)}{G(0,t)}q^{-|t|}. \tag{8}$$

Recalling the notation $\overline{\Pi(b_j)} = \Pi(b_j) - \Pi(b_{j+1})$, we have

$$\begin{aligned}\sum_{t\in\mathbf{T}-\Pi(b_i)} G(s,t)f(t) &= \frac{q}{q-1}\sum_{j=0}^{i-1}\sum_{t\in\overline{\Pi(b_j)}}\frac{G(s,t)}{G(0,t)}q^{-|t|}f(t)\\ &\leqslant c\sum_{j=0}^{i-1}\sum_{t\in\overline{\Pi(b_j)}} q^{2j-|s|}q^{-|t|}f(t)\\ &\leqslant c\sum_{j=0}^{i-1}q^{2j-|s|}\sum_{t\in\mathbf{T}}q^{-|t|}f(t)\\ &= cq^{2i}Gf(0)q^{-|s|}\end{aligned}$$

and the latter goes to 0 as $s \to b$.

Step 2. For $0 < d \leqslant 1$, and $\delta \geqslant 0$, define $B_{d,\delta}$ by

$$B_{d,\delta} = \left\{b \in \Delta : \limsup_{i\to\infty} q^{id}\sum_{\Pi(b_i)} f^p(t)q^{-\gamma|t|} > \delta\right\}.$$

In case $\delta = 0$, we denote $B_{d,\delta}$ by B_d. It is obvious that

$$B_{d,\delta} \subset \{b \in \Delta : M_d h(b) > \delta\},$$

where $h(t) = f^p(t)q^{-\gamma|t|}$. Since $B_{d,\delta}$ is unchanged if f is set equal to zero at any finite number of vertices it follows from Lemma 2 that $H_d(B_{d,\delta}) = 0$, and so by countable subadditivity, $H_d(B_d) = 0$. We conclude that $H_{\tau\gamma}(B_{\tau\gamma}) = 0$.

Step 3. Let β be a number such that $\gamma < \beta < 1$. Define, for each $\delta > 0$,

$$A_{p,\beta,\delta} = \left\{b \in \Delta : \limsup_{i\to\infty}\sum_{\Pi(b_i)} f^p(t)q^{-\beta(|t|-|b\wedge t|)} > \delta\right\}.$$

Since for each i, the function $b \mapsto \sum_{\Pi(b_i)} f^p(t)q^{-\beta(|t|-|b\wedge t|)}$ is locally constant on Δ, it is continuous and hence $b \mapsto \limsup_{i\to\infty}\sum_{\Pi(b_i)} f^p(t)q^{-\beta(|t|-|b\wedge t|)}$ is a Borel function.

We now proceed to show that $H_\gamma(A_{p,\beta,\delta}) = 0$. Suppose, on the contrary that $H_\gamma(A_{p,\beta,\delta}) > 0$. Then by Remark 1 and Lemma 1, there exists a non-trivial Radon measure ν with support in a compact subset of $A_{p,\beta,\delta}$ satisfying for all $t \in \mathbf{T}$, $\nu(E(t)) \leqslant q^{-\gamma|t|}$. We claim that for every $t \in \mathbf{T}$,

$$\int_{\Delta} q^{\beta|b\wedge t|}\,\mathrm{d}\nu(b) \leqslant cq^{(\beta-\gamma)|t|}. \tag{9}$$

Indeed,

$$\int_{\Delta} q^{\beta|b\wedge t|}\,\mathrm{d}\nu(b) = \sum_{j=0}^{|t|-1}\int_{E(t_j)-E(t_{j+1})} q^{\beta|b\wedge t|}\,\mathrm{d}\nu(b) + \int_{E(t)} q^{\beta|b\wedge t|}\,\mathrm{d}\nu(b),$$

where $t_0 = 0, t_1, \ldots, t_{|t|} = t$ are the vertices forming the geodesic from 0 to t. Thus

