CARLESON AND VANISHING CARLESON MEASURES ON RADIAL TREES

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Abstract. We extend a discrete version of an extension of Carleson's theorem proved in [5] to a large class of trees T that have certain radial properties. We introduce the geometric notion of s-vanishing Carleson measure on such a tree T (with $s \geq 1$) and give several characterizations of such measures. Given a measure σ on T and $p \geq 1$, let $L^p(\sigma)$ denote the space of functions g defined on T such that $|g|^p$ is integrable with respect to σ and let $L^p(\partial T)$ be the space of functions f defined on the boundary of T such that $|f|^p$ is integrable with respect to the representing measure of the harmonic function 1. We prove the following extension of the discrete version of a classical theorem in the unit disk proved by Power.

A finite measure σ on T is an s-vanishing Carleson measure if and only if for any real number $p > 1$, the Poisson operator $P : L^p(\partial T) \to L^{sp}(\sigma)$ is compact.

Characterizations of weak type for the case $p = 1$ and in terms of the tree analogue of the extended Poisson kernel are also given. Finally, we show that our results hold for homogeneous trees whose forward probabilities are radial and whose backward probabilities are constant, as well as for semihomogeneous trees.

1. Introduction

For $s \geq 1$, a positive measure σ on the open unit disk $\mathbb{D}$ in the complex plane is said to be an *s-Carleson measure* if there exists $C > 0$ such that for each $\theta_0 \in \mathbb{R}$ and $h \in (0, 1)$,

$$\sigma(S_{\theta_0}(h)) \leq Ch^s,$$

where

$$S_{\theta_0}(h) := \{re^{i\theta} : 1 - h \leq r < 1, |\theta - \theta_0| \leq h/2\}.$$

In [2], Carleson proved that a positive measure σ is 1-Carleson if and only if for each $p > 1$, the Poisson operator P, which associates to each function $f \in L^p(\partial\mathbb{D})$ its Poisson integral Pf, is bounded from $L^p(\partial\mathbb{D})$ to $L^p(\sigma)$, where $g \in L^p(\sigma)$ means

$$\int_{\mathbb{D}} |g(z)|^p \, d\sigma(z) < \infty.$$

In [3], Carleson obtained a similar characterization of such measures in terms of the boundedness of the inclusion map from the Hardy space H^p $(0 < p < \infty)$ into $L^p(\sigma)$. In [7], Duren extended the latter result by showing that for $s \geq 1$ and

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$0 < p < \infty$, σ is an s-Carleson measure if and only if for all $f \in H^p$,

$$\int_{\mathbb{D}} |f(z)|^{sp}\, d\sigma(z) \leq C\|f\|_{H^p}^{sp}.$$

A positive measure σ on $\mathbb{D}$ is said to be an *s-vanishing Carleson measure* if

$$\lim_{h\to 0} \sup_{\theta\in\mathbb{R}} \frac{\sigma(S_\theta(h))}{h^s} = 0.$$

The 1-vanishing Carleson measures are commonly referred to as vanishing Carleson measures.

For $z \in \mathbb{D}$ and $w \in \overline{\mathbb{D}}$, the extended Poisson kernel at (z, w) is defined by

$$\mathcal{P}(z, w) = \frac{1 - |z|^2}{|1 - \overline{z}w|^2}.$$

Note that the formula for $\mathcal{P}(z, w)$ coincides with that of the ordinary Poisson kernel when $|w| = 1$.

In [11], Theorem 8.2.5., vanishing Carleson measures were characterized in terms of the compactness of the Poisson operator and of the above extension of the Poisson kernel $\mathcal{P}$.

Theorem 1.1. [11] *For a positive measure σ on $\mathbb{D}$, the following conditions are equivalent:*

(a) *σ is a vanishing Carleson measure.*

(b) *For $1 < p < \infty$, the Poisson operator P is compact as a mapping from $L^p(\partial\mathbb{D})$ into $L^p(\sigma)$.*

(c) $\displaystyle\lim_{|z|\to 1} \int_{\mathbb{D}} \mathcal{P}(z, w)\, d\sigma(w) = 0.$

The equivalence of (a) and (b) first appeared in [10] (Theorem 2).

In [5], motivated by the important role Carleson measures play in analytic function space theory and operator theory, we introduced the notion of s-Carleson measure on an isotropic homogeneous tree and gave several characterizations of such measures. In particular, we proved a discrete version of a characterization of s-Carleson measures due to Duren ([7]). Carleson-type measures in a discrete setting have previously been considered in [1].

In this work we define what are *radial trees*, and we extend the definitions and the theorems derived in [5] to radial trees satisfying some additional axioms. Furthermore, in this more general setting we develop a discrete notion of s-vanishing Carleson measure (where $s \geq 1$) and provide several characterizations of such measures, some of which are the discrete analogues of Theorem 1.1 in the special case $s = 1$. We conclude the paper by showing that the results given in the present work hold in particular on non isotropic homogeneous trees whose forward probabilities are radial and whose backward probabilities are constant, as well as on semi-homogeneous trees.

We are not aware of any references involving vanishing Carleson measures in a discrete setting, nor of any results similar to Theorem 1.1 for s-Carleson measures in the classical setting of the unit disk for the case $s > 1$.

2. Preliminaries on trees

By a *tree* T we mean a locally finite, connected, and simply-connected graph, which, as a set, we identify with the collection of its vertices. Two vertices u and

v are called *neighbors* if there is an edge connecting them, and we use the notation $u \sim v$. The *degree* of a vertex is the number of its neighbors. The tree is called *homogeneous* if all its vertices have the same degree. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \dots]$ such that $v_k \sim v_{k+1}$, for all k. It is a *geodesic path* if in addition $v_{k-1} \neq v_{k+1}$, for all k. For any pair of vertices u, v there is a unique geodesic path from u to v, which we denote by $[u, v]$.

Given a tree T rooted at e and a vertex $u \in T$, a vertex v is called a *descendant* of u if u lies in the unique path from e to v. The vertex u is then called an *ancestor* of v. Given a vertex $v \neq e$, we denote by v^- the unique neighbor which is an ancestor of v. For $v \in T$, The set S_v consisting of v and all its descendants is called the *sector* determined by v. The sectors are the sets of vertices that will play the role of Carleson squares in the tree setting.

Define the *length* of the finite path $[u_0, u_1, \dots, u_n]$ to be n. The *distance*, $d(u, v)$, between vertices u and v is the length of the geodesic path $[u, v]$.

The tree T is a metric space under the distance d. Fixing e as the root of the tree, we define the *length* of a vertex v, by $|v| = d(e, v)$.

The *boundary* ∂T of T is the set of infinite geodesic paths ω of the form $[e = \omega_0, \omega_1, \omega_2, \dots]$. We denote by $[e, \omega)$ the set of vertices in the path ω. Then, ∂T is a compact space under the topology generated by the sets

$$I_v = \{\omega \in \partial T : v \in [e, \omega)\},$$

which yields a compactification of T. Clearly, $\partial T = I_e$. Furthermore, for any $n \in \mathbb{N}$, ∂T is the disjoint union of the sets I_v over the vertices v of length n. Under this topology, $\partial S_v = I_v$ for each $v \in T$.

For $v \in T$ with $0 \leq n \leq |v|$, define v_n to be the vertex of length n in the path $[e, v]$.

Define a partial order $\leq$ on $T \cup \partial T$ as follows: For $v \in T$ and $x \in T \cup \partial T$, $v \leq x$ if $v \in [e, x]$. Since $T \cup \partial T$ has e as the minimum, for any $x, y \in T \cup \partial T$ the greatest lower bound of x and y is well defined. We denote this greatest lower bound by $x \wedge y$.

A nearest-neighbor transition probability on the vertices of T is a function $p : T \times T \to [0, 1]$ such $p(v, u) > 0$ if and only if $v \sim u$, and $\sum_{v \sim u} p(v, u) = 1$ for each vertex v. The tree is called *isotropic* if for every vertex u, the transition probabilities $p(u, v)$ all equal the reciprocal of the degree of u.

By a *function on a tree* we mean a real-valued function on the set of its vertices.

The Laplacian operator Δ is defined as the averaging operator minus the identity operator; that is, for a function f on T,

$$\Delta f(v) = \sum_{w \sim v} p(v, w) f(w) - f(v),\ v \in T.$$

A function on T is harmonic at v if $\Delta f(v) = 0$, and it is harmonic on T if it is harmonic at every $v \in T$. It is superharmonic on T if $\Delta f(v) \leq 0$ for all $v \in T$.

For $u, v \in T$, define $F(u, v)$ to be the probability that the associated random walk starting at u hits v in positive time, and $G(u, v)$ the expected number of visits to vertex v by the random walk which begins at u. The tree T is called *transient* provided the Green function G is finite at least at one (hence necessarily at every) pair of vertices. This occurs if and only if $F(v, v) < 1$ for at least one (and so necessarily at every) vertex v. Transience can be shown to be equivalent to the existence of a positive superharmonic, nonharmonic function on T. In the transient case, we have the formulas

$$G(v,w) = \begin{cases} (1-F(v,v))^{-1} & \text{if } v = w, \\ F(v,w)G(w,w) & \text{if } v \neq w. \end{cases} \tag{1}$$

The function F is multiplicative in the sense that if $[v_0, v_1, \ldots, v_n]$ is the geodesic from v_0 to v_n, then $F(v_0, v_n) = \prod_{k=0}^{n-1} F(v_j, v_{j+1})$.

We shall assume henceforth that T is a transient tree.

