

# BIHARMONIC EXTENSIONS ON TREES WITHOUT POSITIVE POTENTIALS

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*In honor of Kohur GowriSankaran for his retirement*

ABSTRACT. Let  $T$  be a tree rooted at  $e$  endowed with a nearest-neighbor transition probability that yields a recurrent random walk. We show that there exists a function  $K$  biharmonic off  $e$  whose Laplacian has potential theoretic importance and, in addition, has the following property: Any function  $f$  on  $T$  which is biharmonic outside a finite set has a representation, unique up to addition of a harmonic function, of the form  $f = \beta K + B + L$ , where  $\beta$  a constant,  $B$  is a biharmonic function on  $T$ , and  $L$  is a function, subject to certain normalization conditions, whose Laplacian is constant on all sectors sufficiently far from the root. We obtain a characterization of the functions biharmonic outside a finite set whose Laplacian has 0 flux similar to one that holds for a function biharmonic outside a compact set in  $\mathbb{R}^n$  for  $n = 2, 3$ , and 4 proved by Bajunaid and Anandam. Moreover, we extend the definition of flux and, under certain restrictions on the tree, we characterize the functions biharmonic outside a finite set that have finite flux in this extended sense.

## 1. INTRODUCTION

A basic question in potential theory is: Given a function harmonic outside a compact set, is there a natural way to associate to it a global harmonic function? The following classical theorem answers this question in the case when the ambient space is an open subset of  $\mathbb{R}^n$ , for  $n \geq 2$ .

**Theorem 1.1.** ([1], Theorem 9.7) *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $K$  a compact subset of  $\Omega$  and let  $h$  be a harmonic function on  $\Omega \setminus K$ .*

(a) *If  $n = 2$ , then  $h$  has a unique decomposition*

$$h(x) = \alpha \log |x| + h_\Omega(x) + b(x) \text{ for } x \notin K,$$

where  $\alpha \in \mathbb{R}$ ,  $h_\Omega$  is harmonic on  $\Omega$ , and  $b$  is a harmonic function on  $\mathbb{R}^2 \setminus K$  such that  $\lim_{|x| \rightarrow \infty} b(x) = 0$ .

(b) *If  $n > 2$ , then  $h$  has a unique decomposition*

$$h(x) = h_\Omega(x) + b(x) \text{ for } x \notin K,$$

where  $h_\Omega$  is harmonic on  $\Omega$ , and  $b$  is a harmonic function on  $\mathbb{R}^n \setminus K$  such that  $\lim_{|x| \rightarrow \infty} b(x) = 0$ .

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2000 *Mathematics Subject Classification.* Primary: 31C20.

*Key words and phrases.* biharmonic, trees, harmonic.

This work was supported in part by the Deanship of Academic Research of King Saud University center for Women Students Medical Studies and Sciences Research Center.

In [12], the last three authors answered the analogous question in the discrete setting of a tree without positive potentials. Such a tree can be viewed as a discrete analogue of  $\mathbb{R}^2$ . This result is outlined in Section 3 below.

In this paper we consider the analogous question for biharmonic functions on trees without positive potentials.

We are motivated by the first author's doctoral dissertation [3], in which the problem of extending biharmonic functions across a compact set was studied. Given a function  $f$  on a tree  $T$  without terminal vertices,  $f$  biharmonic outside a finite set, we provide some representations of  $f$  (unique up to the addition of a harmonic function) as sums of a biharmonic function on the whole tree and two functions with special properties that are biharmonic outside a finite set. For a related result on Riemannian manifolds, see the theorem in Section 3 of [10].

In Section 2, we give some preliminary definitions and notation on trees. In Section 3, we outline the concepts of standard and flux of a superharmonic function developed in [6] and [12] and extend the notion of flux to a more general class of functions on the tree. Furthermore, we give a probabilistic interpretation of biharmonicity and of the flux of a biharmonic function.

In Section 4, we present our main results in a series of theorems which generalize the results in [12] to functions biharmonic outside a finite set of vertices. Finally, in Section 5, we introduce the notion of principal value (P.V.) of the flux, and, under certain restrictions on the transition probabilities, we determine conditions under which functions biharmonic outside a finite set have finite P.V. flux. We then characterize the trees for which every bounded function has vanishing P.V. flux.

## 2. PRELIMINARIES

By a *tree*  $T$  we mean a locally finite connected graph with no loops, which, as a set, we identify with the collection of its vertices. We identify one vertex  $e$  as the *root* of  $T$ . Two vertices  $v$  and  $w$  are called *neighbors* if there is an edge  $[v, w]$  connecting them, and we use the notation  $v \sim w$ . A vertex is called *terminal* if it has a unique neighbor. A *path* is a finite or infinite sequence of vertices  $[v_0, v_1, \dots]$  such that  $v_k \sim v_{k+1}$  for all  $k$ . It is called a *geodesic path* if in addition  $v_{k-1} \neq v_{k+1}$ , for all  $k$ . For each pair of vertices  $v$  and  $w$ , let  $[v, w]$  denote the unique geodesic path from  $v$  to  $w$ . Given a vertex  $v \in T$ , a vertex  $w$  is called a *descendant* of  $v$  if  $v$  lies in  $[e, w]$ . The vertex  $v$  is then called an *ancestor* of  $w$ . We call the *parent* of a vertex  $v \neq e$  the only neighbor  $v^-$  of  $v$  which is an ancestor of  $v$ . The vertex  $v$  is then called a *child* of  $v^-$ . Two vertices are called *siblings* if they have the same parent. For  $v \in T$ , the set  $S_v$  consisting of  $v$  and all its descendants is called the *sector* determined by  $v$ . Denote by  $W_v$  the set  $S_v - S_v$ , for  $v \neq e$ . Define the *length* of a finite path  $[v = v_0, v_1, \dots, w = v_n]$  (with  $v_k \sim v_{k+1}$  for  $k = 0, \dots, n$ ) to be the number  $n$  of edges connecting  $v$  to  $w$ . The *distance*,  $d(v, w)$ , between vertices  $v$  and  $w$  is the length of the unique geodesic path connecting  $v$  to  $w$ . We define the *length* of a vertex  $v$ , by  $|v| = d(e, v)$ .

A *function on a tree* is a real-valued function on the set of its vertices. A real-valued function  $f$  on a tree  $T$  is said to be *radial* on a set  $W$  if, for each  $v \in W$ , the value  $f(v)$  depends only on  $|v|$ .

In this paper, we shall assume the tree to be without terminal vertices, and so necessarily infinite.

The trees we will be considering will always be equipped with a nearest-neighbor transition probability matrix  $p$  defined on pairs of vertices as follows: For each  $v, w \in T$ ,  $p(v, w) \geq 0$ , and  $p(v, w) > 0$  if and only if  $v$  and  $w$  are neighbors, and for each  $v \in T$ ,  $\sum_{w \in T} p(v, w) = 1$ .

The *Laplacian operator* on the set of functions  $f$  on  $T$  is defined by

$$\Delta f(v) = \sum_{w \sim v} p(v, w) f(w) - f(v),$$

for each  $v \in T$ .

A function  $f$  on  $T$  is *harmonic* (respectively, *superharmonic*, *subharmonic*, *biharmonic*) at  $v \in T$  if  $\Delta f(v) = 0$  (respectively,  $\Delta f(v) \leq 0$ ,  $\Delta f(v) \geq 0$ ,  $\Delta^2 f(v) = \Delta(\Delta f)(v) = 0$ ). A function  $f$  is said to be harmonic (respectively, superharmonic, subharmonic, biharmonic) on a set  $E$  of vertices if it is harmonic (respectively, superharmonic, subharmonic, biharmonic) at each  $v \in E$ . If  $E = T$ , we refer to  $f$  as harmonic (respectively, superharmonic, subharmonic, biharmonic).

