# THE DISTRIBUTION OF RADIAL EIGENVALUES OF THE EUCLIDEAN LAPLACIAN ON HOMOGENEOUS ISOTROPIC TREES 

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#### Abstract

Let $\Omega$ be a bounded open subset of $\mathbb{R}^{n}$ and $\Delta$ the Euclidean Laplace operator $\sum_{i=1}^{n} \partial^{2} / \partial x_{i}^{2}$. Let $\beta(x)$ denote the number of eigenvalues less or equal to $x$ with respect to the eigenvalue problem $\Delta f=-x f$ on $\Omega$ with $f=0$ on the boundary of $\Omega$. A well-known result due to Hermann Weyl gives the asymptotic formula $\beta(x)=(2 \pi)^{-n} B_{n} m_{n}(\Omega) x^{n / 2}+o\left(x^{n / 2}\right)$ as $x \rightarrow \infty$, where $B_{n}$ is the volume of the unit ball in $\mathbb{R}^{n}$ and $m_{n}(\Omega)$ is the volume of $\Omega$. In this work, we consider the analogous problem for radial functions in the discrete setting of the homogeneous isotropic tree $T$ of homogeneity $q+1(q \geq 2)$. As the volume of $T$ with respect to the hyperbolic metric is infinite, we don't expect and indeed we show that there is no analogous result for the commonly-used hyperbolic Laplacian on $T$. We consider instead the eigenvalue problem for radial functions on $T$ with respect to the Euclidean Laplacian on $T$ introduced in [6], where the boundary condition $f=0$ means that $f$ converges radially to 0 at $\infty$. We prove that $\beta(x)$ is within 2 of $\log _{q} \sqrt{x}$. We also consider other boundary conditions and pose some open questions.


## 1. Introduction

1.1. Some history. In 1900 Lord Rayleigh derived the Rayleigh-Jeans law which gives a formula for the amount of radiation energy emitted by a blackbody (taken in the shape of a cube $C$ in $\mathbb{R}^{3}$ ) at a given frequency. The possible frequencies of the spatial components of the normal modes of the electric field inside the blackbody are proportional to $\sqrt{x}$ where $x$ is an eigenvalue of the negative of the Laplacian on the cube, and the eigenfunctions $f$ are constrained to be 0 on the boundary. Thus $\Delta f=-x f$ in the cube and $f=0$ on the boundary of the cube, where $\Delta=\sum_{i=1}^{3} \partial^{2} / \partial x_{i}^{2}$. In his derivation, Lord Rayleigh made use of the equipartition theorem of thermodynamics together with an asymptotic formula he derived for the number of eigenvalues $\beta(x)$ less or equal to $x$ for large $x$, namely

$$
\beta(x) \sim \frac{1}{6 \pi^{2}} \operatorname{Volume}(C) x^{3 / 2}
$$

At low frequencies the Rayleigh-Jeans law works well, but at high frequencies it gives predictions at odds with the principle of conservation of energy. This problem, known as the ultraviolet catastrophe, led some to conjecture that the above asymptotic formula for $\beta(x)$ was incorrect. However, the formula for $\beta(x)$ turned out to be correct and the problem was really with the equipartition theorem. These issues were resolved after the introduction of quantum theory by Max Planck. In considering the same problem in different shaped regions, it was conjectured in 1910 by Sommerfeld and Lorentz that the formula for $\beta(x)$ depends on the geometry of the region only through its volume, not its specific shape. The conjecture was resolved by Hermann Weyl in 1911 with his formula, known now as Weyl's law, that for any bounded region $\Omega$ in $\mathbb{R}^{n}$,

$$
\beta(x) \sim(2 \pi)^{-n} B_{n} m_{n}(\Omega) x^{n / 2} \text { as } x \rightarrow \infty
$$

where $B_{n}$ is the volume of the unit ball in $\mathbb{R}^{n}$ and $m_{n}(\Omega)$ is the volume of $\Omega$.
Since then, analogues of Weyl's law have been studied in many areas of mathematics including number theory, homogeneous spaces, many classes of manifolds, and on fractals. For connections with physics see [1], [12], [14]. For proofs in various settings see [17], [18], [7], [9],[13], [16].

In recent years considerable attention has been given to solving classical problems in diverse branches of analysis on discrete structures. In this paper, we consider analogues of Weyl's law for radial functions defined on the vertices of a homogeneous isotropic tree.

[^0]1.2. Preliminaries on trees, motivation, and statement of the problem. Let $T$ denote a homogeneous rooted tree of degree $q+1, q \geq 2$. Thus $T$ is an infinite graph in which each vertex of $T$ has exactly $q+1$ neighbours. If $v$ and $u$ are vertices which are neighbours we write $v \sim u$, and refer to $[v, u]$ as the edge from $v$ to $u$. A geodesic path, is a finite or infinite sequence $\left[v_{0}, v_{1}, v_{2}, \ldots\right]$ of vertices in $T$ such that $v_{i} \sim v_{i+1}$ and $v_{i} \neq v_{i+2}$ for each $i \geq 0$. Fixing a vertex $e$ which we call the root of $T$, we view a boundary point $\omega$ of $T$ as being an infinite geodesic path $\omega=\left[e, \omega_{1}, \omega_{2}, \ldots\right)$ starting at the root. Denote the set of boundary points by $\partial T$. If $u, v$ are vertices, we define $d_{H}(u, v)=|u-v|$ to be the number of edges in the geodesic path from $u$ to $v$. If $v=e$, we just write $|u|$, and call it the modulus of $u$. Define $u \wedge v$ to be the vertex of greatest modulus common to the geodesic paths from $e$ to $u$ and from $e$ to $v$. If $\omega=\left[e, v_{1}, \ldots\right)$ is a boundary point, define $u \wedge \omega$ to be $v_{n}$, where $n=\max \left\{i: v_{i}\right.$ is in the geodesic path from $e$ to $\left.u\right\}$. We define $d_{E}(u, v)=q^{-|u \wedge v|}$ and $d_{E}(u, \omega)=q^{-|u \wedge \omega|}$. Observe that for any vertex $v, d_{E}(v, \partial T):=\min \left\{d_{E}(v, \omega): \omega \in \partial T\right\}=q^{-|v|}$.

If $u$ and $v$ are vertices, we write $u \leq v$ if $u$ lies in the geodesic path from $e$ to $v$. If $v \neq e$ we denote by $v^{-}$ the unique neighbour of $v$ with $v^{-} \leq v$.

We can think of $T$ as being a discrete model for $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$, the unit disc in the complex plane where there is both the hyperbolic and Euclidean geometry. Since $d_{H}(u, v)$ can be arbitrarily large and both $d_{E}(u, v)$ and $d_{E}(u, \omega)$ are bounded by 1 , we think of $d_{H}$ as measuring hyperbolic distance and $d_{E}$ as measuring Euclidean distance on $T$. Note that for the boundary point $\omega=\left[e, \omega_{1}, \omega_{2}, \ldots\right)$ the sequence of vertices $\omega_{n}$ converges with respect to the Euclidean distance to $\omega$. We say that the convergence is radial.

If we think of the vertices of $T$ as the states of a stochastic process in which the probability of transition from a vertex $v$ to any one of its neighbours $u$ is $p(v, u)$, where $p(v, u) \geq 0$ and $\sum_{u \sim v} p(v, u)=1$, then the associated Laplace operator for a real-valued function $f$ defined on the vertices is $\Delta_{H} f(v)=$ $\sum_{u \sim v} p(v, u) f(u)-f(v)$. In this formula we are averaging $f$ over the boundary of the hyperbolic ball centered at $v$ of radius 1 , which is the smallest nontrivial such ball, and so for this reason we call $\Delta_{H}$ the hyperbolic Laplacian. We shall assume throughout that the corresponding random walk is isotropic, that is $p(v, u)=1 /(q+1)$ for all vertices $u$ and $v$.