For $v \in T$, $\omega \in \partial T$, denote the value of the Poisson kernel at (ω, v) by $P_\omega(v)$. Every positive harmonic function on T can be written as

$$P\mu(\cdot) := \int_{\partial T} P_\omega(\cdot)\, d\mu(\omega)$$

for a unique Borel measure μ on ∂T.

The Poisson kernel is known to be related to the Green function by means of the formula

$$P_\omega(v) = \frac{G(v, v\wedge\omega)}{G(e, v\wedge\omega)}.$$

Using (1), we see that the Poisson kernel can be written in terms of F as follows:

$$P_\omega(v) = \begin{cases} \frac{F(v,v\wedge\omega)}{F(e,v\wedge\omega)} & \text{if } v\wedge\omega \neq v, \\ \frac{1}{F(e,v)} & \text{if } v\wedge\omega = v \neq e, \\ 1 & \text{if } v = e. \end{cases}$$

For a general reference on the potential theory on trees, see [4].

In the following section, we shall consider a class of trees with radial properties and shall give axioms that will allow us to extend to such trees the theory of Hardy spaces and of Carleson measures developed in [5].

3. Radial trees

A tree is termed *radial* if for all neighboring vertices v, w, the transition probabilities $p(v,w)$ depend only on $|v|$ and $|w|$. This condition places a restriction on the degree of each vertex. Indeed, if $w^- = v$ and q is the number of forward neighbors of v, then the total forward probability is $q\,p(v,w) = 1 - p(v,v^-)$, so $q = (1-p(v,v^-))/p(v,w)$. Thus the degree of vertex v is $1 + \frac{1-p(v,v^-)}{p(v,w)}$, and so it depends only on $|v|$.

For each $k \geq 0$, let q_k denote the number of forward neighbors of any vertex v with $|v| = k$. For each $n \geq 1$, the number of vertices of T of length n is

$$c_n = \prod_{k=0}^{n-1} q_k. \tag{2}$$

Let m be the probability measure on ∂T for which

$$m(I_v) = \begin{cases} 1 & \text{if } v = e, \\ \frac{1}{c_{|v|}} & \text{if } v \neq e. \end{cases}$$

If μ is absolutely continuous with respect to m with density function f, we shall adopt the notation Pf instead of $P\mu$. In particular, m is the representing measure of the harmonic function 1. We shall refer to it as the normalized Lebesgue measure on ∂T.

In addition to assuming that the tree is radial and transient, we shall suppose that there exist constants $q, C_1, C_2 > 0$ and $\delta_1, r \in (0,1)$ such that the following axioms hold:

A1 $$2 \le q_j \le q \text{ for each } j;$$

A2 $$P_\omega(v) \le C_1(m(I_{v\wedge\omega}))^{-1} r^{|v|-|v\wedge\omega|}, \text{ for all } v \in T, \omega \in \partial T;$$

A3 $$P_\omega(v) \ge C_2(m(I_v))^{-1}, \text{ for all } \ v \in T, \omega \in I_v;$$

A4 $$F(v, v^-) \le 1 - \delta_1, \text{ for all } \ v \ne e.$$

From A1 we obtain

$$q^{-|v|} \le m(I_v) = \frac{1}{c_{|v|}} \le 2^{-|v|}. \tag{3}$$

Since for $v \ne e$ we have $m(I_{v^-}) = q_{|v|-1}\, m(I_v)$, the measure m satisfies the "doubling" condition

$$m(I_v) \le m(I_{v^-}) \le q\, m(I_v). \tag{4}$$

Observe that if h is harmonic on T, then

$$\sum_{|w|=n} h(w) = c_n h(e), \ n \ge 1. \tag{5}$$

This can be proved by induction on n, but follows more easily by symmetry and the fact that $h(e)$ is the weighted average of its boundary values on $|v| = n$, where the weights are the hitting probabilities for the first hitting time on $|v| = n$ of the random walk beginning at e.

We remark that for T any radial tree satisfying A4, T is necessarily transient. In fact, in the next proposition we provide a sufficient condition for transience (which clearly holds under the assumption of the axiom A4) that to the best of our knowledge has not appeared in the literature.

Proposition 3.1. *If there is a vertex $v \ne e$ such that $F(v, v^-) < 1$ and $F(u, v) < 1$ for all children u of v, then T is transient.*

Proof. Suppose v is a vertex satisfying the assumption. Choose $\delta > 0$ such that $F(v, v^-) \le 1 - \delta$ and $F(u, v) \le 1 - \delta$ for all children u of v. Since $p(v, v^-) \le F(v, v^-) \le 1 - \delta$, then

$$\begin{aligned}
F(v, v) &= p(v, v^-)F(v^-, v) + \sum_{u^-=v} p(v, u)F(u, v) \\
&\le p(v, v^-) + \sum_{u^-=v} p(v, u)(1 - \delta) \\
&= p(v, v^-) + (1 - p(v, v^-))(1 - \delta) = 1 - \delta(1 - p(v, v^-)) \\
&= 1 - \delta + \delta p(v, v^-) \le 1 - \delta + \delta(1 - \delta) \\
&= 1 - \delta^2,
\end{aligned}$$

proving that T is transient. $\square$

For the remainder of the paper, we shall assume T is a radial tree satisfying axioms A1-A4.

In [5], we proved all of the theorems that appear below in Sections 4 and 5 for the case of a homogeneous isotropic tree. It turns out that all of these results hold in the case of a radial tree satisfying axioms A1-A4. In fact, all of the proofs are virtually the same, except for the few parts below in which we give the needed details.

4. The harmonic Hardy space $\mathcal{H}^p$

For $1 \le p < \infty$ we let $L^p(\partial T)$ denote the functions $f : \partial T \to [-\infty, \infty]$ such that $|f|^p$ is m-integrable.We let $L^\infty(\partial T)$ be the set of bounded functions on ∂T.

For $1 \le p < \infty$, $n \ge 1$, and a function f on T, let $M_p(f, n)$ be the average value of $|f|^p$ over the vertices of length n, namely,

$$M_p(f,n) = \frac{\sum_{|v|=n} |f(v)|^p}{c_{|v|}} = m(I_v) \sum_{|v|=n} |f(v)|^p.$$

We now define the harmonic Hardy space $\mathcal{H}^p$ on T for $p \ge 1$.

Definition 4.1. Let $1 \le p < \infty$ and let h be harmonic on T. Then $h \in \mathcal{H}^p$ provided that

$$\|h\|^p_{\mathcal{H}^p} := \sup_{n \in \mathbb{N}} M_p(h, n) < \infty.$$

Definition 4.2. Let $f \in L^1(\partial T)$. The *Hardy-Littlewood maximal function* of f is the function Mf on ∂T defined as

$$Mf(\omega) = \sup_{\{v \in T:\, \omega \in I_v\}} \frac{\int_{I_v} |f(\tau)|\, dm(\tau)}{m(I_v)} = \sup_{\{v \in T:\, \omega \in I_v\}} c_{|v|} \int_{I_v} |f(\tau)|\, dm(\tau).$$

Lemma 4.1. (Covering Lemma) *Let $A \subseteq T$. Then there exists $\widehat{A} \subseteq A$ such that $\bigcup_{v \in \widehat{A}} S_v = \bigcup_{v \in A} S_v$, $\bigcup_{v \in \widehat{A}} I_v = \bigcup_{v \in A} I_v$, and for each pair of distinct vertices $v, w \in \widehat{A}$, $S_v \cap S_w = \emptyset$ and $I_v \cap I_w = \emptyset$.*

Theorem 4.1. *Let $1 \le p \le \infty$ and $f \in L^p(\partial T)$. Then $Mf < \infty$ m-a.e. and the following inequalities hold:*

(a) *If $p = 1$, then for every $\lambda > 0$,*

$$m\{\omega \in \partial T :\, Mf(\omega) > \lambda\} \le \frac{1}{\lambda} \|f\|_{L^1(\partial T)}.$$

(b) *If $1 < p \le \infty$, there exists a constant $C > 0$ such that for each $f \in L^p(\partial T)$,*

$$\|Mf\|_{L^p(\partial T)} \le C \|f\|_{L^p(\partial T)}.$$

Definition 4.3. Let h be harmonic on T. The *radial maximal function* of h is the function h^* defined on ∂T by $h^*(\omega) = \sup_n |h(\omega_n)|$.

Theorem 4.2. *For every $f \in L^1(\partial T)$ and $\omega \in \partial T$,*

$$(Pf)^*(\omega) \le \frac{C_1}{1 - r} Mf(\omega).$$

Proof. Let $f \in L^1(\partial T)$ and $\omega \in \partial T$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} |Pf(\omega_n)| &\leq \int P_\eta(\omega_n)|f(\eta)|\,dm(\eta) \\ &\leq \sum_{k=0}^{n} C_1\,(m(I_{\omega_k}))^{-1}\,r^{n-k}\int_{I_{\omega_k}}|f(\eta)|dm(\eta) \\ &\leq \sum_{k=0}^{n} C_1\,r^{n-k}Mf(\omega) \\ &= C_1\,\frac{1-r^{n+1}}{1-r}\,Mf(\omega) \\ &\leq \frac{C_1}{1-r}Mf(\omega). \end{aligned}$$

Taking the supremum over all $n \in \mathbb{N}$ yields the result. □

Lemma 4.2. *For a fixed $n \geq 1$, let h be harmonic on $\{v \in T : |v| \leq n\}$ and let f_n be the function defined on ∂T by*

$$f_n = \sum_{|v|=n}\left(\frac{h(v) - F(v,v^-)h(v^-)}{1 - F(v,v^-)}\right)\chi_{I_v}, \tag{6}$$

where χ_E denotes the characteristic function of the set E. Then $Pf_n = h$ on $|v| \leq n$.