A superharmonic function  $f$  is called a positive *potential* if it has no positive harmonic minorant, i.e. if  $h$  is harmonic and  $h \leq f$ , then  $h \leq 0$ . We call a tree *transient* (respectively, *recurrent*) if it has (respectively, does not have) positive potentials. These notions are equivalent to those that arise in connection with the random walk generated by the nearest-neighbor transition probability [13].

A tree is recurrent if and only if there are no nonconstant positive superharmonic functions (see Section 1.1 of [12]) and is transient if and only if  $G(v, w) < \infty$  for any pair of vertices  $v, w$ , where  $G$  is the Green function. Here

$$(1) \quad G(v, w) = \sum_{n=0}^{\infty} p^n(v, w),$$

which in terms of random walk, is the expected number of times a walk beginning at  $v$  visits  $w$ .

Arguing by induction on the length of a vertex  $v$ , it is straightforward to show that if  $g$  is any function on  $T$ , then there exists a function  $f$ , unique up to the addition of a harmonic function, such that  $\Delta f = g$ . In fact, such a function  $f$  can be chosen to be constant on siblings.

For  $n \in \mathbb{N}$ , denote by  $B(n)$  the closed ball  $\{v \in T : |v| \leq n\}$ . Given a function  $f$  defined on the sphere  $S(n) = B(n) \setminus B(n-1)$ , the *solution to the Dirichlet problem with boundary values  $f$  on  $S(n)$*  is a function  $h$  defined on  $B(n)$ , harmonic on  $B(n-1)$  and such that  $h = f$  on  $S(n)$ . The solution always exists and is unique (see Lemma 4.3 of [2]). We shall denote it by  $\mathcal{H}_n f$ .

### 3. STANDARD AND FLUX ON A RECURRENT TREE

Given a tree  $T$ , for  $v \in T$ , let  $[e, v] = [v_0 = e, v_1, \dots, v_n = v]$  be the unique geodesic path from  $e$  to  $v$ . Let  $\alpha_e = 1$  and for  $v \neq e$ , define

$$\alpha_v = \prod_{j=1}^n \frac{p(v_{j-1}, v_j)}{p(v_j, v_{j-1})},$$

the product of the forward probabilities divided by the product of the backward probabilities from  $e$  to  $v$ . Viewing  $\alpha$  as a measure on  $T$  and  $\alpha p$  as the measure

defined by

$$(\alpha p)_v = \sum_{w \in T} \alpha_w p(w, v),$$

a straightforward calculation shows that for all  $v \in T$ ,  $(\alpha p)_v = \alpha_v$ , that is,  $\alpha$  is a  $p$ -invariant measure [13].

By Theorem 6.1 of [13],

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} p^k = L,$$

a matrix all of whose rows are identical. Since the columns are constant,  $pL = L$ . If  $T$  is transient, then by (1) and (2), we have

$$L \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{\infty} p^k = \lim_{n \rightarrow \infty} \frac{1}{n} G.$$

Thus,  $L$  is identically zero.

**Definition 3.1.** If  $T$  is recurrent, we say that  $T$  is *null* if  $L$  is identically 0 and is *ergodic* otherwise.

It is well-known that if  $T$  is recurrent, then  $T$  is ergodic if and only if  $\sum_{v \in T} \alpha_v < \infty$  ([13], Theorem 6.9). In this case, each row of  $L$  is the unique  $p$ -invariant probability measure and so it is a non-zero multiple of  $\alpha$ . The entry in the column corresponding to  $v$  is  $\alpha_v / \sum_{w \in T} \alpha_w$  and represents the expected fraction of time a random walk which continues indefinitely spends at the vertex  $v$ .

The following proposition shows that null trees have a property in common with transient trees not shared by ergodic trees.

**Proposition 3.2.** *If  $T$  is transient, then  $\sum_{v \in T} \alpha_v = \infty$ .*

*Proof.* Arguing by contradiction, suppose that  $\sum_{v \in T} \alpha_v < \infty$ . For a function  $f$  on  $T$ , define  $\alpha f = \sum_{v \in T} \alpha_v f(v)$ , the integral of  $f$  with respect to the measure  $\alpha$ . If  $\chi_T$  is the function that is identically 1 on  $T$ , then  $\alpha \chi_T = \sum_{v \in T} \alpha_v$ , so our assumption is that  $\chi_T \in L^1(\alpha)$ .

For any kernel  $U$  on  $T$  (i.e.  $U$  a function on  $T \times T$  with values in  $[0, \infty]$ ), we define  $\alpha U$  to be the measure  $(\alpha U)_w = \sum_{v \in T} \alpha_v U(v, w)$ . From the  $p$ -invariance of  $\alpha$ , it follows by induction that for any  $n \geq 0$ ,  $\alpha p^n = \alpha$ . In particular,

$$(3) \quad \alpha p^n(\cdot, e) = \alpha_e = 1 \text{ for any } n \geq 0.$$

Consider the sequences of functions  $\{G_n\}$  and  $\{T_n\}$  defined on  $T$  by

$$G_n(v) = \sum_{k=0}^{n-1} p^k(v, e), \quad \text{and } T_n(v) = \frac{G_n(v)}{n}.$$

Then  $G_n(v)$  (respectively,  $T_n(v)$ ) is the expected number of times (respectively, the expected fraction of the time) a random walk which begins at  $v$  visits vertex  $e$  in the first  $n$  steps. Since  $G_n(v) \rightarrow G(v, e)$  as  $n \rightarrow \infty$  and, by the transience hypothesis,  $G$  is finite,  $T_n(v)$  converges to 0 as  $n \rightarrow \infty$ . Since  $0 \leq T_n(v) \leq \chi_T(v)$  for all  $n$  and

$v$ , and  $\chi_T \in L^1(\alpha)$ , it follows by the Dominated Convergence Theorem that  $\alpha T_n$  converges to 0 as  $n \rightarrow \infty$ . On the other hand, by linearity and (3), we obtain

$$\alpha T_n = \frac{1}{n} \sum_{k=0}^{n-1} \alpha p^k(\cdot, e) = 1,$$

and so  $\alpha T_n$  cannot converge to 0 as  $n \rightarrow \infty$ . This contradiction shows that  $\sum_{v \in T} \alpha_v = \infty$ .  $\square$

**Definition 3.3.** A function  $f$  is said to be of *finite type* if there exists  $M \in \mathbb{N}$  such that  $f$  is constant on  $S_v$  for each vertex  $v$  with  $|v| = M$ . A function  $f$  is of  $\Delta$ -*finite type* if its Laplacian is of finite type.

In the remainder of this paper, we shall assume that  $T$  is a recurrent tree.

In [12] we introduced the following concept of standard on a recurrent tree  $T$ .

**Definition 3.4.** A function  $H$  on  $T$  is a *standard* if, given any function  $h$  harmonic outside a finite set, there exists a function  $h_T$  harmonic on  $T$ , a unique real number  $\alpha$ , and a bounded function  $b$  such that  $h = \alpha H + h_T + b$ .

The above notion of standard was motivated by part (a) of Theorem 1.1 when  $\Omega = \mathbb{R}^2$ .