The eigenvalue problem on $T$ with respect to the negative of the hyperbolic Laplacian is that of finding all pairs $(f, x)$ where $f$ is a nontrivial function on $T$ satisfying $\Delta_{H} f(v)=-x f(v)$ for all $v \in T$ with the constraint that $\lim _{n \rightarrow \infty} f\left(\omega_{n}\right)=0$ for all $\omega=\left[e, \omega_{1}, \omega_{2}, \ldots\right) \in \partial T$. We consider only the radial eigenfunctions, i.e. functions $f$ such that $f(v)$ depends only on $|v|$. Denoting an eigenfunction-eigenvalue pair by $(f, x)$, the problem becomes

$$
\begin{equation*}
f_{1}=(1-x) f_{0}, \frac{q}{q+1} f_{n+1}+\frac{1}{q+1} f_{n-1}-f_{n}=-x f_{n} \text { for } n \geq 1, \lim _{n \rightarrow \infty} f_{n}=0 \tag{1}
\end{equation*}
$$

where $f_{n}$ denotes $f(v)$ for $|v|=n$. Since $T$ is unbounded and so has infinite hyperbolic volume, we do not expect an analogue of Weyl's law for $-\Delta_{H}$. We next show that this expectation is correct by proving that the number of eigenvalues in a bounded interval need not be finite.

Theorem 1.1. Let $x$ be a real number. Then $x$ is an eigenvalue with radial eigenfunction $\left\{f_{n}\right\}_{n \geq 0}$ if and only if $|1-x|<1$.

Proof. Observe that if $(f, x)$ is an eigenfunction-eigenvalue pair for $\Delta_{H}$, then since $1-x=(2-x)-1$ it follows that $(g, 2-x)$ is an eigenfunction-eigenvalue pair for $\Delta_{H}$, where $g$ is defined by $v \mapsto(-1)^{|v|} f(v)$. Thus the eigenvalues are symmetric about $x=1$, and so it is enough to prove that $x$ is an eigenvalue if $0<x \leq 1$ and $x$ is not an eigenvalue if $x \leq 0$. Thus for the rest of the proof we will assume that $x \leq 1$.

It is clear from (1) that if $f_{0}=0$, then $f_{n}=0$ for all $n$ and so without loss of generality, we may assume that $f_{0}=1$. The characteristic polynomial associated with $(1)$ is $q r^{2}-(q+1)(1-x) r+1$.

Suppose first that $x<1-2 \sqrt{q} /(q+1)$. Then $(q+1)^{2}(1-x)^{2}>4 q$, so the roots of the characteristic polynomial are real, distinct, and given by $r_{1}=\frac{(q+1)(1-x)+\sqrt{(q+1)^{2}(1-x)^{2}-4 q}}{2 q}$ and $r_{2}=\frac{(q+1)(1-x)-\sqrt{(q+1)^{2}(1-x)^{2}-4 q}}{2 q}$. Note that $r_{1} r_{2}=1 / q$. There are constants $A, B$ such that $f_{n}=A r_{1}^{n}+B r_{2}^{n}$. If $x=0$, then $r_{1}=1$ and $r_{2}=1 / q$. Thus if $x \leq 0$, then $r_{1} \geq 1$ and $0<r_{2} \leq 1 / q<1$, and if $0<x<1-\frac{2 \sqrt{q}}{q+1}$, then both $r_{1}$ and $r_{2}$ are strictly between 0 and 1 . Thus if $0<x<1-\frac{2 \sqrt{q}}{q+1}$, then $f_{n} \rightarrow 0$, and so any such $x$ is an eigenvalue. Suppose next that $x \leq 0$. The initial conditions give

$$
A+B=1 \text { and } A r_{1}+B r_{2}=1-x
$$

Solving for $r_{1}$, we get $\left(r_{1}-r_{2}\right) A=1-x-r_{2}>1-r_{2}>0$. Thus since $A>0$, if $x=0$ then $f_{n} \rightarrow A$ and if $x<0$ then $f_{n} \rightarrow \infty$. We've thus shown that $x$ is not an eigenvalue if $x \leq 0$. This completes the proof in case $x<1-2 \sqrt{q} /(q+1)$.

Suppose next that $x=1-\frac{2 \sqrt{q}}{q+1}$. Then the roots of the characteristic polynomial are $r_{1}=r_{2}=1 / \sqrt{q}$. Thus there exist constants $A, B$ such that $f_{n}=(A+B n) q^{-n / 2}$. Since $f_{n} \rightarrow 0$, then $1-\frac{2 \sqrt{q}}{q+1}$ is an eigenvalue.

Finally suppose that $1-\frac{2 \sqrt{q}}{q+1}<x \leq 1$. Then the roots of the associated characteristic polynomial are

$$
r_{1}=\frac{(q+1)(1-x)+i \sqrt{4 q-(q+1)^{2}(1-x)^{2}}}{2 q}=q^{-1 / 2} e^{i \theta}, r_{2}=\overline{r_{1}}
$$

where $\cos \theta=\frac{(q+1)(1-x)}{2 \sqrt{q}}$ and $\sin \theta=\frac{\sqrt{4 q-(q+1)^{2}(1-x)^{2}}}{2 \sqrt{q}}$ for $\theta \in(0, \pi / 2]$. Then there exists constants $A, B$ such that $f_{n}=q^{-n / 2}(A \cos n \theta+B \sin n \theta)$, so $f_{n} \rightarrow 0$. This shows that any such $x$ is an eigenvalue.

Recall that the Euclidean and hyperbolic Laplacians on $\mathbb{D}$ are related by $\Delta_{E} f(z)=\left(1-|z|^{2}\right)^{-2} \Delta_{H} f(z)$. The factor $\left(1-|z|^{2}\right)^{-2}$ is essentially the reciprocal square distance of $z$ to the boundary in the unit disk, so motivated by this, we defined in [6] the Euclidean Laplacian on $T$ by

$$
\Delta_{E} f(v)=q^{2|v|} \Delta_{H} f(v)=q^{2|v|}\left(\frac{1}{q+1} \sum_{w \sim v} f(w)-f(v)\right)
$$

Since $T$ is bounded with respect to the Euclidean distance we do expect there to be an analogue of Weyl's law with respect to $-\Delta_{E}$. Accordingly, in this paper, we study analogues of Weyl's law on $T$ for radial eigenfunctions of $-\Delta_{E}$. For a radial function $f$ on $T$ we will denote the value of $f$ at vertices of modulus $n$ by $p_{n}$. The equation $\Delta_{E} f(v)=-x f(v)$ becomes $\frac{q}{q+1} p_{n+1}+\frac{1}{q+1} p_{n-1}-p_{n}=-x q^{-2 n} p_{n}, \quad n \geq 1$, with the Laplacian condition at the root being $p_{1}=(1-x) p_{0}$. If $p_{0}=0$, this together with the above recurrence relation implies that $p_{n}=0$ for all $n$, and so no radial eigenfunction can have $p_{0}=0$. Normalizing so that $p_{0}=1$ the eigenvalue problem becomes that of finding pairs $(f, x)$ with $f$ a nontrivial radial function on $T$ given by the sequence $\left\{p_{n}\right\}_{n \geq 0}$ such that

$$
\begin{align*}
& \frac{q}{q+1} p_{n+1}+\frac{1}{q+1} p_{n-1}-p_{n}=-x q^{-2 n} p_{n}, n \geq 1,  \tag{2}\\
& p_{0}=1, p_{1}=1-x, \text { and }  \tag{3}\\
& p_{n} \rightarrow 0 \text { as } n \rightarrow \infty . \tag{4}
\end{align*}
$$

Then our task in formulating Weyl's law is to come up with an asymptotic formula for $\beta(x)$ where the radial eigenvalue counting function $\beta(x)$ is defined by :

$$
\begin{equation*}
\beta(x):=\#\{y: y \text { is a radial eigenvalue and } y \leq x\} \tag{5}
\end{equation*}
$$

1.3. Outline of results. We summarize here briefly the organization of the paper. Observe that we can view each $p_{n}$ satisfying (2) and (3), but not necessarily (4), as a polynomial of degree $n$ in the indeterminate $x$. For any $x$, we can identify the sequence $\left[p_{0}(x), p_{1}(x), \ldots, p_{n}(x), \ldots\right]$ with a radial function $f_{x}$ on $T$, namely $f_{x}(v)=p_{|v|}(x)$. Then at every vertex $v, f_{x}$ satisfies the eigenvalue equation $\Delta_{E} f_{x}(v)=-x f(v)$. Thus the radial eigenvalues whose corresponding eigenfunctions have vanishing radial limits on $\partial T$ are precisely those numbers $x$ for which $\lim _{n \rightarrow \infty} p_{n}(x)=0$.