Proof. Extend h harmonically to T so that for each v with $|v| = n$, h is radial on S_v. Fix such a v. Consider the function on T defined by

$$w \mapsto \left[\frac{h(v)F(v^-,e) - h(v^-)F(v,e)}{F(v^-,e) - F(v,e)}\right] + \left[\frac{h(v^-) - h(v)}{F(v^-,e) - F(v,e)}\right]F(w,e).$$

It is radial, harmonic outside of e, and it is easily verified that its value at v is $h(v)$ and its value at v^- is $h(v^-)$. Thus this function equals h on S_v. It is clear that h is bounded and its representing measure is absolutely continuous with respect to m. By the Fatou radial limit theorem, the density of the representing measure is m-a.e. equal to the radial limit function of h. By A4 and the multiplicative property of F, $F(w,e) \to 0$ as $|w| \to \infty$, so the radial limit of h is

$$\frac{h(v)F(v^-,e) - h(v^-)F(v,e)}{F(v^-,e) - F(v,e)}.$$

Factoring out $F(v^-,e)$ on top and bottom and using the fact that $F(v,e) = F(v,v^-)F(v^-,e)$ yields the desired result. □

In the following theorem, we extend the characterizations of the harmonic functions to be in the Hardy space $\mathcal{H}^p$ (for $1 < p < \infty$) shown in [5] (Theorem 2.3) for the homogeneous isotropic case. The proof of this extension, based on the same argument provided in [5], makes use of Lemma 4.2 and the fact that, due to A4, the function $v \mapsto F(v,v^-)$ is bounded away from 0.

Theorem 4.3. *For a harmonic function h on T and $1 < p < \infty$, the following propositions are equivalent:*

(a) $h \in \mathcal{H}^p$.

(b) $h = Pf$ *for some function* $f \in L^p(\partial T)$.

(c) $\|h^*\|_{L^p(\partial T)} < \infty$.
(d) $|h|^p$ *has a harmonic majorant.*

Theorem 4.4. *Let h be harmonic. Then $h \in \mathcal{H}^1$ if and only if $h = P(\mu)$ for some signed measure μ on ∂T.*

Theorem 4.5. *There exists $C > 0$ such that for all $f \in L^p(\partial T)$,*

$$C\|f\|_{L^p(\partial T)} \leq \|Pf\|_{\mathcal{H}^p} \leq \|f\|_{L^p(\partial T)}.$$

5. Discrete version of Duren's extension of Carleson's theorem

We note again that in this and the previous section we are generalizing results on homogeneous isotropic trees considered in [5] to radial trees satisfying A1-A4. Proofs are only given here in case they differ from those in [5].

Definition 5.1. For a measure σ on T and $s \geq 1$, define

$$\|\sigma\|_* = \sup\left\{\frac{\sigma(S_v)}{m(I_v)^s} : v \in T\right\}.$$

A measure σ on T is said to be *s-Carleson* if $\|\sigma\|_* < \infty$.

Lemma 5.1. *Given $m \in \mathbb{N}$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$, $\mu_1, \ldots, \mu_m > 0$, and $s > 0$,*

$$\sum_{k=1}^{m}(\alpha_k - \alpha_{k-1})(\mu_k + \cdots + \mu_m)^s \leq \left(\sum_{k=1}^{m}\alpha_k^{1/s}\mu_k\right)^s, \tag{7}$$

with equality occurring only if $m = 1$.

As a consequence, we deduce the following result.

Proposition 5.1. *Let $p > 1$, $s \geq 1$, $m \in \mathbb{N}$, $0 = a_0 < a_1 < \cdots < a_m$, and $\mu_1, \ldots, \mu_m > 0$. Then*

$$\sum_{k=1}^{m}(a_k^{sp} - a_{k-1}^{sp})(\mu_k + \cdots + \mu_m)^s \leq \left(\sum_{k=1}^{m}a_k^p\mu_k\right)^s.$$

Theorem 5.1. *Let (Ω, μ) be a finite measure space, $p > 1$, $s \geq 1$, and g a nonnegative measurable function on Ω. Then*

$$\int_0^\infty sp\lambda^{sp-1}\mu\{\omega \in \Omega : g(\omega) > \lambda\}^s\, d\lambda \leq \left(\int_\Omega g^p\, d\mu\right)^s.$$

The *extended Poisson kernel* on the tree is defined by

$$\mathcal{P}(v,w) \quad = \quad \begin{cases} 1 & \text{if } v, w \in T, v = e \\ \frac{1}{F(e,v\wedge w)} & \text{if } v, w \in T,\ v \wedge w = v \neq e, \\ \frac{F(v,v\wedge w)}{F(e,v\wedge w)} & \text{if } v, w \in T,\ v \wedge w \neq v, \\ P_\omega(v) & \text{if } v \in T,\ w = \omega \in \partial T. \end{cases}$$

The relation between the Poisson kernel and the extended Poisson kernel is given in the following proposition. The equality in the proposition is proved using the multiplicative property of F. The inequalities in the proposition follow by applying A2 and A3 with any $\omega \in I_w$ and recalling that $F \leq 1$.

Proposition 5.2. *Let $v, w \in T$, and $\omega \in I_w$. Then*

$$\mathcal{P}(v,w) \quad = \quad \begin{cases} P_\omega(v) & \text{if } v \wedge w = v \wedge \omega, \\ P_\omega(v) F(w, v\wedge\omega) F(v\wedge\omega, w) & \text{if } v \wedge w \neq v \wedge \omega. \end{cases}$$

In particular,

$$\mathcal{P}(v,w) \leq P_\omega(v).$$

If $v \in \{w\} \cup (T \backslash S_w)$, then

$$C_2 (m(I_v))^{-1} \leq \mathcal{P}(v,w) \leq C_1 (m(I_{v\wedge w}))^{-1} r^{|v|-|v\wedge w|}$$

Theorem 5.2. *Let $1 < p < \infty$, $1 \leq s < \infty$, and σ a finite measure on T. The following statements are equivalent.*

(a) *σ is an s-Carleson measure.*
(b) *There exists $C > 0$ such that for all harmonic functions h and $\lambda > 0$,*
$$\sigma\{v \in T : |h(v)| > \lambda\} \leq C\,(m\{\omega : h^*(\omega) > \lambda\})^s.$$
(c) *There exists $C > 0$ such that for all $f \in L^p(\partial T)$,*
$$\|Pf\|_{L^{sp}(\sigma)} \leq C \|f\|_{L^p(\partial T)}.$$
(d) *There exists $C > 0$ such that for all $f \in L^1(\partial T)$ and $\lambda > 0$,*
$$\sigma(\{v \in T : |Pf(v)| > \lambda\}) \leq \frac{C}{\lambda^s} \|f\|^s_{L^1(\partial T)}.$$
(e) $\displaystyle\sup_{v \in T} \sum_{w \in T} \mathcal{P}(v,w)^s \, \sigma(\{w\}) < \infty.$

Proof. (b) $\Longrightarrow$ (a): Suppose that σ is a measure on T satisfying (b). We must show that $\sigma(S_v)/(m(I_v))^s$ is bounded. It is enough to prove this for all $|v|$ sufficiently large, since it obviously holds if v lies in any finite set of vertices.

Let C_1, C_2, r be the positive constants of A2 and A3. Choose a positive integer k such that

$$\frac{C_2}{C_1} > r^k. \tag{8}$$

We will get an upper bound on $\sigma(S_v)/(m(I_v))^s$ for all v with $|v| > k$. This will prove that σ is $s-$Carleson.

For the balance of the proof of (b) $\Rightarrow$ (a), fix $v \in T$, with $n = |v|$ and $n > k$. Let $f = \chi_{I_v}$ and $h = Pf$. Define $S_v^k := S_{v_{n-k}}$, and $I_v^k := I_{v_{n-k}}$.

Let $u \in S_v$. If $\omega \in I_u$, then by A3,

$$h(u) = \int_{I_v} P_\omega(u)\, dm(\omega) \geq \int_{I_u} P_\omega(u)\, dm(\omega) \geq C_2,$$

proving that

$$S_v \subset \{u \in T : h(u) \geq C_2\}. \tag{9}$$

Notice that (9) holds for all v, whether or not $|v| > k$.

We claim that

$$\{\omega \in \partial T : h^*(\omega) > C_2\} \subset I_v^k. \tag{10}$$

To see this, let $u \notin S_v^k$. For any $\omega \in I_v$ we have $u \wedge \omega = u \wedge v$, so by A2,

$$\begin{aligned} h(u) &= \int_{I_v} P_\omega(u)\, dm(\omega) \le C_1\, (m(I_{u\wedge v}))^{-1} r^{|u|-|u\wedge v|} m(I_v) \\ &\le C_1 r^{|v|-|u\wedge v|} \le C_1 r^k < C_2, \end{aligned}$$

establishing the claim.

Thus, by (9), the hypothesis, (10), and (4) we have

$$\begin{aligned} \sigma(S_v) &\le \sigma\{u \in T : h(u) > C_2\} \\ &\le C\, (m\{\omega : h^*(\omega) > C_2\})^s \\ &\le C\, (m(I_v^k))^s \\ &\le C\, q^{ks} (m(I_v))^s, \end{aligned}$$

completing the proof.

(c) $\Longrightarrow$ (a): Let v, f, and h be as in the proof of (b) $\Rightarrow$ (a). Then using (9),

$$\begin{aligned} C_2^{sp} \sigma(S_v) &\le \int_{S_v} |Pf|^{sp}\, d\sigma \le \int_T |Pf|^{sp}\, d\sigma \\ &\le C\|f\|^{sp}_{L^p(\partial T)} = C\,(m(I_v))^s, \end{aligned}$$

proving σ is an s - Carleson measure.