Our definition of standard was implicit in [6], where we proved that there is a unique function  $H$  satisfying the properties  $H \geq 0$ ,  $H$  is harmonic except at  $e$ , normalized by setting  $H(e) = 0$ ,  $\Delta H(e) = 1$ , and  $H$  constant on siblings. An explicit formula is given by

$$(4) \quad H(v) = \sum_{k=0}^{n-1} \epsilon_0(v) \epsilon_1(v) \cdots \epsilon_k(v),$$

where  $[e, v] = [v_0, \dots, v_n]$ ,  $\epsilon_0(v) = 1$ , and  $\epsilon_k(v) = \frac{p(v_k, v_{k-1})}{1-p(v_k, v_{k-1})}$ , for  $k = 1, \dots, n-1$ .

Moreover in [6], for each  $v \in T$ , we determined a function  $H_v$  satisfying the conditions  $H_v \geq 0$ ,  $H_v(v) = 0$ ,  $H_v$  harmonic except at  $v$ ,  $\Delta H_v(v) = 1$ , and  $\alpha_v H - H_v$  of finite type. Such a function is necessarily unique. Indeed, any two choices would differ by a harmonic function of finite type (in particular bounded, hence constant) vanishing at  $v$ .

For each  $v \in T$ , let  $K_v$  be a function constant on siblings such that  $\Delta K_v = H_v$ . Thus,  $K_v - \alpha_v K_e$  is of  $\Delta$ -finite type. Denote  $K_e$  by  $K$ . The function  $K$  is made unique by prescribing  $K(e) = K(w) = 1$  for each vertex  $w$  of length 1. The function  $K$  is the solution to the recurrence relation

$$K(v_{n+1}) = \frac{H(v_n) + K(v_n) - r_n K(v_{n-1})}{1 - r_n}, \text{ for } n \in \mathbb{N},$$

where  $r_n = p(v_n, v_{n-1})$ . Similarly, the functions  $K_v$  can be determined uniquely by adding analogous normalization conditions, and explicit formulas can be derived. However, such normalized functions  $K_v$  are not needed in this work.

In [12], we defined the *flux* of a function  $s$  superharmonic except for finitely many vertices by

$$\text{flux}(s) = \sup A_s,$$

where

$$A_s = \{\alpha \in \mathbb{R} : s - \alpha H \text{ has a harmonic minorant on } T\}$$

and the supremum is defined to be  $-\infty$  if  $A_s$  is the empty set. In that work many properties of the flux were derived. In particular,  $\text{flux}(s)$  is finite if and only if  $s$  has a minorant harmonic outside a finite set of vertices, in which case we say that  $s$  is *admissible*. Furthermore, we showed that for a function  $h$  harmonic outside a finite set, the constant  $\alpha$  that appears in Definition 3.4 is the flux of  $h$ .

In Theorem 4.2 of [12], it was shown that the flux of  $s$  can be calculated by means of the formula

$$\text{flux}(s) = \sum_{v \in T} \Delta s(v) \alpha_v.$$

We use this formula to extend the definition of flux as follows.

**Definition 3.5.** Let  $f$  be any function such that at least one of the quantities  $\sum_{v \in T} (\Delta f(v))^+ \alpha_v$  and  $\sum_{v \in T} (\Delta f(v))^- \alpha_v$  is finite, where for a function  $g$  on  $T$ ,  $g^+ = \max\{g, 0\}$  and  $g^- = \max\{-g, 0\} = g^+ - g$ . We define the *flux* of  $f$  by

$$\text{flux}(f) = \sum_{v \in T} \Delta f(v) \alpha_v.$$

We say that  $f$  is *admissible* if its flux exists and is finite.

**Theorem 3.6.** *Let  $f$  be a function on  $T$ .*

(a) *The function  $f$  is admissible if and only if there exist  $s_1$  and  $s_2$  admissible superharmonic on  $T$  such that  $f = s_1 - s_2$ .*

(b) *If  $f$  is bounded and admissible, then the flux of  $f$  is 0.*

(c) *The function  $f$  is not admissible and the flux of  $f$  is  $\pm\infty$  if and only if there exist  $s_1$  and  $s_2$  superharmonic on  $T$  such that  $f = s_1 - s_2$  and precisely one of the functions  $s_1$  and  $s_2$  is admissible.*

*Proof.* To prove (a), assume  $f$  is admissible. Let  $g_1$  and  $g_2$  be functions such that  $\Delta g_1 = -(\Delta f)^-$  and  $\Delta g_2 = -(\Delta f)^+$ . Then  $\Delta f = \Delta(g_1 - g_2)$ , so that  $f - g_1 + g_2$  is a harmonic function  $h$  on  $T$ . The functions  $s_1 = g_1 + h$  and  $s_2 = g_2$  are admissible superharmonic and such that  $f = s_1 - s_2$ . The converse follows immediately by the linearity of the flux.

If  $f$  is bounded and admissible, then by part (a),  $f = s_1 - s_2$  with  $s_1$  and  $s_2$  admissible superharmonic. Since  $s_1 - s_2$  is bounded, it follows that  $A_{s_1} = A_{s_2}$ , so  $\text{flux}(s_1) = \text{flux}(s_2)$ , proving (b).

Let  $f$  be a non-admissible function with flux  $\infty$ . Arguing as in the proof of part (a),  $f = s_1 - s_2$  with  $\Delta s_1 = -(\Delta f)^-$ ,  $\Delta s_2 = -(\Delta f)^+$ . Then  $s_1$  is admissible and  $s_2$  is not admissible. One argues similarly when  $f$  has flux  $-\infty$ . The converse follows from the linearity of the flux.  $\square$

In Theorem 5.6, we shall see to what extent we can eliminate the admissibility condition in part (b) of Theorem 3.6.

**Remark 3.7.** If  $T$  is an ergodic tree and  $f$  is bounded on  $T$ , then  $f$  is necessarily admissible and by part (b) of Theorem 3.6, the flux of  $f$  is zero. If  $T$  is null, then the function  $f$  which is 1 at the vertices of even length and  $-1$  at the vertices of odd length is bounded but not admissible.

**Remark 3.8.** Here we give a probabilistic interpretation of biharmonicity. Let  $h$  be harmonic on  $T$ . Consider a random walk that begins at vertex  $v$  and continues until the first time after time 0 that it visits  $e$ . Say you start with a fortune of 0

and each time you visit a vertex  $w$  you add  $h(w)$  to your fortune. Let  $B(v)$  be your expected fortune at the first time after time 0 that you visit  $e$ . Then

$$B(v) = \sum_{w \sim v} p(v, w)[B(w) + h(w)] = \sum_{w \sim v} p(v, w)B(w) + h(v),$$

so  $\Delta B(v) = -h(v)$ . Thus  $B$  is biharmonic. In particular if  $h$  is the constant function 1, then  $B(v)$  is the expected time that a walk which begins at  $v$  takes to return to  $e$ .

We can also give a probabilistic interpretation of the flux of  $B$  in case the tree is ergodic. Let  $\beta_v = \alpha_v/|\alpha|$ , where  $|\alpha| = \sum_{v \in T} \alpha_v$ . Then  $\beta_v$  represents the expected fraction of time a random walk spends at  $v$  if the walk goes on forever. Thus

$$-\frac{1}{|\alpha|} \text{flux}(B) = \sum_{v \in T} \beta_v h(v),$$

and this sum represents our expected long term fortune if we let the random walk go on forever.

#### 4. FUNCTIONS BIHARMONIC OUTSIDE A FINITE SET

In this section, motivated by the following theorem (Theorem 16 of [4]), we wish to explore to what extent this result holds on a recurrent tree.