In Section 2 we study the coefficients of the polynomials $p_{n}(x)$, and we also show that when viewed as functions of a complex variable the sequence $p_{n}$ converges locally uniformly in $\mathbb{C}$ to an entire function $p_{\infty}$. A natural guess would be that the roots of $p_{\infty}$ are precisely the eigenvalues of $-\Delta_{E}$, and indeed we shall prove this in Section 4.

In Section 3 we use operator theory to prove that the roots of each $p_{n}$ are real, positive, and simple and they have certain monotonicity properties. We also give formulas for the sum and products of the roots of $p_{n}$ and give some upper and lower bounds for them.

In Section 4 we prove interlacing properties of the roots of pairs $p_{n}$ and $p_{m}$. If $P$ is a polynomial of degree $m$ and $Q$ a polynomial of degree $n$ with $m<n$, then the roots of $P$ and $Q$ are said to interlace if each root of $P$ lies between a pair of consecutive roots of $Q$ and between any pair of consecutive roots of $Q$ there is at most one root of $P$. If $\left\{P_{n}\right\}_{n \geq 0}$ is a sequence of polynomials in which each $P_{n}$ has degree $n$,
the roots of the family are said to be Stieltjes interlacing if the roots of $P_{n}$ and $P_{m}$ interlace for every pair $m, n$ with $m<n$. In [2] it is proved that sequences $\left\{P_{n}\right\}_{n \geq 0}$ which satisfy certain three-term recurrence relations automatically have roots which are Stieltjes interlacing. We show that with the proper choice of parameters and multiplication by a certain function of $n$, our sequence of polynomials $\left\{p_{n}\right\}_{n \geq 0}$ fits within this framework. However for our purposes this is not sufficient. We prove in addition that the roots of our family of polynomials satisfies strong Stieltjes interlacing, by which we mean that for each $p_{n}$ and $p_{m+n}$, the smallest $n+1$ roots of $p_{m+n}$ interlace with the $n$ roots of $p_{n}$. We deduce from this that the roots of $p_{\infty}$ are precisely the radial eigenvalues of $-\Delta_{E}$ with radial eigenfunctions being 0 on $\partial T$.

In Section 5 we use results of the previous sections to prove Theorem 5.1 which shows that the eigenvalue counting function $\beta(x)$ is at most 2 away from $\log _{q} \sqrt{x}$.

In Section 6 we discuss the proof of Favard's theorem as given in [2]. Favard's theorem says that whenever we have a sequence of polynomials $\left\{P_{n}\right\}_{n \geq 0}$ satisfying a certain three-term recurrence relation, then there exists a Borel probability measure $\mu$ on $\mathbb{R}$ such that the sequence is orthogonal in the Hilbert space $L^{2}(\mu)$. We also discuss Favard's theorem in the framework of our polynomials $\left\{p_{n}\right\}_{n \geq 0}$.

Finally, in Section 7 we present some open questions as expressed by several conjectures involving the polynomials $p_{n}(x)$ and $p_{\infty}$ all of which have strong numerical evidence for being true. One of the conjectures says that the roots of $p_{n}$ interlace with the sequence $1, q, q^{2}, q^{3}, q^{4}, \ldots$. We describe some consequences in case this conjecture is true, including a more elegant formula for the eigenvalue counting function $\beta(x)$. We also study the eigenvalue counting function associated with the eigenvalue problem for $-\Delta_{E}$ in case the boundary function is taken to be a nonzero constant. This problem gives rise to other conjectures, and a few theorems are proved which describe the relation between some of those conjectures.

## 2. Associated polynomials $p_{n}(x)$

2.1. The polynomials $p_{n}$. As we mentioned in the outline of results, we can view each $p_{n}$ satisfying (2) and (3), but not necessarily (4), as a polynomial of degree $n$ in the indeterminate $x$. For any $x$, we can identify the sequence $\left[p_{0}(x), p_{1}(x), \ldots, p_{n}(x), \ldots\right]$ with a radial function $f_{x}$ on $T$, namely $f_{x}(v)=p_{|v|}(x)$. Then at every vertex $v, f_{x}$ satisfies the eigenvalue equation $\Delta_{E} f_{x}(v)=-x f(v)$. Thus the radial eigenvalues whose corresponding eigenfunctions vanish at $\infty$ are those numbers $x$ for which $\lim _{n \rightarrow \infty} p_{n}(x)=0$.

We list a few of these polynomials (calculated using Mathematica):

$$
\begin{aligned}
& p_{0}(x)=1 \\
& p_{1}(x)=1-x, \\
& p_{2}(x)=1-\frac{(q+1)\left(1+q^{2}\right)}{q^{3}} x+\frac{q+1}{q^{3}} x^{2}, \\
& p_{3}(x)=1-\frac{\left(q^{3}-1\right)\left(1+q+q^{3}\right)}{q^{5}(q-1)} x+\frac{(q+1)\left(q^{6}-1\right)}{q^{8}(q-1)} x^{2}-\frac{(1+q)^{2}}{q^{8}} x^{3}, \\
& p_{4}(x)=1-\frac{(q+1)\left(q^{2}+1\right)\left(1+q+q^{2}+q^{4}\right)}{q^{7}} x+\frac{(q+1)\left(q^{5}-1\right)\left(1+q+q^{3}+q^{5}\right)}{q^{12}(q-1)} x^{2} \\
&-\frac{(q+1)^{3}\left(1+q^{2}\right)\left(1+q^{4}\right)}{q^{15}} x^{3}+\frac{(1+q)^{3}}{q^{15}} x^{4}, \\
& p_{5}(x)=1-\frac{\left(q^{5}-1\right)\left(1+q+q^{2}+q^{3}+q^{5}\right)}{q^{9}(q-1)} x+\frac{(q+1)\left(q^{3}+1\right)\left(q^{5}-1\right)\left(1+q+q^{2}+q^{4}+q^{6}\right)}{q^{16}(q-1)} x^{2} \\
&-\frac{(q+1)^{2}\left(q^{7}-1\right)\left(1+q+q^{3}+q^{5}+q^{7}\right)}{q^{21}(q-1)} x^{3}+\frac{(q+1)^{3}\left(q^{10}-1\right)}{q^{24}(q-1)} x^{4}-\frac{(1+q)^{4}}{q^{24}} x^{5}, \\
& p_{6}(x)=1--\frac{\left(q^{2}+1\right)\left(q^{6}-1\right)\left(1+q+q^{4}\right)}{q^{11}(q-1)} x+\frac{\left(q^{5}-1\right)\left(q^{6}-1\right)\left(q^{7}-1\right)\left(1-q^{2}+q^{3}\right)}{q^{20}(q-1)^{3}} x^{2} \\
&-\frac{(1+q)^{3}\left(1+q^{2}\right)\left(1+q^{4}\right)\left(1+q^{4}-q^{5}+q^{6}\right)\left(q^{7}-1\right)}{q^{27}(q-1)} x^{3} \\
&+\frac{(q+1)^{3}\left(q^{9}-1\right)\left(1+q+q^{3}+q^{5}+q^{7}+q^{9}\right)}{q^{32}(q-1)} x^{4}-\frac{(q+1)^{4}\left(q^{12}-1\right)}{q^{35}(q-1)} x^{5}+\frac{(1+q)^{5}}{q^{35}} x^{6} .
\end{aligned}
$$

Using Excel and the recurrence relation in (2), it is a simple matter to calculate several terms of the sequence $\left\{p_{n}(x)\right\}_{n=0}^{\infty}$ for specific choices of $q$ and $x$. Such calculations suggested the following statements to be proved later.