(d) $\Longrightarrow$ (a): The proof is done by choosing f and C_2 as in the proof of b $\Rightarrow$ a, evaluating σ on both sides of (9), and applying the inequality in (d) to $f = \chi_{I_v}$ with $\lambda = C_2$.

(a) $\Longrightarrow$ (e): Assume (a) holds. Let $C > 0$ be such that $\sigma(S_v) \le C(m(I_v))^s$ for each $v \in T$. Fix $v \in T$ and let $n = |v|$. Let $v_0 = e, v_1, \dots, v_n = v$ be the sequence of vertices of the path $[e, v]$ with $|v_k| = k$ for $k = 0, \dots, n$. Then we may decompose T into the disjoint union of the sets $W_k = S_{v_k} \backslash S_{v_{k+1}}$ $(0 \le k \le n-1)$ and S_v. Thus, by Proposition 5.2,

$$\begin{aligned} \sum_{w\in T} \mathcal{P}(v,w)^s \sigma(\{w\}) &= \sum_{k=0}^{n-1} \sum_{w\in W_k} \mathcal{P}(v,w)^s \sigma(\{w\}) + \sum_{w\in S_v} \mathcal{P}(v,w)^s \sigma(\{w\}) \\ &\le \sum_{k=0}^{n-1} (C_1\, m(I_{v_k}))^{-s} r^{(n-k)s} \sigma(S_{v_k}) + (C_1\, m(I_v))^{-s} \sigma(S_v) \\ &\le C\left[\sum_{k=0}^{n-1} (m(I_{v_k}))^{-s} r^{(n-k)s} (m(I_{v_k}))^s + (m(I_v))^{-s}(m(I_v))^s\right] \\ &\le C\left[r^{ns} \sum_{k=0}^{n-1} r^{-ks} + 1\right] \\ &\le C. \end{aligned}$$

Condition (e) follows by taking the supremum over all $v \in T$.

(e) $\Longrightarrow$ (a): Suppose (e) holds. Let B be the supremum in (e). Let $v \in T$. Then, by Proposition 5.2 we obtain

$$B \ge \sum_{w\in S_v} \mathcal{P}(v,w)^s \sigma(\{w\}) \ge \sum_{w\in S_v} (C_2 m(I_v))^{-s} \sigma(\{w\}) = (C_2 m(I_v))^{-s} \sigma(S_v)$$

□

Remark 5.1. If σ is s-Carleson, let $\sigma' = \sigma/\|\sigma\|_*$. Applying Theorem 5.2 to σ', we deduce that in part (c) of the theorem, the constant C may be replaced by a multiple of $\|\sigma\|_*^{1/sp}$, and in parts (b) and (d), C may be replaced by a multiple of $\|\sigma\|_*$. Furthermore, the proof of Theorem 5.2 shows that the above multiple is independent of σ, that is, if $\|\sigma\|_* = 1$, then C depends only on universal constants and parameters associated with the tree.

6. Vanishing Carleson measures

We now introduce the notion of s-vanishing Carleson measure on a homogeneous tree T.

Definition 6.1. Let $s \geq 1$. A measure σ on T is said to be an *s-vanishing Carleson measure* if

$$\lim_{|v|\to\infty} \frac{\sigma(S_v)}{m(I_v)^s} = 0.$$

Lemma 6.1. *Let σ be an s-vanishing Carleson measure. For $N \in \mathbb{N}$ let σ_N be σ times the characteristic function of the N-ball, $\{v \in T : |v| \leq N\}$. Then $\lim_{N\to\infty} \|\sigma - \sigma_N\|_* = 0$.*

Proof. Let $\varepsilon > 0$. Choose M such that for all $|v| > M$, $\sigma(S_v) < \varepsilon(m(I_v))^s$. Since σ is a finite measure, we can choose N_1 such that for all $N > N_1$, $(\sigma - \sigma_N)(T) < \varepsilon \min_{|v|\leq M}(m(I_v))^s$. Let $N > N_1$. For any $v \in T$, if $|v| > M$, then $\frac{(\sigma-\sigma_N)(S_v)}{(m(I_v))^s} \leq \frac{\sigma(S_v)}{(m(I_v))^s} < \varepsilon$, and if $|v| \leq M$, then $\frac{(\sigma-\sigma_N)(S_v)}{(m(I_v))^s} \leq \frac{(\sigma-\sigma_N)(T)}{(m(I_v))^s} < \varepsilon$. Thus $\|\sigma - \sigma_N\|_* < \varepsilon$, proving that $\lim_{N\to\infty} \|\sigma - \sigma_N\|_* = 0$. □

Our main result of this paper is the little "oh" version of Theorem 5.2.

Theorem 6.1. *Let σ be a finite measure on T and $s \geq 1$. Then the following statements are equivalent.*

(a) *σ is an s-vanishing Carleson measure.*
(b) *For $1 < p < \infty$, the Poisson operator $P : L^p(\partial T) \to L^{sp}(\sigma)$ is compact.*
(c) $\lim_{|v|\to\infty} \sum_{w\in T} \mathcal{P}(v, w)^s \sigma(\{w\}) = 0.$
(d) *For any sequence $\{f_n\}$ in $L^1(\partial T)$ converging to 0 weakly and for all $\lambda > 0$,*
$$\lim_{n\to\infty} \frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda\})}{\|f_n\|^s_{L^1(\partial T)}} = 0.$$
(e) *For any sequence $\{h_n\}$ of harmonic functions on T converging to 0 pointwise and for all $\lambda > 0$,*
$$\lim_{n\to\infty} \frac{\sigma(\{v \in T : |h_n(v)| > \lambda\})}{m(\{\omega : h_n^*(\omega) > \lambda\})^s} = 0.$$
(f) *For any sequence $\{f_n\}$ in $L^1(\partial T)$ converging to 0 weakly and for all $\lambda > 0$,*
$$\lim_{n\to\infty} \frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda\})}{m(\{\omega : (Pf_n)^*(\omega) > \lambda\})^s} = 0.$$

For the proof we need the following lemma.

Recall that a sequence $\{f_n\}$ in $L^1(\partial T)$ converges to 0 weakly if for each $g \in L^\infty(\partial T)$, $\int f_n g\, dm \to 0$, as $n \to \infty$.

Lemma 6.2. *If $\{f_n\}$ is a sequence in $L^1(\partial T)$ converging to 0 weakly, then $\{Pf_n\}$ converges to 0 pointwise as $n \to \infty$.*

Proof. Let $\{f_n\}$ be a sequence converging to 0 weakly in $L^1(\partial T)$. For $n \in \mathbb{N}$ and $v \in T$, decomposing ∂T into the disjoint union of the sets $I_{v_k} \backslash I_{v_{k+1}}$ (for $k \in \{0, \dots, |v|-1)$ and I_v, we have

$$\begin{aligned} |Pf_n(v)| &= \left| \int P_\omega(v) f_n(\omega)\, dm(\omega) \right| \\ &\le C_1\, (m(I_v))^{-1} \left(\sum_{k=0}^{|v|-1} \int_{I_{v_k} \backslash I_{v_{k+1}}} |f_n(\omega)|\, dm(\omega) + \int_{I_v} |f_n(\omega)|\, dm(\omega) \right), \end{aligned}$$

which can be made arbitrarily small due to the weak convergence of $\{f_n\}$. Therefore, Pf_n converges to 0 pointwise. □

Proof of Theorem 5.2. (a) ⇒ (b): Suppose σ is an s-vanishing Carleson measure. Then, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $(m(I_v))^{-s} \sigma(S_v) < \varepsilon$ for $|v| > N$.

By Proposition 3.3(b), Chapter VI in [6], to prove that $P : L^p(\partial T) \to L^{sp}(\sigma)$ is compact, it suffices to show that if $\{f_n\}$ is a sequence in $L^p(\partial T)$ converging to 0 weakly, then $\|Pf_n\|_{L^{sp}(\sigma)} \to 0$ as $n \to \infty$.

For $N \in \mathbb{N}$ let σ_N be σ times the characteristic function of the N-ball, $\{v \in T : |v| \le N\}$. Since σ_N has finite support, the operator $P : L^p(\partial T) \to L^{sp}(\sigma_N)$ is compact. Indeed, let $\{f_n\}$ be a sequence in $L^p(\partial T)$ converging to 0 weakly. Then for each $w \in T$, letting $p' := \frac{p}{p-1}$, the conjugate index of p, and taking as a test function in $L^{p'}(\partial T)$ the characteristic function of I_w, we have

$$\int_{I_w} f_n(\omega)\, dm(\omega) \to 0 \quad \text{as } n \ \to \infty.$$

Thus, since σ is a finite measure, by A2 and (3), we have

$$\begin{aligned} &\|Pf_n\|^{sp}_{L^{sp}(\sigma_N)} = \sum_{|w| \le N} |Pf_n(w)|^{sp} \sigma(\{w\}) \\ &= \sum_{|w| \le N} \left| \int P_\omega(w) f_n(\omega)\, dm(\omega) \right|^{sp} \sigma(\{w\}) \\ &\le \sum_{|w| \le N} C_1\, (m(I_w))^{-sp} \left[\left(\sum_{k=0}^{|w|-1} \int_{I_{w_k} \backslash I_{w_{k+1}}} + \int_{I_w} \right) |f_n(\omega)|\, dm(\omega) \right]^{sp} \sigma(\{w\}) \\ &\le C q^{spN} \left(\sum_{|w| \le N} \sum_{k=0}^{|w|-1} \int_{I_{w_k} \backslash I_{w_{k+1}}} |f_n(\omega)|\, dm(\omega) + \sum_{|w| \le N} \int_{I_w} |f_n(\omega)|\, dm(\omega) \right)^{sp} \end{aligned}$$

which approaches 0 as $n \to \infty$. This proves the compactness of P as an operator mapping into $L^{sp}(\sigma_N)$.