**Theorem 4.1.** [4] *For a biharmonic function  $f$  defined outside a compact set in  $\mathbb{R}^n$ ,  $2 \leq n \leq 4$ , the following statements are equivalent.*

- (a) *The flux at infinity of  $\Delta f$  is 0.*
- (b) *There exist a biharmonic function  $B$  in  $\mathbb{R}^n$ , and a constant  $\alpha$  such that  $f - B - \alpha E_n$  is bounded near infinity, where  $E_n$  is the fundamental solution of Laplace's equation in  $\mathbb{R}^n$  given by*

$$E_n(x) = \begin{cases} -\log |x| & \text{if } n = 2, \\ |x|^{2-n} & \text{if } n = 3, 4. \end{cases}$$

- (c) *For some  $r_0 > 0$ , the mean value of  $\Delta f$  on the sphere centered at 0 of radius  $r$  is independent of  $r$  for all  $r \geq r_0$ .*

*Furthermore, if  $f$  is harmonic, so is  $B$  in (b).*

Since  $\mathbb{R}^3$  and  $\mathbb{R}^4$  are both spaces with potentials,  $\mathbb{R}^2$  is the only space in the above theorem that corresponds to the recurrent-tree setting. The analogue of  $E_2$  in our setting is the negative of the standard  $H$ .

We begin our investigation by considering the following general problem: Given a function  $f$  biharmonic outside a finite set, find a global biharmonic function  $B$  such that  $f - B$  has natural and useful properties, and such that, under some suitable normalization, the ensuing representation is unique up to the addition of a harmonic function. The following two theorems solve this problem in slightly different ways.

**Theorem 4.2.** *Let  $T$  be a recurrent tree. If  $f$  is a function on  $T$  biharmonic outside a finite set, then there exist  $B$  biharmonic on  $T$ , a unique constant  $\beta$ , and a function  $L$  of  $\Delta$ -finite type such that  $\Delta L(e) = 0$  and*

$$f = \beta K + B + L.$$

*The functions  $B$  and  $L$  in the above representation are unique up to the addition of a harmonic function and  $\beta$  is the flux of  $\Delta f$ .*

*Proof.* (Existence) Since  $\Delta f$  is harmonic outside a finite set, there exist constants  $a_1, \dots, a_n$  and vertices  $v_1, \dots, v_n$  such that  $\Delta^2 f = \sum_{k=1}^n a_k \delta_{v_k}$ . Then  $h_T = \Delta f - \sum_{k=1}^n a_k H_{v_k}$  has Laplacian vanishing everywhere, so it is a global harmonic function. Thus,  $\Delta f = h_T + \sum_{k=1}^n a_k \Delta K_{v_k}$ . Let  $b_T$  be a biharmonic function whose Laplacian is  $h_T$ . Then  $\Delta f = \Delta(b_T + \sum_{k=1}^n a_k K_{v_k})$ , so that  $g_T = f - b_T - \sum_{k=1}^n a_k K_{v_k}$  is a global harmonic function. Thus, setting  $B' = b_T + g_T$ ,  $\beta = \sum_{k=1}^n a_k \alpha_{v_k}$ , and  $L' = \sum_{k=1}^n a_k (K_{v_k} - \alpha_{v_k} K)$ , we obtain  $f = \beta K + B' + L'$ . By replacing  $L'$  with the function  $L$  obtained by subtracting from  $L'$  a global biharmonic function  $B''$  whose Laplacian is equal to the constant function  $\Delta L'(e)$ , we obtain  $f = \beta K + B' + B'' + L$ . Letting  $B = B' + B''$ , we obtain the desired representation of  $f$ .

(Uniqueness) Assume  $f = \beta_1 K + B_1 + L_1 = \beta_2 K + B_2 + L_2$ , with  $\beta_j \in \mathbb{R}$ ,  $B_j$  biharmonic on  $T$ , and  $L_j$  of  $\Delta$ -finite type such that  $\Delta L_j(e) = 0$ , for  $j = 1, 2$ . Taking the Laplacians, we obtain

$$(5) \quad \Delta f = \beta_j H + \Delta B_j + \Delta L_j, \quad j = 1, 2.$$

Since the functions  $\Delta L_1$  and  $\Delta L_2$  are bounded and harmonic outside a finite set, they have 0 flux. Moreover, since for  $j = 1, 2$ ,  $\Delta B_j$  is harmonic on  $T$ , its flux is 0 as well. Thus, the flux of  $\Delta f$  is equal to  $\beta_j$ ,  $j = 1, 2$ , so  $\beta_1 = \beta_2$ . Define  $B = B_1 - B_2$ , and  $L = L_1 - L_2$ . Then

$$(6) \quad B + L = 0.$$

Then,  $\Delta B + \Delta L = 0$ , where  $\Delta L$  is bounded. Thus,  $\Delta B$  must be bounded, and being globally harmonic, it must be a constant  $c$ . In particular,  $c + \Delta L = 0$  and so evaluation at  $e$  yields  $c = 0$ . Therefore  $B$  is a global harmonic function. Hence, by (6),  $L$  is also a global harmonic function, completing the proof of the uniqueness of the representation up to a harmonic function.  $\square$

Note that under ergodicity assumptions on the tree, the function  $L$  in the above representation has finite flux, since

$$\text{flux}(L) = \sum_{v \in T} \alpha_v \Delta L(v).$$

We now present a representation of a function biharmonic outside a finite set in the spirit of the representation provided in Theorem 4.2 but with a different normalization. This new representation will be used in Theorem 4.5 to obtain a characterization of a class of functions biharmonic outside a finite set of vertices. This will furnish a discrete version of Theorem 4.1.

**Theorem 4.3.** *Let  $T$  be a recurrent tree. If  $f$  is a function on  $T$  biharmonic outside a finite set, then there exist  $B$  biharmonic on  $T$ , and a function  $L$  of  $\Delta$ -finite type such that  $f = \text{flux}(\Delta f)K + B + L$  and the average of the distinct constant values of  $\Delta L$  on all balls of sufficiently large radius is 0. The functions  $B$  and  $L$  in the above representation are unique up to the addition of a harmonic function.*

*Proof.* (Existence) By Theorem 4.2,  $f = \beta K + B' + L'$  with  $B'$  biharmonic and  $L'$  of  $\Delta$  finite type, where  $\beta$  is the flux of  $\Delta f$ . Since  $\Delta L'$  has finite type, it attains only finitely many distinct values  $c_1, \dots, c_k$  outside all sufficiently large balls centered at  $e$ . Let  $L = L' - \gamma J$ , where  $\gamma = \frac{1}{k} \sum_{j=1}^k c_j$  and  $J$  is a biharmonic function on  $T$  whose Laplacian is identically 1. Then, by construction the average of the distinct



values of  $\Delta L$  on all sufficiently large balls centered at  $e$  is 0 and  $f = \beta K + B + L$ , where  $B = B' + \gamma J$ .