- $p_{\infty}(x):=\lim _{n \rightarrow \infty} p_{n}(x)$ exists and is finite for all $x$;
- if $x<0$ then $p_{\infty}(x)<0$ and decreases as $x$ decreases;
- if $x>0$ then $p_{\infty}(x)$ alternates infinitely often between negative and positive values as $x$ increases to $\infty$. In fact we will show shortly that the convergence of $p_{n}$ to $p_{\infty}$ is locally uniform on $\mathbb{C}$.

From now on we shall denote the roots of $p_{n}$ by $x_{n, k}, k=1, \ldots, n$, listed in increasing order. This is justified by Theorem 3.1 and Theorem 3.3(iii), where we prove that the roots are real, positive, and simple.

We calculated the roots of $p_{n}(x)$ in case $q=2$ for the first few values of $n$ using Mathematica. The results (rounded) are as follows:

| $x_{n, k}$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ | $k=9$ | $k=10$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |  |  |  |
| $n=1$ | 1 |  |  |  |  |  |  |  |  |  |
| $n=2$ | 0.607 | 4.39 |  |  |  |  |  |  |  |  |
| $n=3$ | 0.487 | 3.42 | 17.09 |  |  |  |  |  |  |  |
| $n=4$ | 0.438 | 3.088 | 13.11 | 68.4 |  |  |  |  |  |  |
| $n=5$ | 0.417 | 2.946 | 11.77 | 52.42 | 274 |  |  |  |  |  |
| $n=6$ | 0.407 | 2.880 | 11.194 | 47.0 | 210 | 1094 |  |  |  |  |
| $n=7$ | 0.402 | 2.847 | 10.93 | 44.7 | 188 | 839 | 4375 |  |  |  |
| $n=8$ | 0.399 | 2.832 | 10.80 | 43.7 | 179 | 753 | 3355 | 17501 |  |  |
| $n=9$ | 0.398 | 2.824 | 10.74 | 43.2 | 175 | 716 | 3010 | 13,418 | 70,005 | 280,019 |

Calculations with different values of $q \geq 2$ indicate that except for $x_{1,1}=1$, all the roots are increasing functions of $q$. We are particularly interested in knowing this for the smallest roots, and in Theorem 3.2 we prove that this is indeed the case.

For $q=2$, using Mathematica, we have also found the approximate values of the first few roots of $p_{\infty}(x)$ by considering $p_{n}(x)$ for $n$ large enough that the values stabilize, and observing the first several occurrences in $x$ of transitions from positive to negative or from negative to positive values as $x$ increases. The results are

$$
\begin{array}{llllllll}
x_{\infty, k}: & 0.39683 & 2.816014 & 10.676 & 42.66 & 170.66 & 682.66 & 2730.6666
\end{array}
$$

This calculation suggests that the entries in each column of the above table of roots decrease with $n$ and their limits are the eigenvalues. We shall prove that this is indeed the case.
2.2. Some formulas for the coefficients of $p_{n}(x)$. Let $p_{n, k}$ denote the coefficient of $x^{k}$ in $p_{n}(x)$. Obviously $p_{n, k}=0$ for $k>n$ and $p_{1,1}=-1$. The following two theorems give the remaining entries of the lower triangular matrix $\left\{p_{n, k}\right\}$ which we can identify explicitly.

Theorem 2.1. (i) $p_{n, 0}=p_{n}(0)=1$ for every $n \geq 0$.
(ii) $p_{n, 1}=-\frac{q^{2}-q+1}{(q-1)^{2}}+\frac{q^{2}+1}{(q-1)^{2}} q^{-n}-\frac{q}{(q-1)^{2}} q^{-2 n}, n \geq 1$.
(iii) $p_{n, n}=(-1)^{n} \frac{(q+1)^{n-1}}{q^{n^{2}-1}}, n \geq 1$.
(iv) $p_{n, n-1}=(-1)^{n-1} \frac{(q+1)^{n-1}}{q^{n^{2}-1}} \sum_{j=0}^{n-1} q^{2 j}=(-1)^{n-1} \frac{(q+1)^{n-1}}{q^{n^{2}-1}}\left(\frac{q^{2 n}-1}{q^{2}-1}\right), n \geq 1$.
(v) $p_{n, n-2}=(-1)^{n} \frac{(q+1)^{n-3}\left(q^{2 n-3}-1\right)\left(1+\sum_{k=1}^{n-1} q^{2 k-1}\right)}{(q-1) q^{n^{2}-4}}$ $=(-1)^{n} \frac{(q+1)^{n-4}\left(q^{2 n-3}-1\right)\left(q^{2 n-1}+q^{2}-q-1\right)}{(q-1)^{2} q^{n^{2}-4}}, \quad n \geq 2$.
Proof. (i): Plugging $x=0$ in (3) gives $p_{0}(0)=1=p_{1}(0)$. Then plugging $x=0$ into the recurrence relation (2) and a simple induction shows that $p_{n}(0)=1$ for all $n$.
(ii): It follows from (2) that $p_{n, k}$ is determined by

$$
\begin{equation*}
q p_{n+1, k}+p_{n-1, k}-(q+1) p_{n, k}=-(q+1) q^{-2 n} p_{n, k-1}, \quad 1 \leq k \leq n, \tag{6}
\end{equation*}
$$

and the conditions $p_{n, k}=0$ for $k>n, p_{n, 0}=1$ for every $n \geq 0$, and $p_{1,1}=-1$. In particular,

$$
q p_{n+1,1}+p_{n-1,1}-(q+1) p_{n, 1}=-(q+1) q^{-2 n}
$$

which is a nonhomogeneous linear 2 nd order recurrence relation with constant coefficients. The associated homogeneous equation has solutions 1 and $q^{-n}$, and there is a particular solution of the form $A q^{-2 n}$. Plugging it into
the equation gives $A=-q /(q-1)^{2}$, so $p_{n, k}=c_{1}+c_{2} q^{-n}-\frac{q}{(q-1)^{2}} q^{-2 n}$. Putting $p_{0,1}=0$ and $p_{1,1}=-1$ gives $c_{1}+c_{2}=q /(q-1)^{2}$ and $c_{1}+c_{2} / q=\frac{1}{q(q-1)^{2}}-1$, from which we obtain $c_{1}=-\frac{\left(q^{2}-q+1\right)}{(q-1)^{2}}$ and $c_{2}=\frac{q^{2}+1}{(q-1)^{2}}$, as desired.