Let $\{f_n\}$ be a sequence in $L^p(\partial T)$ converging to 0 weakly. By the uniform boundedness principle, there exists a constant $M > 0$ such that $\|f_n\|^{sp}_{L^p} < M$ for all $n \in \mathbb{N}$. By Lemma 6.1, there exists a positive integer N such that $\|\sigma - \sigma_N\|_* < \varepsilon/M$. Since σ is an s-Carleson measure, so is $\sigma - \sigma_N$. Thus, by Theorem 5.2 and Remark 5.1, there exists $C > 0$ (independent of N) such that

$$\int |Pf_n(w)|^{sp}\, d(\sigma - \sigma_N)(w) \le C \|\sigma - \sigma_N\|_*\ \|f_n\|^{sp}_{L^p(\partial T)} < C\varepsilon.$$

Thus,

$$\begin{aligned}\|Pf_n\|^{sp}_{L^{sp}(\sigma)} &= \int |Pf_n(w)|^{sp}\, d\sigma_N(w) + \int |Pf_n(w)|^{sp}\, d(\sigma-\sigma_N)(w)\\ &\leq \int |Pf_n(w)|^{sp}\, d\sigma_N(w) + C\varepsilon.\end{aligned}$$

Letting $n \to \infty$ and using the compactness of P as an operator mapping into $L^{sp}(\sigma_N)$, we obtain $\limsup_{n\to\infty} \|Pf_n\|^{sp}_{L^{sp}(\sigma)} \leq C\varepsilon$. Since ε is arbitrary, $\lim_{n\to\infty} \|Pf_n\|^{sp}_{L^{sp}(\sigma)} = 0$, and we are done.

(b) $\Rightarrow$ (c): Suppose (b) holds. For $v \in T$, define $f_v : \partial T \to \mathbb{R}$ by

$$f_v(\omega) = P_\omega(v)^{1/p},\ \omega \in \partial T.$$

We now show that f_v converges to 0 weakly in $L^p(\partial T)$ as $|v| \to \infty$, that is, for all $g \in L^{p'}(\partial T)$ with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\int f_v g\, dm \to 0 \text{ as } |v| \to \infty. \tag{11}$$

Fix $v \in T$ and set $|v| = n$. In the special case when g is the constant function 1, using the decomposition of ∂T as the disjoint union of the sets $I_{v_k} \backslash I_{v_{k+1}}$ $(0 \leq k \leq n-1)$ with I_v, and applying A2 and (3), we have

$$\begin{aligned}\int f_v(\omega)\, dm(\omega) &\leq C\left[\sum_{k=0}^{n-1} (m(I_{v_k}))^{-1/p}\, r^{(n-k)/p} m(I_{v_k}) + (m(I_v))^{-1/p} m(I_v)\right]\\ &= C\left[r^{n/p} \sum_{k=0}^{n-1} (m(I_{v_k}))^{1-1/p}\, r^{-k/p} + (m(I_v))^{1-1/p}\right]\\ &\leq C\left[r^{n/p} \sum_{k=0}^{n-1} \left(2^{(1-1/p)}\, r^{1/p}\right)^{-k} + 2^{-(1-1/p)n}\right]\end{aligned}$$

Since $1 < p < \infty$, the last term on the right side goes to 0 as $n \to \infty$. If $2^{1-1/p}\, r^{1/p} < 1$, the first term on the right side is at most $Cr^{n/p}\left(2^{(1-1/p)} r^{1/p}\right)^{-n} = C\left(2^{-(1-1/p)n}\right)$, and if $2^{1-1/p}\, r^{1/p} \geq 1$, it is at most $Cnr^{n/p}$. In either case, we obtain $\int f_v(\omega)\, dm(\omega) \to 0$ as $|v| = n \to \infty$. Thus, (11) holds if g is any simple function.

Next, denote by p' the conjugate index of p and suppose $g \in L^{p'}(\partial T)$. Choose a sequence $\{g_k\}$ of simple functions such that $\|g_k - g\|_{L^{p'}(\partial T)} \to 0$ as $k \to \infty$. Then

$$\begin{aligned}\int |f_v(\omega) g(\omega)|\, dm(\omega) &\leq \int |f_v(\omega) g_k(\omega)|\, dm(\omega)\\ &\quad + \int |f_v(\omega)(g(\omega) - g_k(\omega))|\, dm(\omega) = I + II.\end{aligned}$$

By the above remarks, $I \to 0$ as $|v| \to \infty$, whereas, by Hölder's inequality,

$$II \leq \left(\int P_\omega(v)\, dm(\omega)\right)^{1/p} \|g - g_k\|_{L^{p'}(\partial T)} = \|g - g_k\|_{L^{p'}(\partial T)},$$

which can be made arbitrarily small by choosing k sufficiently large. This proves the weak convergence to 0 of f_v.

Since by assumption, the operator $P : L^p(\partial T) \to L^{sp}(\sigma)$ is compact, it follows that $\|Pf_v\|_{L^{sp}(\sigma)} \to 0$ as $|v| \to \infty$.

Fix v and w in T. Since both the Poisson kernel and the function f_v are positive, by Proposition 5.2 and A3, we have

$$\begin{aligned} Pf_v(w) &= \int P_\omega(w) f_v(\omega)\, dm(\omega) \\ &\geq \int_{I_w} P_\omega(w) f_v(\omega)\, dm(\omega) \\ &\geq \int_{I_w} P_\omega(w) \mathcal{P}(v,w)^{1/p}\, dm(\omega) \\ &\geq C_2\, \mathcal{P}(v,w)^{1/p}. \end{aligned}$$

Hence, raising both sides of the resulting inequality to the power sp, multiplying by $\sigma(\{w\})$, and summing over all $w \in T$, we obtain

$$\sum_{w\in T} \mathcal{P}(v,w)^s \sigma(\{w\}) \quad \leq \quad C \sum_{w \in T} |Pf_v(w)|^{sp} \sigma(\{w\}) = C\, \|Pf_v\|^{sp}_{L^{sp}(\sigma)}$$

which, as we showed above, converges to 0 as $|v| \to \infty$. Thus (c) holds.

(c) $\Rightarrow$ (a): Assume (c) holds and fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for $|v| > N$

$$\sum_{w\in T} \mathcal{P}(v,w)^s\, \sigma(\{w\}) < \varepsilon. \tag{12}$$

For $|v| > N$, we obtain from Proposition 5.2 that

$$\left(C_2 (m(I_v))^{-1}\right)^s \sigma(S_v) \leq \sum_{w \in S_v} \mathcal{P}(v,w)^s \sigma(\{w\}) \leq \sum_{w\in T} \mathcal{P}(v,w)^s \sigma(\{w\}) < \varepsilon.$$

proving that σ is an s-vanishing Carleson measure.

(a) $\Rightarrow$ (d): Assume σ is an s-vanishing Carleson measure and let $\{f_n\}$ be a sequence in $L^1(\partial T)$ converging to 0 weakly. Fix $\lambda, \varepsilon > 0$. For $N \in \mathbb{N}$, let σ_N be the restriction of σ to the ball of radius N. By Lemma 6.1, we can choose N such that $\|\sigma - \sigma_N\|_* < \varepsilon \lambda^s$. We need to show that

$$\frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda, |v| < N\})}{\|f_n\|^s_{L^1(\partial T)}} < \varepsilon \quad \text{and} \tag{13}$$

$$\frac{\sigma(\{v \in T : |Pf_n(v)| > \lambda, |v| \geq N\})}{\|f_n\|^s_{L^1(\partial T)}} < C\, \varepsilon \tag{14}$$

for all n sufficiently large.

Using Lemma 6.2, we have $Pf_n \to 0$ pointwise as $n \to \infty$. Thus, the set $\{v \in T : |Pf_n(v)| > \lambda, |v| < N\}$ is empty for n sufficiently large. Thus, (13) holds.

Next, using the fact that $\sigma - \sigma_N$ is an s-Carleson measure, it follows from Theorem 5.2 and Remark 5.1 that, for each $n \in \mathbb{N}$ we have

$$\sigma(\{v \in T : |Pf_n(v)| > \lambda, |v| \geq N\}) \leq C\, \frac{\|\sigma - \sigma_N\|_*}{\lambda^s} \|f_n\|^s_{L^1(\partial T)} < C\varepsilon \|f_n\|^s_{L^1(\partial T)},$$

where C is independent of n and N. This proves (14).

(d) $\Rightarrow$ (a): Assume (d) holds. To prove (a), it suffices to show that if $\{v_n\}$ is any sequence of vertices such that $|v_n| \to \infty$, then $(m(I_{v_n}))^{-s} \sigma(S_{v_n}) \to 0$ as $n \to \infty$. Let $\{v_n\}$ be such a sequence and for $n \in \mathbb{N}$, define $f_n = \chi_{I_{v_n}}$. Then

$\|f_n\|_{L^1(\partial T)} = m(I_{v_n})$ and $f_n \to 0$ weakly in $L^1(\partial T)$. Indeed, for all $g \in L^\infty(\partial T)$, by (3) we have

$$\left| \int f_n(\omega) g(\omega)\, dm(\omega) \right| = \left| \int_{I_{v_n}} g(\omega)\, dm(\omega) \right| \leq \|g\|_\infty m(I_{v_n}) \to 0,$$

as $n \to \infty$.