$$\begin{aligned}\int_{\Delta} q^{\beta|b\wedge t|}\,\mathrm{d}\nu(b) &\leqslant \sum_{j=0}^{|t|-1}\int_{E(t_j)} q^{\beta j}\,\mathrm{d}\nu(b) + q^{\beta|t|}\nu(E(t))\\ &\leqslant \sum_{j=0}^{|t|-1} q^{\beta j}q^{-\gamma j} + q^{\beta|t|}q^{-\gamma|t|}\\ &\leqslant cq^{(\beta-\gamma)|t|},\end{aligned}$$

proving the claim. Consider a positive integer N such that, for $\mathbf{T}^N = \{t \in \mathbf{T} : |t| \geqslant N\}$, we have

$$\sum_{t\in\mathbf{T}^N} f^p(t)q^{-\gamma|t|} < \frac{\delta}{2c}\|\nu\|, \tag{10}$$

where c is as in (9). By the Monotone Convergence Theorem,

$$\begin{aligned}\int_{\Delta}\sum_{\mathbf{T}^N} q^{\beta|b\wedge t|} f^p(t)q^{-\beta|t|}\,\mathrm{d}\nu(b) &= \sum_{\mathbf{T}^N}\int_{\Delta} q^{\beta|b\wedge t|} f^p(t)q^{-\beta|t|}\,\mathrm{d}\nu(b)\\ &\leqslant c\sum_{\mathbf{T}^N} q^{(\beta-\gamma)|t|} f^p(t)q^{-\beta|t|} \quad \text{(by (9))}\\ &= c\sum_{\mathbf{T}^N} q^{-\gamma|t|} f^p(t)\\ &< \frac{\delta}{2}\|\nu\|.\end{aligned}$$

On the other hand, if we consider any b in $A_{p,\beta,\delta}$, there exists j greater than N such that

$$\sum_{\Pi(b_j)} f^p(t)q^{-\beta(|t|-|b\wedge t|)} > \delta$$

and so

$$\sum_{\mathbf{T}^N} q^{\beta|b\wedge t|} f^p(t) q^{-\beta|t|} \geqslant \sum_{\Pi(b_j)} f^p(t) q^{-\beta(|t|-|b\wedge t|)} > \delta.$$

Integrating both sides with respect to ν, we get the obvious contradiction that $\|\nu\|\delta < \|\nu\|(\delta/2)$. This completes the proof that $H_\gamma(A_{p,\beta,\delta}) = 0$. If we now define

$$A_{p,\beta} = \left\{ b \in \Delta : \limsup_{i\to\infty} \sum_{\Pi(b_i)} f^p(t) q^{-\beta(|t|-|b\wedge t|)} > 0 \right\},$$

we deduce that $H_\gamma(A_{p,\beta}) = 0$. A fortiori, it follows that $H_{\tau\gamma}(A_{p,\beta}) = 0$.

Step 4. Let A be defined by

$$A = \left\{ b \in \Delta : \limsup_{i\to\infty} q^i \sum_{t\in\Pi(b_i)} f(t) q^{-|t|} > 0 \right\}.$$

We now show that A is a subset of $A_{p,\beta}$ for any β strictly between γ and 1.

Assume that b is not an element of $A_{p,\beta}$. Note that if $p' = p/(p-1)$ is the index conjugate to p, then, since $\beta < 1$, $(1-\beta/p)p' > 1$. Applying Lemma 4 and observing $|b \wedge t| \geqslant i$ for $t \in \Pi(b_i)$, we have

$$\begin{aligned} &q^i \sum_{\Pi(b_i)} f(t) q^{-|t|} \\ &= q^i \sum_{\Pi(b_i)} \left[q^{-|t|} q^{-\frac{\beta}{p}(|b\wedge t|-|t|)} \right] \left[f(t) q^{\frac{\beta}{p}(|b\wedge t|-|t|)} \right] \\ &\leqslant q^{i(1-\frac{\beta}{p})} \left(\sum_{\Pi(b_i)} q^{-(1-\frac{\beta}{p})|t|p'} \right)^{1/p'} \left(\sum_{\Pi(b_i)} f^p(t) q^{\beta(|b\wedge t|-|t|)} \right)^{1/p} \\ &\leqslant c q^{i(1-\frac{\beta}{p})} \left(q^{-(1-\frac{\beta}{p})ip'} \right)^{1/p'} \left(\sum_{\Pi(b_i)} f^p(t) q^{\beta(|b\wedge t|-|t|)} \right)^{1/p}. \end{aligned}$$

When $i \to \infty$, by the definition of $A_{p,\beta}$, the right side tends to 0 and therefore b is not an element of A.