(Uniqueness) Assume  $f = \text{flux}(\Delta f)K + B_1 + L_1 = \text{flux}(\Delta f)K + B_2 + L_2$ , with  $B_j$  biharmonic on  $T$ , and  $L_j$  of  $\Delta$ -finite type such that the average of the distinct values of  $\Delta L_j$  on all balls centered at  $e$  of sufficiently large radius is 0,  $j = 1, 2$ . Then  $B_1 - B_2 = L_2 - L_1$  is globally biharmonic and of  $\Delta$ -finite type. Therefore its Laplacian is bounded and harmonic, hence is a constant, say  $C$ . Then  $\Delta L_2 = \Delta L_1 + C$ . Since the average of the distinct values of both  $\Delta L_1$  and  $\Delta L_2$  is 0, the constant  $C$  must be 0. Therefore  $L_2$  and  $L_1$  (and hence  $B_1$  and  $B_2$ ) differ by a global harmonic function, as desired.  $\square$

**Remark 4.4.** If  $f$  is harmonic outside a finite set, then its Laplacian is bounded and vanishes outside a finite set, and thus  $f$  is of  $\Delta$ -finite type and  $\Delta f$  has 0 flux. Then the function  $B$  in the above representation is necessarily harmonic. Indeed, since  $f$  admits the representation  $f = 0 + f$  with 0 biharmonic and  $f$  of  $\Delta$ -finite type whose value on spheres of sufficiently large radius is 0, by the uniqueness of the representation up to the addition of a harmonic function,  $B$  must be harmonic.

The remaining theorems of this section relate to Theorem 4.1.

**Theorem 4.5.** *Let  $T$  be a recurrent tree. For  $f$  biharmonic outside a finite set, the following statements are equivalent:*

- (a) *The flux of  $\Delta f$  is 0.*
- (b) *The function  $f$  admits the representation  $f = B + L$ , where  $B$  is biharmonic on  $T$  and  $L$  is of  $\Delta$ -finite type and such that the average value of the distinct values of  $\Delta L$  on sufficiently large balls is 0.*
- (c) *The sequence  $\{\mathcal{H}_n \Delta f(e)\}$  is bounded.*

*Proof.* (a) $\iff$ (b) follows immediately from Theorem 4.3.

(b) $\implies$ (c): Assume  $f = B + L$  as in (b). Then, by the harmonicity of  $\Delta B$ ,  $\mathcal{H}_n \Delta B(e) = \Delta B(e)$ , while  $\{\mathcal{H}_n \Delta L(e)\}$  is bounded, since  $\Delta L$  itself is bounded. Condition (c) follows immediately by additivity.

(c) $\implies$ (a): Assume (c) valid. By Theorem 4.3,  $f = \beta K + B + L$  with  $\beta$  equal to the flux of  $\Delta f$ ,  $B$  biharmonic and  $L$  of  $\Delta$ -finite type such that the average value of the distinct values of  $\Delta L$  on sufficiently large balls is 0. Since the sequences  $\{\mathcal{H}_n \Delta B(e)\}$ ,  $\{\mathcal{H}_n \Delta f(e)\}$ , and  $\{\mathcal{H}_n \Delta L(e)\}$  are all bounded, the same is true for the sequence  $\{\mathcal{H}_n \beta \Delta K(e)\} = \{\mathcal{H}_n \beta H(e)\}$ . However, the sequence  $\{\mathcal{H}_n H(e)\}$  is necessarily divergent to  $+\infty$ , or else the limit of  $\{\mathcal{H}_n H\}$  would be the least harmonic majorant  $h$  of  $H$ . Then,  $h - H$  would be a nonconstant positive superharmonic function, which is impossible since the tree is recurrent. Thus,  $\beta$  must be 0, as desired.  $\square$

The normalization condition in part (b) of Theorem 4.5 has the disadvantage of not having a probabilistic interpretation. In the next theorem, we impose a stronger restriction on the transition probabilities in order to obtain an improved version of condition (b). Of all our results, this theorem most closely resembles Theorem 4.1.

**Theorem 4.6.** *Let  $T$  be a recurrent tree such that  $p(v, v^-)$  is a constant  $q$  throughout the tree and let  $f$  be biharmonic outside a finite set. Then, the following statements are equivalent:*

- (a) *The flux of  $\Delta f$  is 0.*

(b) The function  $f$  admits the representation  $f = B_0 + L_0$ , where  $B_0$  is biharmonic on  $T$  and  $L_0$  is of  $\Delta$ -finite type and such that  $\mathcal{H}_n \Delta L_0(e) = 0$  for all  $n$  sufficiently large.

(c) The sequence  $\mathcal{H}_n \Delta f(e)$  is eventually constant.

For the proof we need several lemmas.

**Lemma 4.7.** *Let  $T$  be a tree such that  $p(v, v^-)$  is a constant  $q$  for each  $v \neq e$ . Fix  $n \in \mathbb{N}$  and let  $[v_0, \dots, v_n]$  be a geodesic path of length  $n$  and such that each vertex  $v_j$  has only the neighbors  $v_{j-1}$  and  $v_{j+1}$  for  $j = 1, \dots, n-1$ . Then the solution  $h$  to the Dirichlet problem on  $[v_0, \dots, v_n]$  with boundary values  $a_0$  at  $v_0$  and  $a_n$  at  $v_n$  is given by*

$$h(v_j) = a_0 - (a_0 - a_n) \left( \frac{1 - r^j}{1 - r^n} \right),$$

for  $j = 0, \dots, n$ , where  $r = \frac{q}{1-q}$ .

*Proof.* Fix  $j = 1, \dots, n-1$ . The harmonicity at  $v_j$  of the solution to the Dirichlet problem yields the linear second order recurrence relation

$$(1 - q)h(v_{j+1}) - h(v_j) + qh(v_{j-1}) = 0,$$

whose characteristic equation is  $(1 - q)x^2 - x + q = 0$ . Its roots are  $x = 1$  and  $x = \frac{q}{1-q} = r$ . Thus, the general solution is  $h(v_j) = A + Br^j$ , and from the boundary conditions  $h(v_0) = a_0$  and  $h(v_n) = a_n$ , we see that the constants  $A$  and  $B$  must satisfy the relations  $A + B = a_0$  and  $A + Br^n = a_n$ . Hence,  $A = a_0 - \frac{a_0 - a_n}{1 - r^n}$  and  $B = \frac{a_0 - a_n}{1 - r^n}$ . Therefore

$$h(v_j) = a_0 - (a_0 - a_n) \left( \frac{1 - r^j}{1 - r^n} \right),$$

which yields the result.  $\square$

**Lemma 4.8.** *Let  $T$  be a tree such that  $p(v, v^-)$  is a constant  $q$  for each  $v \neq e$ . Given any  $w \in T$  and  $n \in \mathbb{N}$  such that  $n > |w|$ , let  $h$  be a function defined on  $S_w \cap B(n)$ , harmonic on  $S_w \cap B(n-1)$ , and constant on  $S_w \cap S(n)$ . Then  $h$  is radial.*

*Proof.* For  $v \in T$  define the function

$$f(v) = A + B \left( \frac{q}{1-q} \right)^{|v|}$$

where the constants  $A$  and  $B$  are chosen so that  $f(w) = h(w)$  and  $f(v)$  is the constant value of  $h$  for  $v \in S_w \cap S(n)$ . Observe that  $f$  is radial and harmonic on  $S_w \cap B(n-1)$ . Thus, by the uniqueness of the solution to the Dirichlet problem,  $f = h$  on  $S_w \cap B(n)$ . Therefore,  $h$  is radial.  $\square$

**Lemma 4.9.** *Let  $T$  be a recurrent tree such that  $p(v, v^-)$  is a constant  $q$  for each  $v \neq e$ . Fix  $m \in \mathbb{N}$  and a vertex  $w$  of length  $m$ , and let  $n$  be an integer such that  $n \geq m$ . Then the value at  $e$  of the solution to the Dirichlet problem with boundary values the characteristic function of  $S_w \cap S(n)$  on  $S(n)$  is given by*

$$\frac{\prod_{j=0}^{m-1} p_j}{(1-q)^{m-1}},$$

where  $v_0 = e, v_1, \dots, v_m = w$  are the vertices in the path  $[e, w]$  with  $v_{j-1} \sim v_j$ , and  $p_j = p(v_j, v_{j+1})$ , for  $j = 1, \dots, m-1$ .