If $v \in S_{v_n}$, then by A3,

$$Pf_n(v) = \int_{I_{v_n}} P_\omega(v)\, dm(\omega) \geq \int_{I_v} P_\omega(v)\, dm(\omega) \geq C_2.$$

Thus, if $0 < \lambda < C_2$, then $S_{v_n} \subset \{v \in T : Pf_n(v) > \lambda\}$, and so, since $\|f_n\|_{L^1(\partial T)} = m(I_{v_n})$, then

$$(m(I_{v_n}))^{-s} \sigma(S_{v_n}) \leq \frac{\sigma(\{v \in T : Pf_n(v) > \lambda\})}{\|f_n\|^s_{L^1(\partial T)}}.$$

By assumption, the weak convergence of f_n implies that the latter goes to 0 as n goes to ∞, proving the result.

(a) $\Rightarrow$ (e): Suppose (a) holds and fix $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $\sigma(S_v) < \varepsilon\, m(I_v)^s$ for $|v| > N$. Let $\{h_n\}$ be a sequence of harmonic functions converging to 0 pointwise and let $\lambda > 0$. Then for $|v| \leq N$ and n sufficiently large, $|h_n(v)| < \lambda$. In particular, for such an n, the set $\{v \in T : |v| \leq N, |h_n(v)| > \lambda\}$ is empty. Define $A_n = \{v \in T : |h_n(v)| > \lambda\} = \{v \in T : |v| > N, |h_n(v)| > \lambda\}$ and let $\widehat{A}_n$ be as in Lemma 4.1.

Since for $\omega \in I_v$ with $v \in A_n$, $h_n^*(\omega) \geq |h_n(v)| > \lambda$, we have

$$\begin{aligned}
\sigma(\{v : |h_n(v)| > \lambda\}) &\leq \sigma\left(\bigcup_{v \in \widehat{A}_n} S_v\right) = \sum_{v \in \widehat{A}_n} \sigma(S_v) \\
&< \varepsilon \sum_{v \in \widehat{A}_n} m(I_v)^s \leq \varepsilon \left(\sum_{v \in \widehat{A}_n} m(I_v)\right)^s \\
&= \varepsilon \left(m\left(\bigcup_{v \in \widehat{A}_n} I_v\right)\right)^s = \varepsilon \left(m\left(\bigcup_{v \in A_n} I_v\right)\right)^s \\
&\leq \varepsilon\, m(\{\omega : h_n^*(\omega) > \lambda\})^s,
\end{aligned}$$

which yields (e).

(e) $\Rightarrow$ (f) follows immediately from Lemma 6.2.

(f) $\Rightarrow$ (a): Suppose (f) holds. To prove that σ is an s-vanishing Carleson measure, it suffices to show that if $\{v_n\}$ is a sequence of vertices such that $|v_n| \to \infty$, then $\sigma(S_{v_n})/m(I_{v_n})^s \to 0$ as $n \to \infty$. Let $\{v_n\}$ be such a sequence and, for $n \in \mathbb{N}$, let $f_n = \chi_{I_{v_n}}$. As shown in the proof of (d) $\Rightarrow$ (a), $\{f_n\}$ converges to 0 weakly in $L^1(\partial T)$. Let $\varepsilon > 0$. Then

$$\sigma(\{v \in T : |Pf_n(v)| > C_2\}) < \varepsilon\, m(\{\omega : (Pf_n)^*(\omega) > C_2\})^s$$

for all n sufficiently large. The proof then follows from the proof of $(b) \Rightarrow (a)$ in Theorem 5.2, replacing v by v_n and f by f_n.

$\square$

7. Axiomatics in the radial case

The following theorem gives a condition on the transition probabilities which guarantees that A4 holds.

Theorem 7.1. *Let T be a radial, transient tree for which there exists $0<\delta<1/2$ such that $p(v,v^-)<1/2-\delta$ for all $v\neq e$. Then*

$$F(v,v^-)\leq \frac{1/2-\delta}{1/2+\delta},\ \ \textit{for all}\ \ v\neq e.$$

In the appendix of [9], it is proved that a tree is transient if it satisfies the stronger condition that there exists $\delta>0$ such that $\delta<p(v,w)<1/2-\delta$ for all vertices v and w. The proof of Theorem 7.1 is a modification of that proof.

We shall need the following result.

Lemma 7.1. *The sequence $\{\alpha_k\}$ defined inductively by $\alpha_1=1/2-\delta$, and*

$$\alpha_k=\frac{1/2-\delta}{1-(1/2+\delta)\alpha_{k-1}},\ \ \textit{for}\ \ k\geq 2,$$

is increasing and $\lim\limits_{k\to\infty}\alpha_k=\dfrac{1/2-\delta}{1/2+\delta}$.

Proof. We first show, using induction, that

$$\alpha_k<\frac{1/2-\delta}{1/2+\delta},\ \ \text{for}\ \ k\geq 1, \tag{15}$$

or equivalently

$$1-(1/2+\delta)\alpha_k>1/2+\delta,\ \ \text{for}\ \ k\geq 1. \tag{16}$$

Inequality (15) holds for $k=1$ since $1/2+\delta<1$. Now suppose it holds for some $k\geq 1$. Then using the inductive hypothesis stated as in (16),

$$\alpha_{k+1}=\frac{1/2-\delta}{1-(1/2+\delta)\alpha_k}<\frac{1/2-\delta}{1/2+\delta},$$

completing the inductive proof.

Since $\alpha_k<\frac{1/2-\delta}{1/2+\delta}<1$, we see by (15) that

$$(1/2+\delta)\alpha_k^2-\alpha_k+(1/2-\delta)=[(1/2+\delta)\alpha_k-(1/2-\delta)][\alpha_k-1]>0,$$

so

$$1/2-\delta>\alpha_k-(1/2+\delta)\alpha_k^2=\alpha_k(1-(1/2+\delta)\alpha_k).$$

Thus, by (16),

$$\alpha_k<\frac{1/2-\delta}{1-(1/2+\delta)\alpha_k}=\alpha_{k+1}. \tag{17}$$

We have shown that $\{\alpha_k\}$ is increasing and bounded above by $\frac{1/2-\delta}{1/2+\delta}$, which we note is less than 1. Thus the sequence has a finite limit, and letting $k\to\infty$ on both sides of $\alpha_k=\frac{1/2-\delta}{1-(1/2+\delta)\alpha_{k-1}}$ gives that the limiting value is $\frac{1/2-\delta}{1/2+\delta}$. □

For each $k\geq 1$ and $v\in T$ with $|v|=k$, let $p_k=1-p(v,v-)$ denote the total forward probability at v and let $F_k=F(v,v^-)$. For each $m\geq 1$, let $F_{k,m}$ be the conditional probability that the random walk starting at v visits v^- given that the path is never farther than distance m from v^-.

Proof of Theorem 7.1. We claim that for all $k, m \geq 1$, $F_{k,m} < \alpha_m$. We argue by induction on m. Consider the case $m = 1$. By definition, $F_{k,1} = 1 - p_k$, so the fact that the claim holds for $m = 1$ is immediate from our assumption that $1 - p_k < 1/2 - \delta$. Suppose $m \geq 2$ and the formula holds for all $k \geq 1$ and $m - 1$. Then

$$F_{k,m} = 1 - p_k + p_k F_{k+1,m-1} F_{k,m},$$

so by the inductive hypothesis,

$$F_{k,m} = \frac{1 - p_k}{1 - p_k F_{k+1,m-1}} < \frac{1 - p_k}{1 - p_k \alpha_{m-1}} = \frac{1 - p_k}{1 - \alpha_{m-1} + (1 - p_k)\alpha_{m-1}}.$$

Since the function $x \mapsto \frac{x}{1-\alpha_{m-1}+x\alpha_{m-1}}$ is increasing for $x > 0$, and $1 - p_k < 1/2 - \delta$, we obtain

$$F_{k,m} < \frac{1/2 - \delta}{1 - \alpha_{m-1} + (1/2 - \delta)\alpha_{m-1}} = \alpha_m,$$

completing the proof of the claim. Letting $m \to \infty$ and applying Lemma 7.1 gives

$$F_k = \lim_{m\to\infty} F_{k,m} \leq \lim_{m\to\infty} \alpha_m = \frac{1/2 - \delta}{1/2 + \delta}$$

and we are done.

□

8. The P-tree

In this and the next section we give an example of a tree to which we can apply the theory developed in the paper.

A tree is termed *homogeneous* of degree $q + 1$ (with $q \in \mathbb{N}$) if all its vertices have $q + 1$ neighbors. The number of vertices of T of length n is

$$c_n = \begin{cases} (q+1)q^{n-1} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

We fix once and for all a real number P such that $1/2 < P < 1$ and an integer $q \geq 2$. Throughout this section we shall assume that T is a homogeneous tree of degree $q + 1$ with radial forward probabilities $p(e, v) = \frac{1}{q+1}$ for $|v| = 1$, and $p(v-, v) = P/q$ and backward probabilities $p(v, v^-) = 1 - P$ for $v \neq e$.

Define $r = \frac{1-P}{P}$. Then $0 < r < 1$ and $pr^2 - r + 1 - p = 0$. A simple calculation shows that the function $v \mapsto r^{|v|}$ is harmonic on $T \setminus \{e\}$, and the Laplacian at e is negative. Thus, since there exists a positive superharmonic function that is not harmonic, T is transient. The transience of T also follows from Theorem 7.1.