Step 5. We now proceed to complete the proof of the theorem. We shall show that if b is not an element of $A \cup B_{\tau\gamma}$, then $Gf(s)$ tends to zero as $s \to b$ within the indicated approach region. Fix such a b. We consider s in the region $\Omega_{\tau,a}(b)$. Let $\epsilon > 0$. We shall first use the fact that b is not in A to show that we can establish a range of s close enough to b so that the contribution $\sum_{t\in(\Pi(b\wedge s))^c} G(s,t) f(t)$ to $Gf(s)$ can be made $< c\epsilon$. Since b is not in A, we can choose an integer i_0 such that

$$C_{i_0} = \sup_{j\geqslant i_0} q^j \sum_{\Pi(b_j)} f(t) q^{-|t|} < \epsilon.$$

By virtue of Lemma 3, for all s close enough to b,

$$\sum_{t\in(\Pi(b_{i_0}))^c} G(s,t)f(t) < \epsilon.$$

We also have

$$\begin{aligned}\sum_{j=i_0}^{|b\wedge s|-1} \sum_{t\in\overline{\Pi(b_j)}} G(s,t)f(t) &\leqslant c \sum_{j=i_0}^{|b\wedge s|-1} \sum_{t\in\Pi(b_j)} q^{2j-|s|}q^{-|t|}f(t) \\ &= c \sum_{j=i_0}^{|b\wedge s|-1} (q^{j-|s|}) \sum_{t\in\Pi(b_j)} q^j q^{-|t|} f(t) \\ &\leqslant cC_{i_0}q^{|b\wedge s|-|s|} \\ &\leqslant cC_{i_0} \\ &< c\epsilon.\end{aligned}$$

This is true for all s sufficiently close to b whether s is in $\Omega_{\tau,a}(b)$ or not. We have shown $\sum_{t\in(\Pi(b\wedge s))^c} G(s,t)f(t) < c\epsilon$ for all s close enough to b.

Now we consider the contribution $\sum_{\Pi(b\wedge s)} G(s,t)f(t)$ for all $s \in \Omega_{\tau,a}(b)$. We split this sum over $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ as in Section 3. In order to apply Hölder's inequality we rewrite

$$G(s,t)f(t) = \frac{q}{q-1}\frac{G(s,t)}{G(0,t)}q^{-|t|(1-\frac{\gamma}{p})} f(t)q^{-\frac{\gamma}{p}|t|}. \tag{11}$$

Consider the summation over $\mathbf{T}_2$. Denoting the geodesic from $b\wedge s$ to s by $[s_0,\ldots,s_{|s|-|b\wedge s|}]$, putting $n = |s| - |b\wedge s| - 1$, and using Lemma 4, we note

$$\begin{aligned}\sum_{\mathbf{T}_2}\left(\frac{G(s,t)}{G(0,t)}\right)^{p'} q^{-p'|t|(1-\frac{\gamma}{p})} &= \sum_{j=1}^{n}\sum_{\overline{\Pi(s_j)}} q^{([2(|b\wedge s|+j)]-|s|)p'}q^{-p'|t|(1-\frac{\gamma}{p})} \\ &\leqslant c\sum_{j=1}^{n} q^{([2(|b\wedge s|+j)]-|s|)p'}q^{-p'(1-\frac{\gamma}{p})[|b\wedge s|+j]} \\ &= c\sum_{j=1}^{n} q^{p'[(1+\frac{\gamma}{p})|b\wedge s|-|s|+(1+\frac{\gamma}{p})j]} \\ &\leqslant cq^{p'[(1+\frac{\gamma}{p})|b\wedge s|-|s|+(1+\frac{\gamma}{p})(|s|-|b\wedge s|)]} \\ &= cq^{\frac{\gamma}{p}p'|s|}.\end{aligned}$$