*Proof.* We first show that the result can be reduced to proving the case when  $n = m$ , by arguing inductively. By the radially property of the solution to the Dirichlet problem in Lemma 4.8, the problem can be simplified by looking at a tree that has the shape of an uneven comb, that is, the tree consisting of the path  $[v_0 = e, v_1, \dots, v_m = w]$  together with the paths  $\pi_j = [v_j, v_j^{j+1}, v_j^{j+2}, \dots, v_j^n]$ , with  $|v_j^k| = k$ ,  $k = j+1, \dots, n$ ,  $h(v_j^n) = 0$  for  $j = 0, \dots, m-1$ , and  $h(v_m^n) = 1$ . The paths  $\pi_0, \dots, \pi_{m-1}$  are the “teeth of the comb” and each path  $\pi_j$  has length  $n-j$ . See Figure 1 for a representation of the case  $m = 3$  and  $n = 4$ .

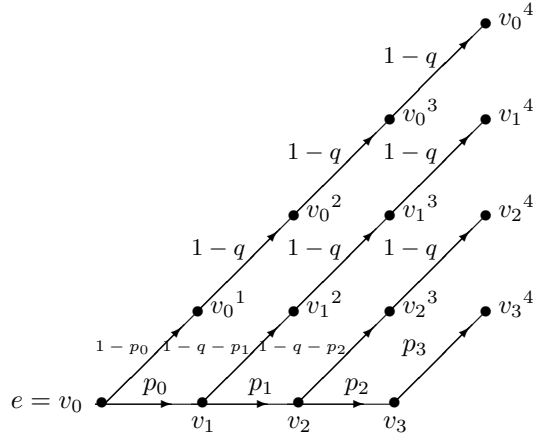


FIGURE 1. Case  $m = 3$  and  $n = 4$ .

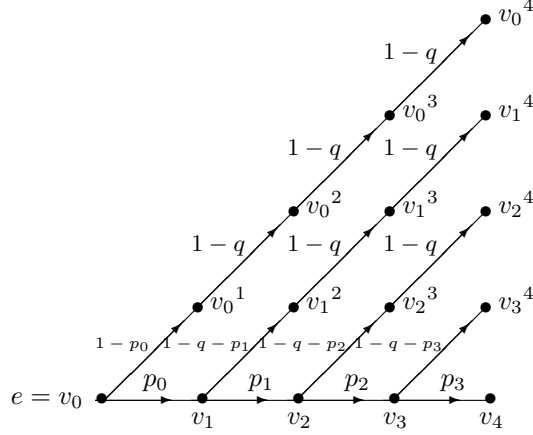
Assume  $n > m$  and let us denote by  $h^{(m-1)}$  and  $h^{(m)}$  the solution to the Dirichlet problem on  $B(n)$  with boundary values the characteristic function of  $S_{v_{m-1}} \cap S(n)$  and  $S_{v_m} \cap S(n)$ , respectively. Then  $h^{(m-1)}(e) = h^{(m)}(e) + \tilde{h}^{(m)}(e)$ , where  $\tilde{h}^{(m)}$  is the solution to the Dirichlet problem with values 0 at  $v_j^n$  for  $j \neq m-1$  and 1 at  $v_{m-1}^n$ . Applying the formula to be proved to the two solutions to the Dirichlet problems for  $B(n)$ , we obtain

$$\begin{aligned} h^{(m-1)}(e) &= \frac{\prod_{j=0}^{m-1} p_j}{(1-q)^{m-1}} + \frac{\left(\prod_{j=0}^{m-2} p_j\right) (1-q-p_{m-1})}{(1-q)^{m-1}} \\ &= \frac{\prod_{j=0}^{m-2} p_j}{(1-q)^{m-1}} (p_{m-1} + 1 - q - p_{m-1}) = \frac{\prod_{j=0}^{m-2} p_j}{(1-q)^{m-2}}. \end{aligned}$$

Thus, if the formula holds for  $m$ , it also holds for  $m-1$ . Consequently, if we show that the formula holds for  $m=n$ , then arguing inductively on  $n-m$ , we obtain the formula for each  $m < n$ .

From now on, assume that  $m = n$ . See Figure 2 for an illustration of the case  $m = n = 4$ .

To simplify the notation, let us set  $a_j^k = h^{(m)}(v_j^k)$  and  $a_j = h^{(m)}(v_j)$ , for  $j = 0, \dots, m-1$ ,  $k = j+1, \dots, m$ . Noting that  $a_{m-j} = 0$  and setting  $r = \frac{q}{1-q}$ , by

FIGURE 2. Case  $m = n = 4$ .

Lemma 4.7, we have

$$(7) \quad a_j^k = a_j \left( 1 - \frac{1 - r^{k-j}}{1 - r^{m-j}} \right).$$

The harmonicity condition at  $v_0 = e$  yields

$$p_0 a_1 + (1 - p_0) a_0^1 = a_0.$$

Hence, using (7) for  $j = 0$  and  $k = 1$ , we obtain

$$(8) \quad \left[ 1 - (1 - p_0) \left( 1 - \frac{1 - r}{1 - r^m} \right) \right] a_0 - p_0 a_1 = 0.$$

The harmonicity condition at  $v_j$ , for  $1 \leq j \leq m - 2$ , yields

$$q a_{j-1} + p_j a_{j+1} + (1 - q - p_j) a_j^{j+1} = a_j.$$

Thus, using (7) for  $k = j + 1$ , we get

$$q a_{j-1} + p_j a_{j+1} + (1 - q - p_j) a_j \left( 1 - \frac{1 - r}{1 - r^{m-j}} \right) = a_j,$$

whence

$$(9) \quad -q a_{j-1} + \left[ 1 - (1 - q - p_j) \left( 1 - \frac{1 - r}{1 - r^{m-j}} \right) \right] a_j - p_j a_{j+1} = 0.$$

Finally, the harmonicity condition at  $v_{m-1}$  yields

$$(10) \quad q a_{m-2} + p_{m-1} = a_{m-1}.$$

Set  $c_0 = 1 - (1 - p_0) \left( 1 - \frac{1-r}{1-r^m} \right)$ , and for  $1 \leq j \leq m - 2$ , let

$$c_j = 1 - (1 - q - p_j) \left( 1 - \frac{1 - r}{1 - r^{m-j}} \right).$$

Combining (8), (9), and (10), we obtain a system of  $m$  linear equations in the unknowns  $a_0, a_1, \dots, a_{m-1}$ , which can be described as the matrix equation

$$\begin{bmatrix} c_0 & -p_0 & 0 & \dots & \dots & 0 \\ -q & c_1 & -p_1 & 0 & \dots & 0 \\ 0 & -q & c_2 & -p_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -q & c_{m-2} & -p_{m-2} \\ 0 & \dots & \dots & 0 & -q & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_{m-1} \end{bmatrix}.$$

Using Cramer's rule to find  $a_0$ , and expanding the determinant in the numerator about the first column, we obtain

$$a_0 = \frac{p_0 p_1 \dots p_{m-1}}{\det A},$$

where  $A$  is the coefficient matrix.