Before we prove the remaining formulas, note that for constants $A$ and $B$, the solution of the recurrence relation

$$
\begin{equation*}
z_{n+1}=A \cdot B^{n} z_{n}, n \geq n_{0} \tag{7}
\end{equation*}
$$

is given by

$$
\begin{equation*}
z_{n}=A^{n-n_{0}} B^{\frac{\left(n-n_{0}\right)\left(n-n_{0}+1\right)}{2}} z_{n_{0}}, n \geq n_{0} . \tag{8}
\end{equation*}
$$

(iii): If we replace $k$ by $n+1$ in formula (6), we get

$$
p_{n+1, n+1}=-\left(\frac{q+1}{q}\right) q^{-2 n} p_{n, n}, n \geq 1 .
$$

Applying (7) and (8) with $A=-(q+1) / q, B=q^{-2}, n_{0}=1$, and $z_{n_{0}}=p_{1,1}=-1$ gives result (iii).
(iv): To calculate $x_{n, n-1}$ we again use (7) and (8). From (6), if we replace $k$ with $n$ and then $x_{n, n-1}$ with $y_{n}$, using the result of part (iii) we get

$$
\begin{equation*}
y_{n+1}=\left(\frac{q+1}{q}\right) \frac{(-1)^{n}(q+1)^{n-1}}{q^{n^{2}-1}}-\left(\frac{q+1}{q}\right) q^{-2 n} y_{n}, n \geq 1, y_{1}=1 . \tag{9}
\end{equation*}
$$

Letting $y_{n}^{h}$ denote the general solution of the associated homogeneous problem obtained by ignoring the first term on the right side of (9), its solution by (7) and (8), is

$$
y_{n}^{h}=c \frac{(-1)^{n-1}(q+1)^{n-1}}{q^{n^{2}-1}}
$$

for some constant $c$. Observe that (9) can now be written as

$$
\begin{equation*}
y_{n+1}=-\frac{1}{c}\left(\frac{q+1}{q}\right) y_{n}^{h}-\left(\frac{q+1}{q}\right) q^{-2 n} y_{n} . \tag{10}
\end{equation*}
$$

Also note that $\frac{y_{n+1}^{h}}{y_{n}^{h}}=-\left(\frac{q+1}{q}\right) q^{-2 n}$. We now look for a particular solution $y_{n}^{p}$ of (10) of the form

$$
y_{n}^{p}=y_{n}^{h} g(n)=\frac{(-1)^{n-1}(q+1)^{n-1}}{q^{n^{2}-1}} g(n)
$$

for some nonconstant function $g(n)$. Plugging this into (10) and then dividing by $-\left(\frac{q+1}{q}\right) y_{n}^{h} q^{-2 n}$ gives

$$
\begin{equation*}
g(n+1)=q^{2 n}+g(n) \tag{11}
\end{equation*}
$$

We look for solutions of this of the form $g(n)=A q^{2 n}$. Plugging this into (11) gives $A=1 /\left(q^{2}-1\right)$, so $g(n)=$ $q^{2 n} /\left(q^{2}-1\right)$. Thus $y_{n}^{p}=y_{n}^{h} q^{2 n} /\left(q^{2}-1\right)$. Finally,

$$
p_{n, n-1}=y_{n}=y_{n}^{h}+y_{n}^{p}=\frac{(-1)^{n-1}(q+1)^{n-1}}{q^{n^{2}-1}}\left(c+\frac{q^{2 n}}{q^{2}-1}\right)
$$

and using $p_{1,0}=1$, gives $c=-1 /\left(q^{2}-1\right)$, and so

$$
p_{n, n-1}=\frac{(-1)^{n-1}(q+1)^{n-1}}{q^{n^{2}-1}}\left(\frac{q^{2 n}-1}{q^{2}-1}\right)
$$

which proves (iv).
The formula in (v) is proved by induction on $n$. For $n=2$, the right side is

$$
\frac{(-1)^{2}(q+1)^{-1}(q-1)(q+1)}{(q-1) q^{0}}=1=p_{n, 0} .
$$

For the inductive step, let $n \geq 2$ and assume the formula holds for $n$. We rewrite (6) as

$$
\begin{equation*}
q x_{n+1, n-1}+(q+1) q^{-2 n} x_{n, n-2}=(q+1) x_{n, n-1}-x_{n-1, n-1} . \tag{12}
\end{equation*}
$$

The result will follow if we can confirm that both sides come out the same if we replace the first term on the left side with the desired formula and apply the inductive hypothesis to the second term on the left side. First we deal with the right side. Applying formulas (iii) and (iv) of the theorem, the right side becomes

$$
\begin{align*}
\text { RHS } & =\frac{(-1)^{n-1}(q+1)^{n}\left(q^{2 n}-1\right)}{q^{n^{2}-1}\left(q^{2}-1\right)}-\frac{(-1)^{n-1}(q+1)^{n-2}}{q^{n^{2}-1} q^{-2 n+1}}  \tag{13}\\
& =\frac{(-1)^{n-1}(q+1)^{n-2}}{q^{n^{2}-1}}\left(\frac{(q+1)^{2}\left(q^{2 n}-1\right)}{q^{2}-1}-q^{2 n-1}\right) \\
& =\frac{(-1)^{n-1}(q+1)^{n-2}}{q^{n^{2}-1}}\left(\frac{q^{2 n+1}+q^{2 n-1}-q-1}{q-1}\right)
\end{align*}
$$

Next, working on the left side of (12) gives

$$
\begin{aligned}
& \text { LHS }=q \frac{(-1)^{n-1}(q+1)^{n-2}\left(q^{2(n+1)-3}-1\right)\left(1+\sum_{k=1}^{n} q^{2 k-1}\right)}{(q-1) q^{(n+1)^{2}-4}} \\
& \quad+\frac{(-1)^{n}(q+1)^{n-2} q^{-2 n}\left(q^{2 n-3}-1\right)\left(1+\sum_{k=1}^{n-1} q^{2 k-1}\right)}{(q-1) q^{n^{2}-4}} \\
&=(q+1)^{n-2}\left[\frac{(-1)^{n+1}\left(q^{2 n-1}-1\right)\left(1+\sum_{k=1}^{n-1} q^{2 k-1}+q^{2 n-1}\right)+(-1)^{n}\left(q^{2 n-3}-1\right)\left(1+\sum_{k=1}^{n-1} q^{2 k-1}\right)}{(q-1) q^{n^{2}+2 n-4}}\right] \\
&=\frac{(-1)^{n+1}(q+1)^{n-2}}{(q-1) q^{n^{2}+2 n-4}}\left[\left(1+\sum_{k=1}^{n-1} q^{2 k-1}\right)\left(q^{2 n-1}-1-q^{2 n-3}+1\right)+q^{2 n-1}\left(q^{2 n-1}-1\right)\right] \\
&=\frac{(-1)^{n+1}(q+1)^{n-2}}{(q-1) q^{n^{2}+2 n-4}} q^{2 n-3}\left[\left(q^{2}-1\right)\left(1+\sum_{k=1}^{n-1} q^{2 k-1}\right)+q^{2}\left(q^{2 n-1}-1\right)\right] \\
&=\frac{(-1)^{n+1}(q+1)^{n-2}}{(q-1) q^{n^{2}-1}} q^{2 n-3}\left[q^{2}-1+\sum_{k=1}^{n-1} q^{2 k+1}-\sum_{k=1}^{n-1} q^{2 k-1}+q^{2 n+1}-q^{2}\right] \\
&=\frac{(-1)^{n+1}(q+1)^{n-2}}{(q-1) q^{n^{2}-1}}\left[-1+q^{2 n-1}+q^{2 n+1}-q\right],
\end{aligned}
$$

in agreement with (13). This completes the proof.
Theorem 2.2. The following formula holds:

$$
\begin{aligned}
p_{n, 2} & =\frac{q^{2}\left(q^{5}+1\right)}{(q-1)^{4}(q+1)\left(1+q+2 q^{2}+q^{3}+q^{4}\right)}-\frac{q\left(q^{4}+1\right)}{(q-1)^{3}\left(q^{3}-1\right)} q^{-n} \\
& +\frac{q\left(q^{3}+1\right)}{(q-1)^{4}(q+1)} q^{-2 n}-\frac{q^{2}\left(q^{2}+1\right)}{(q-1)^{3}\left(q^{3}-1\right)} q^{-3 n}+\frac{q^{4}}{(q-1)^{3}\left(q^{2}+1\right)\left(q^{3}-1\right)} q^{-4 n}
\end{aligned}
$$