Hence, since $s\in\Omega_{\tau,a}(b)$, we have by Hölder's inequality and (11) that the contribution to $Gf(s)$ from $\mathbf{T}_2$ satisfies

$$\sum_{\mathbf{T}_2} G(s,t)f(t) \leqslant c\left(q^{\gamma|s|}\sum_{t\in\mathbf{T}_2} f^p(t)q^{-\gamma|t|}\right)^{1/p}$$

$$\leqslant c\left(q^{\tau\gamma|b\wedge s|}\sum_{\Pi(b\wedge s)} f^p(t)q^{-\gamma|t|}\right)^{1/p}$$

and this tends to 0 as $s \to b$ since b is not in $B_{\tau\gamma}$.

Let us now consider the contribution to $Gf(s)$ from summing over $t \in \mathbf{T}_3$. We have

$$\begin{aligned}\sum_{\mathbf{T}_3}\left(\frac{G(s,t)}{G(0,t)}\right)^{p'} q^{-p'|t|(1-\frac{\gamma}{p})} &= \sum_{t\in\Pi(s)} q^{|s|p'}q^{-p'|t|(1-\frac{\gamma}{p})}\\ &\leqslant cq^{|s|p'}q^{-(1-\frac{\gamma}{p})p'|s|}\\ &= cq^{\frac{\gamma}{p}p'|s|}.\end{aligned}$$

As in the summation over $\mathbf{T}_2$, we deduce the contribution to $Gf(s)$ from $t \in \mathbf{T}_3$ goes to zero as $s \to b$, $s \in \Omega_{\tau,a}(b)$.

Lastly, consider the contribution to $Gf(s)$ from $t \in \mathbf{T}_4$. Again

$$\begin{aligned}\sum_{\mathbf{T}_4}\left(\frac{G(s,t)}{G(0,t)}\right)^{p'} q^{-p'|t|(1-\frac{\gamma}{p})} &\leqslant \sum_{\Pi(b\wedge s)} q^{(2|b\wedge s|-|s|)p'}q^{-p'|t|(1-\frac{\gamma}{p})}\\ &= cq^{(2|b\wedge s|-|s|)p'}q^{-p'(1-\frac{\gamma}{p})|b\wedge s|}\\ &\leqslant cq^{\frac{\gamma}{p}|s|p'}.\end{aligned}$$

Exactly as shown above, the contribution to $Gf(s)$ from $t \in \mathbf{T}_4$ tends to zero as $s \to b$ and $s \in \Omega_{\tau,a}(b)$. We have clearly demonstrated that $\lim_{\Omega_{\tau,a}(b)\ni s\to b} Gf(s) = 0$ if b is not an element of $A \cup B_{\tau\gamma}$. Since by Steps 2, 3, and 4 we have $H_{\tau\gamma}(A \cup B_{\tau\gamma}) = 0$, the proof is complete. □

5. Converse Results

We now show that we cannot improve on the exceptional set in Theorem 1.

THEOREM 2. *Let* $0 < \gamma \leqslant 1$, $1 \leqslant \tau \leqslant 1/\gamma$, $p \geqslant 1$, *and suppose* $H_{\tau\gamma}(E) = 0$.

(A) *Then there exists a function* f *on* $\mathbf{T}$ *such that* $\sum_{t\in\mathbf{T}} f(t)q^{-\gamma|t|} < \infty$ *and* $\limsup_{\Omega_\tau(b)\ni s\to b} Gf(s) = \infty$ *for every* $b \in E$.

(B) *If in addition* $\gamma < 1$ *and* E *is compact, then there exists a function* f *on* $\mathbf{T}$ *such that* $\|f\|_\infty \leqslant 1$, $\sum_{t\in\mathbf{T}} f^p(t)q^{-\gamma|t|} < \infty$ *and* $\liminf_{\Omega_\tau(b)\ni s\to b} Gf(s) < \limsup_{\Omega_\tau(b)\ni s\to b} Gf(s)$ *for every* $b \in E$.