For each  $j = 0, \dots, m-2$ , define  $d_j$  as the number obtain from  $c_j$  by replacing  $p_j$  with 0. Then  $d_j$  is given by

$$(11) \quad d_j = \begin{cases} \frac{1-r}{1-r^m} & \text{if } j = 0, \\ \frac{1-r^{m-j+1}}{(1+r)(1-r^{m-j})} & \text{if } j = 1, \dots, m-2. \end{cases}$$

In addition, we have

$$(12) \quad c_j = d_j + p_j \left( \frac{r - r^{m-j}}{1 - r^{m-j}} \right), \quad \text{for } j = 0, \dots, m-2.$$

We claim that row reducing  $A$  by using the diagonal entries in turn from the bottom right hand corner upward yields a lower-triangular matrix in which the diagonal entries are  $d_j$ . To prove the claim, it suffices to check the conditions

$$\begin{aligned} -qp_{m-2} + c_{m-2} &= d_{m-2}, \quad \text{and} \\ -\frac{q}{d_j} p_{j-1} + c_{j-1} &= d_{j-1}, \quad j = 1, \dots, m-2, \end{aligned}$$

and these follow easily from (11) and (12).

Thus  $\det A$  equals the product of the  $d_j$ 's. Recalling that  $q = \frac{r}{1+r}$  and using (11), we obtain

$$\begin{aligned} \prod_{j=0}^{m-2} d_j &= \left( \frac{1-r}{1-r^m} \right) \frac{\prod_{j=1}^{m-2} (1-r^{m-j+1})}{(1+r)^{m-2} \prod_{j=1}^{m-2} (1-r^{m-j})} \\ &= \frac{1-r}{(1+r)^{m-2} (1-r^2)} = \frac{1}{(1+r)^{m-1}} = (1-q)^{m-1}. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Theorem 4.6.* (a)  $\implies$  (c): As observed in Theorem 4.5,  $f = B + L$  with  $B$  biharmonic on  $T$  and  $L$  of  $\Delta$ -finite type. Then  $\Delta B$  is globally harmonic, and thus,  $\mathcal{H}_n \Delta B(e)$  is precisely the constant  $\Delta B(e)$ . On the other hand,  $\Delta L$  is of finite type and its values are constant on sectors determined by vertices of sufficiently large length. By linearity, it suffices to show the result when  $\Delta L$  restricted to  $S(n)$  is the characteristic function  $\chi$  of the intersection of  $S(n)$  with the sector determined by a fixed vertex of length  $m$ , where  $m \leq n$ . By Lemma 4.9,  $\mathcal{H}_n \chi(e)$  is a number which is independent of  $n$ , for all  $n \geq m$ .

(c)  $\implies$  (b): By Theorem 4.5, there exist  $B$  biharmonic on  $T$ ,  $L$  of  $\Delta$ -finite type such that the average of the distinct values of  $\Delta L$  on sufficiently large balls is 0 and  $f = B + L$ . Let  $J$  be a biharmonic function whose Laplacian is identically 1 and let  $\gamma = \mathcal{H}_n \Delta L(e)$  for  $n$  sufficiently large. Such a value is independent of  $n$  for  $n$  large because this is true for both  $\mathcal{H}_n \Delta f(e)$  (by assumption) and  $\mathcal{H}_n \Delta B(e) = \Delta B(e)$  (since  $\Delta B$  is harmonic). Set  $B_0 = B + \gamma J$  and  $L_0 = L - \gamma J$ . Then,  $B_0$  is biharmonic on  $T$  and  $L_0$  satisfies the desired requirements.

(b)  $\implies$  (a): Since  $\Delta B_0$  is harmonic and  $\Delta L_0$  is bounded and harmonic outside a finite set, both  $\Delta B_0$  and  $\Delta L_0$  have 0 flux. By additivity, it follows that the flux of  $\Delta f$  is 0.  $\square$

We end the section by giving an example which shows that in the recurrent tree setting, even in the restrictive case of constant inward probabilities, there is no analogue of part (b) of Theorem 4.1.

**Example 4.10.** Let  $T$  be the homogeneous tree of degree 2, which we identify with  $\mathbb{Z}$ , where  $p(n, n+1) = \frac{1}{2}$  for all  $n \in \mathbb{Z}$ . Then the biharmonic functions are the polynomials of degree no greater than 3 and the standard is the function  $H(n) = |n|$ , for  $n \in \mathbb{Z}$ . The function  $f$  defined by

$$f(n) = \begin{cases} n^3 + 2n^2 & \text{if } n \geq 0, \\ n^3 & \text{if } n < 0. \end{cases}$$

is biharmonic off  $\{-1, 1\}$  and  $\Delta f$  has zero flux. Yet there exist no biharmonic function on  $\mathbb{Z}$  and no multiple of  $H$  which subtracted from  $f$  could yield a bounded function. In the context of Theorem 4.6, we may let  $B(n) = n^3 + n^2$  and

$$L(n) = \begin{cases} n^2 & \text{if } n \geq 0, \\ -n^2 & \text{if } n < 0. \end{cases}$$

## 5. PRINCIPAL VALUE OF THE FLUX

In this section, our goal is to determine under what conditions functions biharmonic outside a finite set exhibit properties similar to admissible superharmonic functions, i.e. have finite flux. The example below, however, shows that even in the ergodic case global biharmonic functions need not have finite flux. For this reason, we introduce the notion of principal value of the flux.

**Definition 5.1.** Given a function  $f$  on  $T$ , define the *principal value of the flux* of  $f$  to be

$$\text{P.V. flux}(f) = \text{P.V.} \sum_{v \in T} \alpha_v \Delta f(v) := \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{v \in S(k)} \alpha_v \Delta f(v),$$

provided that this limit exists in the extended sense (i.e. the limit is either finite or  $+\infty$  or  $-\infty$ ).

Note that this definition agrees with the definition of flux in Definition 3.5 if the flux exists in the extended sense.

The function  $L$  in Example 4.10 has P.V. flux equal to 0, but the flux of  $L$  is not defined in the extended sense. In the following example, we give an ergodic tree and a global biharmonic function with no flux but with finite P.V. flux. Theorem 5.3 will show that this is not accidental.

**Example 5.2.** Let  $T$  be  $\mathbb{Z}$  with  $p(n, n + \operatorname{sgn} n) = 1/4$ ,  $p(n, n - \operatorname{sgn} n) = 3/4$ , for  $n \neq 0$ , and  $p(0, \pm 1) = 1/2$ . Let  $B$  be a biharmonic function such that  $h = \Delta B$  is given by  $h(n) = 3^n - 1$  and  $h(-n) = -h(n)$  for  $n \geq 0$ . Note that for  $n \neq 0$ ,  $\alpha_n = \frac{2}{3^{|n|}}$ , and  $\alpha_0 = 1$ . The P.V. flux of  $B$  is clearly 0 but, since  $h(n)\alpha_n$  is approximately  $2(\operatorname{sign} n)$ , the flux of  $B$  is not defined.