For $u, v \in T$, recall that $F(u, v)$ is the probability that the random walk starting at u hits v in positive time. We calculate F as follows. First observe that since all backward probabilities are equal, for $v \neq e$, $F(v, v^-)$ is independent of v and $F(v, v^-) = 1 - P + PF(v, v^-)^2$, which easily yields the factorization

$$(F(v, v^-) - 1)(PF(v, v^-) - (1 - P)) = 0.$$

Since $F(v, v^-) < 1$, due to the transience of the random walk on T, it follows that $F(v, v^-) = \frac{1-P}{P}$ or

$$F(v, v^-) = r. \tag{18}$$

We next calculate $F(v,w)$ for $|v|=n$ and $w^-=v$. Due to the radiality of the transition probabilities, $F(v,w)$ only depends on n. Thus, we shall denote its value by F_n. Observe that $F_0=\frac{1}{q+1}+\frac{q}{q+1}rF_0$, so that

$$F_0=\frac{1}{q+1-qr}.$$

For $n\geq 1$, we have

$$F_n=\frac{P}{q}+(q-1)\frac{P}{q}rF_n+(1-P)F_{n-1}F_n,$$

whence, a straightforward computation yields

$$F_n=\frac{1}{q+r-qrF_{n-1}}. \tag{19}$$

Proposition 8.1. *For each $n\in\mathbb{N}$, $F_n=\dfrac{x_n}{x_{n+1}}$ where*

$$x_n=(q+1)(1-r)q^n+(qr-1)r^n. \tag{20}$$

Proof. We argue by induction on n. For $n=0$, we have

$$\frac{x_0}{x_1}=\frac{q-r}{qr^2-(q^2+q+1)r+q(q+1)}=\frac{1}{q+1-qr}=F_0.$$

Assume the result holds for some integer $n\geq 0$. Then, by (19),

$$\begin{aligned}F_{n+1}&=\frac{1}{q+r-qrF_n}=\frac{x_{n+1}}{(q+r)x_{n+1}-qrx_n}\\&=\frac{(q+1)(1-r)q^{n+1}+(qr-1)r^{n+1}}{(q+1)(1-r)q^{n+2}+(qr-1)r^{n+2}}\\&=\frac{x_{n+1}}{x_{n+2}},\end{aligned}$$

proving the result. □

We now calculate the Poisson kernel.

Theorem 8.1. *Let $v\in T$ with $|v|=n$, $\omega\in\partial T$, and let $k=|v\wedge\omega|$. Then*

$$P_\omega(v)=\left(\frac{(q+1)(1-r)q^k+(qr-1)r^k}{q-r}\right)r^{n-k}.$$

In particular, if the transition probability on T is isotropic, then

$$P_\omega(v)=q^{2k-n}.$$

Proof. For $v\wedge\omega\neq v$ we have $G(v,v\wedge\omega)=F(v,v\wedge\omega)G(v\wedge\omega,v\wedge\omega)$, so recalling that $P_\omega(v)=\frac{G(v,v\wedge\omega)}{G(e,v\wedge\omega)}$ we obtain

$$P_\omega(v)=\begin{cases}\frac{F(v,v\wedge\omega)}{F(e,v\wedge\omega)} & \text{if } v\wedge\omega\neq v\\ \frac{1}{F(e,v\wedge\omega)} & \text{if } v\wedge\omega=v\end{cases}=\frac{\prod_{j=k+1}^{n}F(v_j,v_{j-1})}{\prod_{j=0}^{k-1}F(v_j,v_{j+1})},$$

where $[v_0,v_1,\ldots,v_n]=[e,v]$ and the product in the numerator is 1 in case $k=n$. In particular, $v_k=v\wedge\omega$. Thus, from (18) and Proposition 8.1, we obtain

$$P_\omega(v)=\frac{r^{n-k}}{\prod_{j=0}^{k-1}\frac{x_j}{x_{j+1}}}=r^{n-k}\frac{x_k}{x_0}.$$

The result follows at once from (20). □

We shall now calculate the Green function using the formulas

$$G(v,v)=\frac{1}{1-F(v,v)} \quad\text{and}\quad G(v,w)=F(v,w)G(w,w), \quad \text{for}\ \ v\neq w.$$

First we need to evaluate $F(v,v)$.

Clearly, $F(e,e)=(q+1)\frac{1}{q+1}r=r$, while for $v\neq e$, $|v|=n$,

$$F(v,v)=\sum_{u\sim v}p(v,u)F(u,v)=Pr+(1-P)F_{n-1}=(1-P)(1+F_{n-1}).$$

Thus, for $|v|=n>0$,

$$G(v,v)=\frac{1}{P-(1-P)F_{n-1}}=\frac{1+r}{1-rF_{n-1}}. \tag{21}$$

In general, for $|v|=n$, $|w|=m$, and $|v\wedge w|=k$, from (20), we have

$$\begin{aligned} G(v,w) &= F(v,w)G(w,w)=r^{n-k}\frac{x_k}{x_m}\frac{1+r}{1-rF_{m-1}}\\ &= r^{n-k}\frac{(1+r)x_k}{x_m-rx_{m-1}}\\ &= \frac{r^{n-k}(1+r)[(q+1)(1-r)q^k+(qr-1)r^k]}{(q+1)(1-r)(q-r)q^{m-1}}. \end{aligned} \tag{22}$$

A direct calculation applied to (21) shows that for $|v|=n$,

$$G(v,v)=\frac{(1+r)[(q+1)(1-r)q^n+(qr-1)r^n]}{(q+1)(1-r)(q-r)q^{n-1}}.$$

Therefore, formula (22) also holds for $v=w$. Thus, we have proved:

Theorem 8.2. *The Green function on a homogeneous tree T with forward radial transition probabilities given by $p(e,v)=\frac{1}{q+1}$ for $|v|=1$, $p(v^-,v)=\frac{P}{q}$ for $|v|>1$, is given by*

$$G(v,w)=\frac{r^{n-k}(1+r)[(q+1)(1-r)q^k+(qr-1)r^k]}{(q+1)(1-r)(q-r)q^{m-1}},$$

where $|v|=n$, $|w|=m$, and $|v\wedge w|=k$. In particular, in the isotropic case, the Green function is given by

$$G(v,w)=\left(\frac{q}{q-1}\right)q^{-(n+m)+2k}.$$

It can be easily verified using Theorem 8.1 and (18) that the P-tree satisfies the required properties A1-A4.

9. The semi-homogeneous tree

Let q_0,q_1 be positive integers and let T be a tree such that for $v\in T$, if $|v|$ is even, then v has q_0+1 neighbors, and if $|v|$ is odd, then v has q_1+1 neighbors. In particular, the root e has q_0+1 neighbors each of which has q_1 children, each if which has q_0 children, etc. Thus, the number of the vertices of T of length n is given by

$$m_n=\begin{cases} 1 & \text{for } n=0,\\ \frac{q_0+1}{q_0}\,\alpha^n & \text{for } 0<n \text{ even},\\ \frac{q_0+1}{\sqrt{q_0q_1}}\,\alpha^n & \text{for } n \text{ odd}, \end{cases} \tag{23}$$

where $\alpha = \sqrt{q_0 q_1}$. We say that T is *semihomogeneous of degrees* q_0 *and* q_1 if T satisfies the above conditions and the transition probabilities of T are isotropic.

Let T be semihomogeneous of degrees q_0 and q_1. Let $\widehat{T}$ be the tree whose set of vertices is $\{v \in T : |v| = 2\}$ and let two vertices v and w in $\widehat{T}$ be neighbors if and only if $d(v,w) = 2$, where d denotes the distance function in T. Then $\widehat{T}$ is a homogeneous tree of degree $Q+1$, where $Q = q_1(q_0+1) - 1$. We define the transition probabilities so that $\widehat{T}$ is isotropic. Clearly, $\partial T = \partial \widehat{T}$.

For any function f on T, let $\widehat{f}$ be its restriction to $\widehat{T}$.

Proposition 9.1. *The mapping $f \mapsto \widehat{f}$ is a bijection between the set of harmonic functions on T and the set of harmonic functions on $\widehat{T}$.*

Proof. We first show that if f is harmonic on T, then $\widehat{f}$ is harmonic on $\widehat{T}$. Fix $v \in \widehat{T}$ and reserve the notation $v \sim w$ for neighboring vertices v and w in T and $u^\sim$ for the neighbor of u closest to v in T. Since v has q_0+1 neighbors, by the harmonicity of f at v and at its neighbors in T, we have

$$\begin{aligned} f(v) &= \frac{1}{q_0+1} \sum_{w \sim v} f(w), \quad \text{and for each } w \sim v, \\ f(w) &= \frac{1}{q_1+1} \sum_{u \sim w} f(u) = \frac{1}{q_1+1} \left[\sum_{u^\sim = w} f(u) + f(v) \right], \end{aligned}$$

Thus, eliminating the intermediate step and substituting the value of $f(w)$ from the second equation into the first equation, we have

$$\begin{aligned} f(v) &= \frac{1}{(q_0+1)(q_1+1)} \left[\sum_{w \sim v} f(v) + \sum_{w \sim v} \sum_{u^\sim = w} f(u) \right] \\ &= \frac{1}{(q_0+1)(q_1+1)} \left[(q_0+1) f(v) + \sum_{d(u,v)=2} f(u) \right] \\ &= \frac{1}{q_1+1} f(v) + \frac{1}{(q_0+1)(q_1+1)} \sum_{d(u,v)=2} f(u). \end{aligned}$$

Combining the terms in $f(v)$ in the resulting equation and multiplying both sides by (q_1+1), we obtain

$$q_1 f(v) = \frac{1}{q_0+1} \sum_{d(u,v)=2} f(u),$$

whence

$$\widehat{f}(v) = f(v) = \frac{1}{q_1(q_0+1)} \sum_{d(u,v)=2} f(u) = \frac{1}{Q+1} \sum_{\widehat{d}(u,v)=1} \widehat{f}(u),$$

where $\widehat{d}$ denotes the distance function in $\widehat{T}$. Therefore, $\widehat{f}$ is harmonic at v in $\widehat{T}$.