Proof. (A) There exists $\{t^{i,j}\} \subset \mathbf{T}$ such that for each positive integer j, $E \subset \bigcup_i E(t^{i,j})$; $\{E(t^{ij})\}_i$ is a disjoint family; for each i, $|t^{i,j}| > j$; and $\sum_i q^{-\tau\gamma|t^{i,j}|} < 2^{-j}$. For each i and j, let $s^{i,j} \in \Pi\{t^{i,j}\}$ such that $|s^{i,j}| = [\tau|t^{i,j}|]$ ([...] denotes the greatest integer function). Define f on $\mathbf{T}$ by $f(s^{i,j}) = j$ and 0 otherwise. Let $b \in E$. For each positive integer j there exists $i(j)$ such that $b \in$

$E(t^{i(j),j})$ and hence $s^{i(j),j} \in \Omega_\tau(b)$, $|s^{i(j),j}| > j$ and $f(s^{i(j),j}) = j$. It follows that $\limsup_{\Omega_\tau(b)\ni s\to b} Gf(s) = \infty$. Finally

$$\begin{aligned}\sum_{t\in\mathbf{T}} f(t)q^{-\gamma|t|} &= \sum_j \sum_i jq^{-\gamma|s^{i,j}|}\\ &\leqslant \sum_j q^\gamma j \sum_i q^{-\tau\gamma|t^{i,j}|}\\ &\leqslant \sum_j q^\gamma j 2^{-j} < \infty.\end{aligned}$$

(B) For each positive integer j we shall inductively define a positive integer $i(j)$, vertices $t^{i,j}$, $s^{i,j}$, and function values $f(s^{i,j})$ for each i from 1 to $i(j)$. First take $j = 1$. Since E is compact and $H_{\tau\gamma}(E) = 0$, there exists a finite set of vertices $t^{1,1}, t^{2,1}, \ldots, t^{i(1),1}$ such that, if $i \neq i'$ then $E(t^{i,1}) \cap E(t^{i',1}) = \emptyset$, $E \subset \bigcup_{i=1}^{i(1)} E(t^{i,1})$, and $\sum_{i=1}^{i(1)} q^{-\gamma\tau|t^{i,1}|} < 2^{-1}$. For each $t^{i,1}$, let $s^{i,1}$ be any vertex in $\Pi(t^{i,1})$ such that $|s^{i,1}| = [\tau|t^{i,1}|]$. Define $f(s^{i,1}) = 0$. Now suppose that, for some $j \geqslant 1$ we have chosen $i(j), t^{i,j}, s^{i,j}$, and $f(s^{i,j})$ for each $i \leqslant i(j)$. There exists a finite set of vertices $t^{1,j+1}, t^{2,j+1}, \ldots, t^{i(j+1),j+1}$ such that (1) if $i \neq i'$ then $E(t^{i,j+1}) \cap E(t^{i',j+1}) = \emptyset$; (2) $E \subset \bigcup_{i=1}^{i(j+1)} E(t^{i,j+1})$; (3) $\sum_{i=1}^{i(j+1)} q^{-\gamma\tau|t^{i,j+1}|} < 2^{-(j+1)}$; (4) for each $i = 1, \ldots, i(j+1)$, $|t^{i,j+1}| \geqslant (1-\gamma)^{-1} \max\{|t^{m,k}|, k \leqslant j, m \leqslant i(k)\}$; and (5) for all s such that $|s| \geqslant \min\{|t^{i,j+1}| : i \leqslant i(j+1)\}$, the potential at s due to the values of $f(s^{m,k})$, $k \leqslant j, m \leqslant i(k)$ is at most $1/16$. Lemma 3 allows us to arrange for property (5). For each i from 1 to $i(j+1)$, let $s^{i,j+1}$ be any vertex in $\Pi(t^{i,j+1})$ such that $|s^{i,j+1}| = [\tau|t^{i,j+1}|]$. Define $f(s^{i,j+1})$ to be 0 if j is even and $3/16$ if j is odd.