In order to prove this theorem we make use of the following lemma.
Lemma 2.1. For each $k$ there exist $A_{k, 0}, A_{k, 1}, \ldots, A_{k, 2 k}$ depending only on $q$ and $k$ such that

$$
\begin{equation*}
p_{n, k}=\sum_{j=0}^{2 k} \frac{A_{k, j}}{q^{j n}} \text { for every } n \geq k \tag{14}
\end{equation*}
$$

Proof. We prove (14) by induction on $k$. By Theorem 2.1(ii) the result holds for $k=1$. Now let $k \geq 2$ and assume (14) holds for $k-1$. Then we have

$$
q p_{n+1, k}+p_{n-1, k}-(q+1) p_{n, k}=-(q+1) q^{-2 n} p_{n, k-1}=-(q+1) \sum_{j=0}^{2(k-1)} \frac{A_{k-1, j}}{q^{(j+2) n}}
$$

where the last equality follows from the inductive hypothesis. Letting $y_{n}:=p_{n, k}$, this becomes

$$
\begin{equation*}
q y_{n+1}+y_{n-1}-(q+1) y_{n}=-(q+1) \sum_{j=0}^{2(k-1)} \frac{A_{k-1, j}}{q^{(j+2) n}} \tag{15}
\end{equation*}
$$

This is a second order nonhomogeneous linear recurrence relation with constant coefficients. The general solution of the corresponding homogeneous equation is $c_{1}+c_{2} / q^{n}$. A simple substitution shows that a particular solution of the equation

$$
q y_{n+1}+y_{n-1}-(q+1) y_{n}=-(q+1) A_{k-1, j} q^{-(j+2) n}
$$

is

$$
y_{n}=\frac{q^{j+2}(-1)(q+1) A_{k-1, j}}{\left(q^{j+2}-q\right)\left(q^{j+2}-1\right)} q^{-(j+2) n}
$$

By summing all of these particular solutions with the solution of the homogeneous equation we get that the general solution of equation (15) can be written in the form

$$
y_{n}=c_{1}+\frac{c_{2}}{q^{n}}+\sum_{j=0}^{2(k-1)} \frac{q^{j+2}(-1)(q+1) A_{k-1, j}}{\left(q^{j+2}-q\right)\left(q^{j+2}-1\right)} q^{-(j+2) n}=c_{1}+\frac{c_{2}}{q^{n}}+\sum_{j=2}^{2 k} \frac{q^{j}(-1)(q+1) A_{k-1, j-2}}{\left(q^{j}-q\right)\left(q^{j}-1\right)} q^{-j n} .
$$

Then

$$
\begin{equation*}
y_{n}=c_{1}+\frac{c_{2}}{q^{n}}+\sum_{j=2}^{2 k} \frac{A_{k, j}}{q^{j n}}, \quad \text { where } \quad A_{k, j}=\frac{q^{j}(-1)(q+1) A_{k-1, j-2}}{\left(q^{j}-q\right)\left(q^{j}-1\right)}, 2 \leq j \leq 2 k . \tag{16}
\end{equation*}
$$

Note that the quantities $A_{k, j}$ 's do not depend on $n$. Finally, from (16), we get

$$
y_{n}=A_{k, 0}+\frac{A_{k, 1}}{q^{n}}+\sum_{j=2}^{2 k} \frac{A_{k, j}}{q^{j n}}
$$

where $A_{k, 0}$ is the choice of $c_{1}$ and $A_{k, 1}$ the choice of $c_{2}$ such that that $y_{n}$ satisfies the initial conditions $y_{k-1}=0$, $y_{k}=p_{k, k}=\frac{(-1)^{k}(q+1)^{k-1}}{q^{k^{2}-1}}$ (by Theorem 2.1(iii)). The result is

$$
\begin{aligned}
& A_{k, 0}=\frac{1}{q-1}\left(\sum_{j=2}^{2 k} A_{k, j} q^{-j k}\left(q^{j}-q\right)+(-1)^{k}(q+1)^{k-1} q^{-k^{2}+2}\right) \\
& A_{k, 1}=\frac{1}{q-1}\left(-\sum_{j=2}^{2 k} A_{k, j} q^{k} q^{-j k}\left(q^{j}-1\right)+(-1)^{k+1}(q+1)^{k-1} q^{k} q^{1-k^{2}}\right)
\end{aligned}
$$

This completes the proof.
Proof of Theorem 2.2. Applying Lemma 2.1 with $k=2$ we see that $p_{n, 2}$ can be written in the form

$$
\begin{equation*}
p_{n, 2}=\sum_{j=0}^{4} \frac{A_{2, j}}{q^{j n}} . \tag{17}
\end{equation*}
$$

Using explicit formulas for $p_{2,2}, \ldots, p_{6,2}$ (which we can read off directly from the formulas we provided for $\left.p_{2}(x), \ldots, p_{6}(x)\right)$, we get the following linear equations for $A_{2,0}, A_{2,1}, A_{2,2}, A_{2,3}, A_{2,4}$ :

$$
\left(\begin{array}{ccccc}
1 & q^{-2} & q^{-4} & q^{-6} & q^{-8} \\
1 & q^{-3} & q^{-6} & q^{-9} & q^{-12} \\
1 & q^{-4} & q^{-8} & q^{-12} & q^{-16} \\
1 & q^{-5} & q^{-10} & q^{-15} & q^{-20} \\
1 & q^{-6} & q^{-12} & q^{-18} & q^{-24}
\end{array}\right)\left(\begin{array}{c}
A_{2,0} \\
A_{2,1} \\
A_{2,2} \\
A_{2,3} \\
A_{2,4}
\end{array}\right)=\left(\begin{array}{c}
p_{2,2} \\
p_{3,2} \\
p_{4,2} \\
p_{5,2} \\
p_{6,2}
\end{array}\right),
$$

where for the right side we use the following values:

$$
\begin{aligned}
& p_{2,2}=\frac{q+1}{q^{3}} \\
& p_{3,2}=\frac{(q+1)\left(q^{6}-1\right)}{q^{8}(q-1)} \\
& p_{4,2}=\frac{(q+1)\left(q^{5}-1\right)\left(1+q+q^{3}+q^{5}\right)}{q^{12}(q-1)} \\
& p_{5,2}=\frac{(q+1)\left(q^{3}+1\right)\left(q^{5}-1\right)\left(1+q+q^{2}+q^{4}+q^{6}\right)}{q^{16}(q-1)} \\
& p_{6,2}=\frac{\left(q^{5}-1\right)\left(q^{6}-1\right)\left(q^{7}-1\right)\left(1-q^{2}+q^{3}\right)}{q^{20}(q-1)^{3}}
\end{aligned}
$$

Solving this system for $A_{2,0}, \ldots, A_{2,4}$ using Mathematica and plugging the solutions into (17) yields the result.
2.3. Convergence of $\left\{p_{n}\right\}_{n=0}^{\infty}$ to the entire function $p_{\infty}$.

Theorem 2.3. For $z \in \mathbb{C}$, let $p_{n}=p_{n}(z)$ be the sequence of polynomials determined by the recurrence relation in (2) and the initial conditions in (3). Let $r_{n}=p_{n}-p_{n-1}, n \geq 1$.
(i) The sequence $p_{n}(z)$ converges locally uniformly in $\mathbb{C}$ to an entire function $p_{\infty}(z)$.
(ii) The sequence $q^{n} r_{n}$ is locally uniformly bounded.
(iii) The roots of $p_{\infty}$, i.e. the radial eigenvalues of $-\Delta_{E}$, are necessarily real and positive.