Conversely, suppose $\tilde{f}$ is harmonic on $\widehat{T}$. For all $v \in \widehat{T}$, let $f(v) = \tilde{f}(v)$ and for all $w \in T \backslash \widehat{T}$, let

$$f(w) = \frac{1}{q_1+1} \sum_{v \sim w} \tilde{f}(v).$$

Then, by construction, f is harmonic at each $w \in T \backslash \widehat{T}$. Moreover, for $v \in \widehat{T}$,

$$\begin{aligned} \frac{1}{q_0+1}\sum_{w\sim v} f(w) &= \frac{1}{q_0+1}\sum_{w\sim v}\frac{1}{q_1+1}\left[\sum_{u\sim w}\tilde{f}(u)\right] \\ &= \frac{1}{q_0+1}\sum_{w\sim v}\frac{1}{q_1+1}\left[\sum_{u^\sim = w}\tilde{f}(u)+\tilde{f}(v)\right] \\ &= \frac{1}{(q_0+1)(q_1+1)}\sum_{w\sim v}\sum_{u^\sim = w}\tilde{f}(u)+\frac{1}{q_1+1}f(v) \\ &= \frac{1}{(q_0+1)(q_1+1)}(Q+1)\tilde{f}(v)+\frac{1}{q_1+1}f(v) \\ &= \frac{q_1}{q_1+1}\tilde{f}(v)+\frac{1}{q_1+1}f(v) = f(v), \end{aligned}$$

proving the harmonicity of f on T. This proves the result. □

Theorem 9.1. *Let T be a semihomogeneous tree of degrees q_0 and q_1 with $q_j \geq 2$ for $j = 0, 1$. Then T satisfies axioms* A1-A4.

Proof. Axiom A1 clearly holds. Next note that since $q_j \geq 2$ for $j = 0, 1$, then for any vertex $v \in T$, $v \neq e$, $p(v, v^-) \leq \frac{1}{3} = \frac{1}{2} - \frac{1}{6}$. Thus, Theorem 7.1 holds with $\delta = \frac{1}{6}$. Therefore A4 holds.

For any $v \in T$, $\omega \in \partial T$, let $P_\omega(v)$ be the Poisson kernel in T. If $v \in \widehat{T}$, let $\widehat{P}_\omega(v)$ be the Poisson kernel in $\widehat{T}$. By Proposition 9.1 and the uniqueness of the representation of a positive harmonic function as a Poisson integral, it follows that $\widehat{P}_\omega$ is just the restriction of P_ω to $\widehat{T}$. Let $n(v,\omega) = 2|v \wedge \omega| - |v|$ measured in T and let $\widehat{n}(v,\omega) = 2|v \wedge \omega| - |v|$ measured in $\widehat{T}$. If $v \wedge \omega \in \widehat{T}$, then $n(v,\omega) = 2\widehat{n}(v,\omega)$, but if $v \wedge \omega \notin \widehat{T}$, then $n(v,\omega) = 2\widehat{n}(v,\omega) + 2$. Thus, for $v \in \widehat{T}$, if $v \wedge \omega \in \widehat{T}$, then

$$P_\omega(v) = \widehat{P}_\omega(v) = Q^{\widehat{n}(v,\omega)}.$$

Let $\beta = Q^{1/2}$ and note that $\beta > \alpha > 1$. Then

$$P_\omega(v) = \beta^{2\widehat{n}(v,\omega)} = \begin{cases} \beta^{n(v,\omega)} & \text{if } v \wedge \omega \in \widehat{T}, \\ \beta^{n(v,\omega)-2} & \text{if } v \wedge \omega \notin \widehat{T}. \end{cases}$$

Setting $k = |v \wedge \omega|$ and $n = |v|$ measured in T, we have

$$P_\omega(v) = \beta^{2k-n}\varepsilon, \quad \text{where } \varepsilon = \begin{cases} 1 & \text{if } v \wedge \omega \in \widehat{T}, \\ \beta^{-2} & \text{if } v \wedge \omega \notin \widehat{T}. \end{cases}$$

Since $m(I_{v\wedge\omega}) = \frac{1}{m_k}$, by (23) we may write

$$c_{q_0,q_1} m(I_{v\wedge\omega}) = \alpha^{-k}$$

for an appropriate constant c_{q_0,q_1}. Therefore,

$$\begin{aligned} c_{q_0,q_1} P_\omega(v) m(I_{v\wedge\omega}) &= \varepsilon\beta^{2k-n}\alpha^{-k} = \varepsilon\left(\frac{1}{\beta}\right)^{n-2k}\left(\frac{1}{\alpha}\right)^{k} \\ &\leq \varepsilon\left(\frac{1}{\alpha}\right)^{n-2k}\left(\frac{1}{\alpha}\right)^{k} = \varepsilon\left(\frac{1}{\alpha}\right)^{n-k}. \end{aligned} \tag{24}$$

If $\omega \in I_v$, then

$$c_{q_0,q_1} P_\omega(v) m(I_v) = \varepsilon \beta^{2n-n} \alpha^{-n} = \varepsilon \left(\frac{\beta}{\alpha}\right)^n > \varepsilon.$$

Next, assume $w \in T \backslash \widehat{T}$ and $\omega \in \partial T$. Then,

$$c_{q_0,q_1} P_\omega(w) = \frac{c_{q_0,q_1}}{q_1+1} \sum_{v \sim w} P_\omega(v).$$

Therefore, if $\omega \notin I_w$, then $v \wedge \omega = w \wedge \omega$ for all $v \sim w$, so by (24), letting $k = |w \wedge \omega|$, we have

$$\begin{aligned}
c_{q_0,q_1} P_\omega(w) m(I_{w\wedge\omega}) &= \frac{c_{q_0,q_1}}{q_1+1} \sum_{v \sim w} P_\omega(v) m(I_{v\wedge\omega}) \\
&\leq \frac{\varepsilon}{q_1+1} \sum_{v \sim w} \left(\frac{1}{\alpha}\right)^{|v|-k} \\
&\leq \frac{\varepsilon}{q_1+1} \left[q_1 \left(\frac{1}{\alpha}\right)^{|w|+1-k} + \left(\frac{1}{\alpha}\right)^{|w|-1-k} \right] \\
&= \frac{\varepsilon}{q_1+1} \left(\frac{q_1}{\alpha} + \alpha\right) \left(\frac{1}{\alpha}\right)^{|w|-k} \\
&= \frac{q_1(1+q_0)\varepsilon}{(q_1+1)\alpha} \left(\frac{1}{\alpha}\right)^{|w|-k}.
\end{aligned}$$

On the other hand, if $\omega \in I_w$, then $w \wedge \omega = w$, $w^- \wedge \omega = w^-$ and ω belongs to I_{v_0} for exactly one child v_0 of w, so that $v_0 \wedge \omega = v_0$, while for $v \sim w$, $v \neq v_0, w^-$, $v \wedge \omega = w \wedge \omega = w$. Thus, by (24),

$$\begin{aligned}
c_{q_0,q_1} P_\omega(w) m(I_{w\wedge\omega}) &= \frac{c_{q_0,q_1}}{q_1+1} \left[P_\omega(w^-) + P_\omega(v_0) + \sum_{v^-=w, v\neq v_0} P_\omega(v) \right] m(I_w) \\
&= \frac{c_{q_0,q_1}}{q_1+1} \left[P_\omega(w^-) \frac{1}{q_0} m(I_{w^-}) + P_\omega(v_0) m(I_{v_0}) q_1 \right] \\
&\quad + \frac{c_{q_0,q_1}}{q_1+1} \sum_{v^-=w, v\neq v_0} P_\omega(v) m(I_{v\wedge\omega}) \\
&\leq \frac{\varepsilon}{q_1+1} \left(\frac{1}{q_0} + q_1 + \frac{q_1-1}{\alpha} \right) \qquad (25) \\
&= C \left(\frac{1}{\alpha}\right)^{|w|-k},
\end{aligned}$$

where $k = |w \wedge \omega| = |w|$. Moreover,

$$\begin{aligned} c_{q_0,q_1} P_\omega(w) m(I_w) &= \frac{1}{q_1+1}\left(\sum_{v\sim w} P_\omega(v)\right)\alpha^{-|w|} \\ &= \frac{\alpha^{-|w|}}{q_1+1}\left(P_\omega(v_0) + P_\omega(w^-) + \sum_{v^-=w, v\neq v_0, w^-} P_\omega(v)\right) \\ &= \frac{\alpha^{-|w|}\varepsilon}{q_1+1}\left[\beta^{|w|+1} + \beta^{|w|-1} + (q_1-1)\beta^{|w|-1}\right] \\ &= \left(\frac{\beta}{\alpha}\right)^{|w|} \frac{\varepsilon}{q_1+1}\left[\beta + \beta^{-1} + (q_1-1)\beta^{-1}\right] \\ &> \frac{\varepsilon}{q_1+1}\left(\beta + \frac{q_1}{\beta}\right). \end{aligned}$$

This proves that axioms A2 and A3 are satisfied with $r = \dfrac{1}{\alpha} = \dfrac{1}{\sqrt{q_0 q_1}}$. □

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