Fix j so that j is odd, $2^j \geqslant 16q^2/(q-1)$, and let $i \leqslant i(j)$. Then

$$Gf(s^{i,j}) \leqslant \sum_{m=1}^{j-1}\sum_{k=1}^{i(m)} G(s^{i,j}, s^{k,m})f(s^{k,m}) + \sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} G(s^{i,j}, s^{k,m}). \tag{12}$$

We estimate each term in (12) separately. For the first term we have

$$\sum_{m=1}^{j-1}\sum_{k=1}^{i(m)} G(s^{i,j}, s^{k,m})f(s^{k,m}) \leqslant \frac{1}{16} \tag{13}$$

by property (5) in the previous paragraph. Note that for $m \geqslant j+1$ and $k \leqslant i(m)$ we have by property (4) above that $|t^{k,m}| \geqslant (1-\gamma)^{-1}|t^{i,j}|$ so that $|t^{k,m}| - |t^{i,j}| \geqslant \gamma|t^{k,m}|$. Thus, for the second term in (12) we have

$$\begin{aligned}\sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} G(s^{i,j}, s^{k,m}) &= \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} \frac{G(s^{i,j}, s^{k,m})}{G(0, s^{k,m})} q^{-|s^{k,m}|}\\ &= \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} q^{2|s^{i,j}\wedge s^{k,m}|-|s^{i,j}|} q^{-|s^{k,m}|}\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} q^{|s^{i,j}|-|s^{k,m}|}\\
&\leqslant \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{k(m)} q^{-\tau(|t^{k,m}|-|t^{i,j}|)+1}\\
&\leqslant \frac{q^2}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{i(m)} q^{-\tau\gamma|t^{k,m}|}\\
&\leqslant \frac{q^2}{q-1}\sum_{m=j+1}^{\infty} 2^{-m}\\
&= \frac{q^2}{q-1}2^{-j}\\
&\leqslant \frac{1}{16}. \qquad (14)
\end{aligned}$$

It follows from (13) and (14) that if j is odd and larger than $\log_2(16q^2/(q-1))$, we have

$$Gf(s^{i,j}) \leqslant \frac{2}{16}. \qquad (15)$$

On the other hand, if j is even,

$$Gf(s^{i,j}) \geqslant \frac{q}{q-1} f(s^{i,j}) > \frac{3}{16}. \qquad (16)$$

Let $b \in E$. For all j arbitrarily large, there exists $i \leqslant i(j)$ such that $b \in E(t^{i,j})$, hence $s^{i,j} \in \Omega_\tau(b)$. By (15) and (16) we have

$$\liminf_{\Omega_\tau(b)\ni t\to b} Gf(t) \leqslant \frac{2}{16} < \frac{3}{16} \leqslant \limsup_{\Omega_\tau(b)\ni t\to b} Gf(t).$$

Finally,

$$\begin{aligned}
\sum_{t\in\mathbf{T}} f^p(t) q^{-\gamma|t|} &\leqslant \sum_{j=1}^{\infty}\sum_{i=1}^{i(j)} q^{-\gamma|s^{i,j}|}\\
&\leqslant c\sum_{j=1}^{\infty}\sum_{i=1}^{i(j)} q^{-\tau\gamma|t^{i,j}|}\\
&\leqslant c\sum_{j=1}^{\infty} 2^{-j}\\
&< \infty. \qquad \square
\end{aligned}$$

The next result shows that the degree of tangency in Theorem 1 cannot be increased beyond $1/\gamma$.

THEOREM 3. *Let* $0 < \gamma \leqslant 1$, $p \geqslant 1$, *and let* $\tau > 1/\gamma$.

(A) *Then there is a potential* Gf *such that* $\sum_{t\in\mathbf{T}} f(t)q^{-\gamma|t|} < \infty$ *and* $\limsup_{\Omega_\tau(b)\ni s\to b} Gf(s) = \infty$ *for every* $b \in \Delta$.

(B) *If in addition* $\gamma < 1$, *then there is a potential* Gf *such that* $\|f\|_\infty \leqslant 1$, $\sum_{t\in\mathbf{T}} f^p(t)q^{-\gamma|t|} < \infty$, *and* $\liminf_{\Omega_\tau(b)\ni s\to b} Gf(s) < \limsup_{\Omega_\tau(b)\ni s\to b} Gf(s)$ *for every* $b \in \Delta$.