**Theorem 5.3.** *Let  $T$  be a homogeneous tree of degree  $t + 1$  with radial transition probabilities.*

(a) *If  $T$  is ergodic, then a function biharmonic outside a finite set has finite P.V. flux if and only if the flux of its Laplacian is 0.*

(b) *Let  $f$  be biharmonic on  $T$ . If  $T$  is ergodic, then  $f$  has finite P.V. flux. If  $T$  is non-ergodic, then  $f$  has finite P.V. flux if and only if  $\Delta f(e) = 0$ .*

*Proof.* To prove (a), we shall first show that if  $B$  is biharmonic on  $T$ , then P.V. flux( $B$ ) =  $\Delta B(e)M$ , where  $M = \sum_{v \in T} \alpha_v < \infty$  since  $T$  is ergodic. For each nonnegative integer  $n$  let  $c_n$  denote the number of vertices of length  $n$ , so that  $c_0 = 1$  and for  $n > 0$ ,  $c_n = (t + 1)t^{n-1}$ . By the radially assumption on the probabilities, the function  $v \in T \mapsto \alpha_v$  depends only on the length of  $v$ , and so we may set  $\alpha_n = \alpha_v$  for  $|v| = n$ . Then

$$M = \sum_{v \in T} \alpha_v = \sum_{n=0}^{\infty} \sum_{|v|=n} \alpha_v = \sum_{n=0}^{\infty} c_n \alpha_n.$$

Let  $h = \Delta B$  which is harmonic on  $T$ . Then, by the Mean Value Property,  $h(e) = \frac{1}{c_n} \sum_{|v|=n} h(v)$ . Thus

$$\sum_{|v|=n} \alpha_v h(v) = \alpha_n \sum_{|v|=n} h(v) = \alpha_n c_n h(e).$$

Hence

$$\text{P.V.} \sum_{v \in T} \alpha_v h(v) = h(e) \sum_{n=0}^{\infty} \alpha_n c_n = h(e)M.$$

Consequently,

$$(13) \quad \text{P.V. flux}(B) = \text{P.V.} \sum_{v \in T} \alpha_v \Delta B(v) = \Delta B(e)M,$$

which is finite.

Next assume  $f$  is biharmonic outside a finite set. Then  $f = \text{flux}(\Delta f)K + B + L$ , for some biharmonic function  $B$  and a function  $L$  of  $\Delta$ -finite type. Then  $\Delta L$  is bounded, so P.V. flux( $L$ ) =  $\sum_{v \in T} \alpha_v \Delta L(v)$  is convergent.

We now show that  $K$  has infinite flux. By definition of flux and the radially assumption on the tree, we need to show that  $\sum_{v \in T} \alpha_v H(v) = \sum_{n=0}^{\infty} \alpha_n c_n H_n$  is divergent, where  $H_n = H(v)$  for  $|v| = n$ . Letting  $r_j = p(v_j, v_{j-1})$  for each  $j \in \mathbb{N}$ , observe that  $p_j = p(v_j, v_{j+1}) = \frac{1-r_j}{t}$ , and  $p_0 = \frac{1}{t+1}$ , so that, for  $n \in \mathbb{N}$ , we have

$$c_n \alpha_n = (t+1)t^{n-1} \frac{(1-r_1) \cdots (1-r_{n-1})}{(t+1)r_1 \cdots r_n t^{n-1}} = \frac{(1-r_1) \cdots (1-r_{n-1})}{r_1 \cdots r_n}.$$

Thus, using (4), we obtain

$$c_n \alpha_n H_n = c_n \alpha_n \left( 1 + \sum_{k=1}^{n-1} \prod_{j=1}^k \frac{r_j}{(1-r_j)} \right) > c_n \alpha_n \prod_{j=1}^{n-1} \frac{r_j}{(1-r_j)} = \frac{1}{r_n} \geq 1.$$

Thus, the series  $\sum_{n=0}^{\infty} c_n \alpha_n H_n$  diverges.

Consequently, by the additivity of the flux, the P.V. flux of  $f$  is finite if and only if the flux of  $\Delta f$  is 0.

Let us now prove (b). In the ergodic case, the result follows from part (a) and the fact that every harmonic function has 0 flux. In the non-ergodic case the result follows by applying (13) using the fact that  $M = \infty$ .  $\square$

We now give the notion of  $*$ -ergodicity on a tree.

**Definition 5.4.** A tree  $T$  is said to be  $*$ -ergodic if

$$\lim_{n \rightarrow \infty} \sum_{|v|=n} \alpha_v = 0.$$

Clearly, an ergodic tree is  $*$ -ergodic, but the converse is false, as shown in the following example, in which we also describe a bounded non-admissible function with zero P.V. flux. This leads to the characterization of  $*$ -ergodic trees in Theorem 5.6 (to be compared with Theorem 3.6).

**Example 5.5.** Consider  $T = \mathbb{Z}$  endowed with transition probabilities

$$p(n, n+1) = p(-n, -n-1) = \frac{n}{2n+1}$$

for  $n \geq 1$ , and  $p(0, 1) = p(0, -1) = \frac{1}{2}$ . Then, a straightforward computation shows that for  $n \geq 1$ ,  $\alpha_n = \alpha_{-n} = \frac{2n+1}{2n(n+1)}$ , so that  $\alpha_{-n} + \alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ , while  $\sum \alpha_n = \infty$ . Thus,  $T$  is  $*$ -ergodic but not ergodic.

Now consider  $f(n) = \frac{1+(-1)^n}{2}$ , for  $n \in \mathbb{Z}$ . Then  $\Delta f(n) = (-1)^{n+1}$ , so the flux of  $f$  would be

$$-1 + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{2n(n+1)}$$

which converges to 0 but is not absolutely convergent. Thus  $f$  is bounded but not admissible. Of course, the P.V. flux of  $f$  is zero.

We now use the P.V. flux to characterize  $*$ -ergodicity.

**Theorem 5.6.** A tree  $T$  is  $*$ -ergodic if and only if every bounded function on  $T$  has vanishing P.V. flux.

*Proof.* Assume  $T$  is  $*$ -ergodic. Let  $f$  be bounded on  $T$  with  $\|f\|_{\infty} = M$ . By formula (4.3) of [12],

$$\begin{aligned} \text{P.V. flux}(f) &= \lim_{n \rightarrow \infty} \sum_{|v| \leq n-1} \alpha_v \Delta f(v) = \lim_{n \rightarrow \infty} \sum_{|v|=n} (f(v) - f(v^-)) \alpha_v p(v, v^-) \\ &\leq \lim_{n \rightarrow \infty} \sum_{|v|=n} 2M \alpha_v = 0. \end{aligned}$$

On the other hand, arguing as above using the function  $-f$ , we also obtain that  $\text{P.V. flux}(-f) \leq 0$ . Hence,  $\text{P.V. flux}(f) = 0$ .



To prove the converse, assume  $T$  is not  $*$ -ergodic. Then, there exist  $\epsilon > 0$  and a sequence of positive integers  $\{n_i\}_{i \in \mathbb{N}}$  such that  $n_i \geq n_{i-1} + 3$  for all  $i \in \mathbb{N}$  and  $\sum_{|v|=n_i} \alpha_v > \epsilon$ . Let  $A = \bigcup_{i \in \mathbb{N}} S(n_i)$  and define  $f = -\chi_A$ , where  $\chi_A$  is the characteristic function of  $A$ . Then  $\Delta f(v) = 1$  for all vertices  $v$ ,  $|v| = n_i$ ,  $i \in \mathbb{N}$ . Then

$$\sum_{|v| \leq n_i} \Delta f(v) \alpha_v - \sum_{|v| < n_i} \Delta f(v) \alpha_v > \epsilon.$$

Thus, the sequence  $\left\{ \sum_{|v| \leq n_i} \Delta f(v) \alpha_v \right\}_{i \in \mathbb{N}}$  is not Cauchy, and hence, it is not convergent. Therefore

$$\text{P.V. flux}(f) = \lim_{n \rightarrow \infty} \sum_{|n| \leq v} \Delta f(v) \alpha_v$$

cannot exist. □

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