Proof. To obtain (i) we begin with a proof of the preliminary result that for any bounded open subset $U$ of $\mathbb{C}$ and every $z \in \bar{U}$, there exists $A \in \mathbb{R}$ depending only on $U$ and $q$ such that

$$
\begin{equation*}
\left|r_{n}\right| \leq A q^{-n / 2} \text { for all } n \geq 1 \text { and } z \in \bar{U} \tag{18}
\end{equation*}
$$

This will later be used to obtain (ii), a stronger result. The sequence $\left\{r_{n}\right\}_{n=1}^{\infty}$ satisfies

$$
\begin{equation*}
q r_{n+1}-r_{n}=-z(q+1) q^{-2 n} p_{n} \text { and } p_{n}=1+r_{1}+r_{2}+\cdots+r_{n} \tag{19}
\end{equation*}
$$

To prove (18), choose $n_{0}$ such that

$$
\gamma+|z|\left(\frac{q+1}{q}\right) \gamma^{3 n_{0}-1}(1-\gamma)^{-1} \leq 1 \text { for all } z \in \bar{U}
$$

where $\gamma=q^{-1 / 2}$. Then define $A=1+\max \left\{\left|r_{n}\right| \gamma^{-n}: n \leq n_{0}\right\}$. We will prove by strong induction that (18) holds with this choice of $A$. By definition of $A$ the inequality in (18) holds for all $n \leq n_{0}$. Let $n \geq n_{0}$ and assume the inequality holds for $1,2, \ldots, n$. Then

$$
\begin{aligned}
\left|r_{n+1}\right| & =\left|\frac{r_{n}}{q}-z\left(\frac{q+1}{q}\right) q^{-2 n}\left(1+r_{1}+\cdots+r_{n}\right)\right| \\
& \leq\left|\frac{r_{n}}{q}\right|+|z|\left(\frac{q+1}{q}\right) q^{-2 n}\left(1+\left|r_{1}\right|+\cdots+\left|r_{n}\right|\right) \\
& \leq A\left(\gamma^{n+2}+|z|\left(\frac{q+1}{q}\right) \gamma^{4 n}\left(1+\gamma+\gamma^{2}+\cdots+\gamma^{n}\right)\right. \\
& \leq A\left(\gamma^{n+2}+|z|\left(\frac{q+1}{q}\right) \gamma^{4 n}(1-\gamma)^{-1}\right) \\
& =A \gamma^{n+1}\left(\gamma+|z|\left(\frac{q+1}{q}\right) \gamma^{3 n-1}(1-\gamma)^{-1}\right) \\
& \leq A \gamma^{n+1}\left(\gamma+|z|\left(\frac{q+1}{q}\right) \gamma^{3 n_{0}-1}(1-\gamma)^{-1}\right) \\
& \leq A \gamma^{n+1}
\end{aligned}
$$

This completes the proof of (18).
Thus the sequence $\left\{r_{n}(z)\right\}_{n=1}^{\infty}$ is absolutely summable, and the tail end can be made arbitarily small uniformly for all $z \in \bar{U}$. Since $p_{n}=1+r_{1}+\ldots r_{n}$, it follows that the sequence $p_{n}$ converges uniformly on $\bar{U}$. Since each $p_{n}$ is a polynomial, the limit function is analytic on $U$. That this is true for all $U$ implies the limit function $p_{\infty}$ is entire.

To prove (ii) let $U \subset \mathbb{C}$ be open and relatively compact. By part (i) we can choose $M \in \mathbb{R}$ such that

$$
|z|\left(\frac{q+1}{q}\right)\left|p_{n}(z)\right|<M \text { for all } n \geq 0 \text { and } z \in \bar{U}
$$

It follows from this and (19) that

$$
\begin{equation*}
\left|r_{n+1}\right| \leq\left|\frac{r_{n}}{q}\right|+M q^{-2 n} \tag{20}
\end{equation*}
$$

We show next by induction on $n$ that

$$
\begin{equation*}
\left|r_{n}\right| \leq \frac{|z|}{q^{n-1}}+\frac{M}{q^{n}} \sum_{j=0}^{n-2} q^{-j}, n \geq 2 \tag{21}
\end{equation*}
$$

That it holds for $n=2$ is immediate from (20) and that $r_{1}=-z$. Assuming the inequality holds for $r_{n}$, then

$$
\left|r_{n+1}\right| \leq\left|\frac{r_{n}}{q}\right|+M q^{-2 n} \leq \frac{1}{q}\left(\frac{|z|}{q^{n-1}}+\frac{M}{q^{n}} \sum_{j=0}^{n-2} q^{-j}\right)+M q^{-2 n}=\frac{|z|}{q^{n}}+\frac{M}{q^{n+1}} \sum_{j=0}^{n-1} q^{-j}
$$

completing the proof of (21). Thus by (21) we have

$$
\left|q^{n} r_{n}\right| \leq q|z|+M \sum_{j=0}^{\infty} q^{-j}=q\left(|z|+\frac{M}{q-1}\right)
$$

proving (ii).
To prove (iii) we make use of the following statement of Green's theorem which is a special case of the version in ([5], Theorem 4.1, page 12):

$$
\begin{equation*}
\left.\sum_{|v|<N} \Delta_{H} f(v) g(v)+\frac{1}{q+1} \sum_{0<|v|<N}\left(f(v)-f\left(v^{-}\right)\right)\left(g(v)-g\left(v^{-}\right)\right)=\frac{1}{q+1} \sum_{|v|=N}\left(f(v)-f\left(v^{-}\right)\right) g\left(v^{-}\right)\right) . \tag{22}
\end{equation*}
$$

In the general formula in [5], since $T$ is homogeneous and isotropic, we take $\alpha_{v}=1$ and the conductances $c$ to be $1 /(q+1)$. We apply this with $f=\bar{g}$ for $f$ the radial function determined by the sequence $\left\{p_{n}(z)\right\}_{n=0}^{\infty}$ and $z$ an eigenvalue for $\Delta_{E}$. Thus $p_{\infty}(z)=0$ and $\Delta_{H} f(v)=-z q^{-2|v|} f(v)$. For each $N$, using the fact that $f$ is radial, (22) becomes

$$
-z+\sum_{j=1}^{N-1}-z q^{-2 j}\left|p_{j}^{2}\right|(q+1) q^{j-1}+\frac{1}{q+1} \sum_{j=1}^{N-1}\left|p_{j}-p_{j-1}\right|^{2}(q+1) q^{j-1}=\frac{1}{q+1}\left(p_{N}-p_{N-1}\right) \overline{p_{N-1}}(q+1) q^{N-1}
$$

Solving for $z$ and replacing each $p_{j}-p_{j-1}$ with $r_{j}$ gives

$$
z=\frac{-r_{N} \overline{p_{N-1}} q^{N-1}+\sum_{j=1}^{N-1}\left|r_{j}\right|^{2} q^{j-1}}{1+\sum_{j=1}^{N-1}\left|p_{j}\right|^{2}\left(\frac{q+1}{q}\right) q^{-j}} .
$$

Let $N \rightarrow \infty$ in the above display. By part (ii), $r_{N} q^{N}$ is bounded and by assumption $p_{N} \rightarrow 0$ as $N \rightarrow \infty$. Thus the first term in the numerator goes to 0 . We get

$$
z=\frac{\sum_{j=1}^{\infty}\left|r_{j}\right|^{2} q^{j-1}}{1+\sum_{j=1}^{\infty}\left|p_{j}\right|^{2}\left(\frac{q+1}{q}\right) q^{-j}} .
$$

By part (ii) the sum in the numerator converges, and since $p_{j}(z)$ is a bounded sequence, the sum in the denominator also converges. Thus the eigenvalues are necessarily real and positive.