Proof. (A) For each positive integer i, let $t^{1,j}, t^{2,j}, \ldots,$ be the $(q+1)q^{j-1}$ vertices of modulus j. For each such vertex $t^{i,j}$, pick a vertex $s^{i,j} \in \Pi\{t^{i,j}\}$ such that $|s^{i,j}| = [\tau|t^{i,j}|] = [\tau j]$. Let $f(s)$ be j if $s = s^{i,j}$ and 0 otherwise. Clearly the lim sup of $f(s)$ is ∞ at every $b \in \Delta$ if approach is in $\Omega_\tau(b)$ and therefore the same is true of the lim sup of $Gf(s)$. Also

$$\begin{aligned}\sum_{t\in\mathbf{T}} f(t)q^{-\gamma|t|} &= \sum_{i,j} f(s^{i,j})q^{-\gamma|s^{i,j}|}\\ &\leqslant \sum_j jq^{-\gamma(j\tau-1)}(q+1)q^{j-1}\\ &\leqslant c\sum_j jq^{-(\tau\gamma-1)j}\\ &< \infty.\end{aligned}$$

(B) Let $\{d_j\}$ be an increasing sequence of positive integers. We shall soon impose some restrictions on this sequence. For each j, let $t^{1,d_j}, t^{2,d_j}, \ldots,$ be the $(q+1)q^{d_j-1}$ vertices of modulus d_j. For each such vertex t^{i,d_j}, pick a vertex $s^{i,d_j} \in \Pi\{t^{i,d_j}\}$ such that $|s^{i,d_j}| = [\tau|t^{i,d_j}|] = [\tau d_j]$. Define $f(s^{i,d_j})$ to be 0 if j is odd and $3/16$ if j is even. We assume that the sequence $\{d_j\}$ is chosen so that (1) for each j, $d_{j+1} \geqslant 2(1+\tau d_j)/(\tau-1)$, and (2) for all $j \in \mathbb{Z}^+$ and all s such that $|s| \geqslant d_{j+1}$, the potential at s due to all the values of $f(s^{i,d_k})$ for $k \leqslant j$ and $i \leqslant (q+1)q^{d_k-1}$ is at most $1/16$. This is possible by Lemma 3.

Fix j so that j is odd and $\sum_{m=j+1}^{\infty} \frac{q+1}{q-1} q^{-\frac{\tau-1}{2}d_m} < 1/16$. Then

$$Gf(s^{i,d_j}) \leqslant \sum_{m=1}^{j-1}\sum_{k=1}^{k(m)} G(s^{i,d_j}, s^{k,d_m})f(s^{k,d_m}) + \sum_{m=j+1}^{\infty}\sum_{k=1}^{k(m)} G(s^{i,d_j}, s^{k,d_m}), \quad (17)$$

where $k(m) = (q+1)q^{d_m-1}$. By property (2) of the previous paragraph, the first term in (17) is at most $1/16$. A simple calculation and property (1) shows that if $m \geqslant j+1$,

$$d_m - (\tau d_m - 1 - \tau d_j) \leqslant -\frac{\tau-1}{2}d_m. \quad (18)$$

The second term in (17) thus satisfies

$$\begin{aligned}\sum_{m=j+1}^{\infty}\sum_{k=1}^{k(m)} G(s^{i,d_j}, s^{k,d_m}) &= \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{k(m)} q^{-d(s^{i,d_j},s^{k,d_m})}\\ &\leqslant \frac{q}{q-1}\sum_{m=j+1}^{\infty}\sum_{k=1}^{k(m)} q^{-(|s^{k,dm}|-|s^{i,d_j}|)}\\ &\leqslant \frac{q(q+1)}{q-1}\sum_{m=j+1}^{\infty} q^{d_m-1}q^{-(\tau d_m-1-\tau d_j)}\\ &\leqslant \frac{q+1}{q-1}\sum_{m=j+1}^{\infty} q^{-\frac{\tau-1}{2}d_m} \quad \text{(by 18)}\\ &< \frac{1}{16}.\end{aligned}$$

We deduce that for all $b \in \Delta$,

$$\liminf_{\Omega_\tau(b)\ni t\to b} Gf(t) \leqslant \frac{2}{16} < \frac{3}{16} \leqslant \limsup_{\Omega_\tau(b)\ni t\to b} Gf(t).$$

Finally,

$$\begin{aligned}\sum_{t\in\mathbf{T}} f^p(t)q^{-\gamma|t|} &< \sum_{j=1}^{\infty}\sum_{i=1}^{k(j)} q^{-\gamma|s^{i,d_j}|}\\ &\leqslant \sum_{j=1}^{\infty}(q+1)q^{d_j-1}q^{-\gamma(\tau d_j-1)}\\ &= \sum_{j=1}^{\infty} q^{\gamma-1}(q+1)q^{-(\tau\gamma-1)d_j}\\ &< \infty.\end{aligned}$$

□

References

1. Arsove, M. and Huber, A.: 'On the existence of non-tangential limits of subharmonic functions', *J. London Math. Soc.* **421** (1967), 125–132.
2. Berman, R.D. and Cohn, W.S.: 'Littlewood theorems for limits and growth of potentials along level sets of Hölder continuous functions', *Amer. J. Math.* **114** (1991), 185–227.
3. Carleson, L.: On a class of meromorphic functions and its associated exceptional sets, Thesis, Uppsala, 1950.
4. Cartier, P.: 'Fonctions harmoniques sur un arbre', *Symposia Math.* **9** (1972), 203–270.
5. Doob, J.L.: *Classical Potential Theory and its Probabilistic Counterpart*, Springer, New York, 1984.

6. Frostman, O.: ‘Potentiel d’équilibre et capacité des ensembles avec quelques application à la théorie des fonctions’, *Meddel. Lunds. Univ. Mat. Sem.* **3** (1935), 1–118.
7. GowriSankaran, K. and Singman, D.: ‘A generalized Littlewood theorem for Weinstein potentials on a halfspace’, *Illinois J. Math.* **41** (1997), 630–647.
8. GowriSankaran, K. and Singman, D.: ‘Minimal fine limits for a class of potentials’, *Potential Anal.* **13** (2000), 103–114.
9. GowriSankaran, K. and Singman, D.: ‘A projection theorem and boundary behavior of potentials’, *Proc. Amer. Math. Soc.* **129**(2) (2000), 397–405.
10. Hayman, W.K. and Kennedy, P.B.: *Subharmonic Functions, Volume 1*, Academic Press, 1976.
11. Littlewood, J.E.: ‘Mathematical Notes (8): On subharmonic functions in a circle (II)’, *Proc. London Math. Soc. (2)* **28** (1928), 383–394.
12. Lyons, T.J., MacGibbon, K.B. and Taylor, J.C.: ‘Projection theorems for hitting probabilities and a theorem of Littlewood’, *J. Funct. Anal.* **59**(3) (1984), 470–489.
13. Martin, R.S.: ‘Minimal positive harmonic functions’, *Trans. Amer. Math. Soc.* **49** (1941), 137–172.
14. Mizuta, Y.: ‘Boundary limits of Green potentials of order α’, *Hiroshima Math. J.* **11** (1981), 111–123.
15. Mizuta, Y.: ‘On the behavior of potentials near a hyperplane’, *Hiroshima Math. J.* **13** (1983), 529–542.
16. Privalov, N.: ‘Boundary problems of the theory of harmonic and subharmonic functions in space’, *Mat. Sb.* **3** (1938), 3–25 (Russian).
17. Rogers, C.A.: *Hausdorff Measures*, Cambridge University Press, 1970.
18. Salvatori, M. and Vignati, M.: ‘Tangential boundary behaviour of harmonic functions on trees’, *Potential Anal.* **6** (1997), 269–287.
19. Wu, J.-M.: ‘L^p-densities and boundary behaviors of Green potentials’, *Indiana Math. J.* **28** (1979), 895–911.