## 3. Some properties of the roots of $p_{n}$

We recall from Section 2 that the roots of $p_{n}$ are denoted by $x_{n, 1}, \ldots, x_{n, n}$ written in increasing order. That they are real and positive is shown in Theorem 3.1 below.
3.1. The operator $-\Delta_{E}$. Let $B_{n}=\{v \in T:|v| \leq n\}$ denote the ball of radius $n$ centered at the root and let $\mathcal{H}_{n}$ denote the set of radial functions on $B_{n}$ which are 0 on the boundary $|v|=n$. We can identify such functions $f$ with $(n+1)$-tuples $\left(f_{0}, f_{1}, \ldots, f_{n}\right)$ of real numbers with $f_{n}=0$ and we can view $\Delta_{E}$ as operating on these $(n+1)$-tuples as follows:

$$
\begin{aligned}
& \left(\Delta_{E} f\right)_{0}=f_{1}-f_{0} \\
& \left(\Delta_{E} f\right)_{k}=q^{2 k}\left(\frac{q}{q+1} f_{k+1}+\frac{1}{q+1} f_{k-1}-f_{k}\right), 1 \leq k \leq n-1 .
\end{aligned}
$$

We use the notation $f$ to denote either the radial function on $B_{n}$ or the vector in $\mathbb{R}^{n+1}$. Consider the inner product $\langle\cdot, \cdot\rangle_{E}$ defined on $\mathcal{H}_{n}$ by

$$
\begin{equation*}
\langle f, g\rangle_{E}:=\sum_{|v| \leq n} f(v) g(v) q^{-2|v|}=f_{0} g_{0}+\left(\frac{q+1}{q}\right) \sum_{k=1}^{n-1} f_{k} g_{k} q^{-k} . \tag{23}
\end{equation*}
$$

In Theorem 2.3(iii) we showed that the roots of $p_{\infty}$ are real and positive. In the following theorem we prove the analogous result for $p_{n}$.
Theorem 3.1. The operator $-\Delta_{E}$ is positive and symmetric with respect to the above inner product on $\mathcal{H}_{n}$ and so its eigenvalues are real and positive.

Proof. By Green's theorem as stated in (22), we get

$$
\begin{align*}
\left\langle\Delta_{E} f, g\right\rangle_{E} & =\sum_{|v|<n} q^{-2|v|} \Delta_{E} f(v) g(v)=\sum_{|v|<n} \Delta_{H} f(v) g(v) \\
& =-\frac{1}{q+1} \sum_{0<|v|<n}\left(f(v)-f\left(v^{-}\right)\right)\left(g(v)-g\left(v^{-}\right)\right)-\frac{1}{q+1} \sum_{|v|=n} f\left(v^{-}\right) g\left(v^{-}\right) \\
& =-\frac{1}{q+1}\left[\sum_{k=1}^{n-1}(q+1) q^{k-1}\left(f_{k}-f_{k-1}\right)\left(g_{k}-g_{k-1}\right)+(q+1) f_{n-1} g_{n-1} q^{n-1}\right] \\
& =-\sum_{k=1}^{n-1} q^{k-1}\left(f_{k}-f_{k-1}\right)\left(g_{k}-g_{k-1}\right)-f_{n-1} g_{n-1} q^{n-1} . \tag{24}
\end{align*}
$$

The last expression is invariant if we interchange $f$ and $g$, and is negative in case $f \equiv g$. This completes the proof.
3.2. Monotonicity properties of the roots of the sequence $p_{n}$. Since $-\Delta_{E}$ is a positive symmetric operator, it is well known that the biggest and smallest eigenvalues are given by

$$
\begin{align*}
& x_{n, 1}=\min \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n},\|f\|_{E}=1\right\}=\min \left\{\frac{\left\langle-\Delta_{E} f, f\right\rangle_{E}}{\|f\|_{E}^{2}}: f \in \mathcal{H}_{n},\|f\|_{E} \neq 0\right\}  \tag{25}\\
& x_{n, n}=\max \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n},\|f\|_{E}=1\right\}=\max \left\{\frac{\left\langle-\Delta_{E} f, f\right\rangle_{E}}{\|f\|_{E}^{2}}: f \in \mathcal{H}_{n},\|f\|_{E} \neq 0\right\} \tag{26}
\end{align*}
$$

([3], Proposition 6.9). Furthermore, for each $1 \leq k<n$, if $V_{n, k}$ is the subspace of $\mathcal{H}_{n}$ generated by the eigenspaces for $x_{n, 1}, \ldots, x_{n, k}$, and $W_{n, k}$ the subspace generated by the eigenspaces for $x_{n, k+1}, \ldots, x_{n, n}$ then

$$
\begin{align*}
x_{n, k+1} & =\min \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n} \cap V_{n, k}^{\perp},\|f\|_{E}=1\right\}  \tag{27}\\
x_{n, k} & =\max \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n} \cap W_{n, k}^{\perp},\|f\|_{E}=1\right\} . \tag{28}
\end{align*}
$$

Theorem 3.2. For each $n \geq 1$ the smallest root $x_{n, 1}$ and the biggest root $x_{n, n}$ of $p_{n}$ are increasing functions of $q$.
Proof. For any $f \in \mathcal{H}_{n}$ with $\|f\|_{E} \neq 0$, we have from (24) and (23)

$$
\begin{aligned}
& \left\langle-\Delta_{E} f, f\right\rangle=\sum_{k=1}^{n-1} q^{k-1}\left(f_{k}-f_{k-1}\right)^{2}+f_{n-1}^{2} q^{n-1}, \quad \text { and } \\
& \|f\|_{E}^{2}=f_{0}^{2}+\frac{q+1}{q} \sum_{k=1}^{n-1} f_{k}^{2} q^{-k}=f_{0}^{2}+\sum_{k=1}^{n-1} f_{k}^{2} q^{-k}+\sum_{k=1}^{n-1} f_{k}^{2} q^{-(k+1)} .
\end{aligned}
$$

As a function of $q$ the first of these increases and the second decreases, and so the ratio increases with $q$. The result then follows from the expression on the right of (25) and (26).

Theorem 3.3. Let $X$ denote the lower triangular matrix of eigenvalues, i.e. $X_{n, k}=x_{n, k}$ for $n \geq k$.
(i) In every column of $X$ the sequence of entries on or below the diagonal strictly decreases, i.e. $x_{n, k}>x_{n+1, k}$ for all $n \geq k$.
(ii) The entries along each subdiagonal of $X$ strictly increase, that is for each $j<n$ fixed, $x_{n, j}<x_{n+1, j+1}$, $n=1,2, \ldots$.
(iii) The entries in $X$ along the part of each row on or below the diagonal strictly increase, i.e. $x_{n, k}<x_{n, k+1}$ for $k=1,2, \ldots, n-1$.

Proof. (i): We first prove that for any $n \geq 1, p_{n}$ and $p_{n-1}$ have no root in common. We do this by induction on $n$. It is true for $n=1$ since $p_{0}$ has no root, and $p_{1}$ has only the root $x=1$. Suppose the result is true for some $n \geq 1$. Since we have $\frac{q}{q+1} p_{n+1}+\frac{1}{q+1} p_{n-1}=\left(1-x q^{-2 n}\right) p_{n}$, if $x$ is a common root of $p_{n}$ and $p_{n+1}$, then it would also be a root of $p_{n-1}$. This contradiction of the hypothesis completes the induction. In particular, since $x_{n, k}$ is a root of $p_{n}$ and $x_{n+1, k}$ is a root of $p_{n+1}$, it follows $x_{n, k} \neq x_{n+1, k}$ for every $n$ and $k \leq n$. Thus in order to complete the proof of (i), it suffices to show that $x_{n, k} \geq x_{n+1, k}$.

Note that for $1 \leq n<m$, to each $f=\left(f_{0}, f_{1}, \ldots, f_{n}\right)$ in $\mathcal{H}_{n}$ we can associate the vector $\tilde{f}=\left(f_{0}, \ldots, f_{n}, 0, \ldots, 0\right)$ in $\mathcal{H}_{m}$ obtained by inserting $m-n$ additional 0 's. Observe that for two such functions $f$ and $g$, the inner product $\langle f, g\rangle_{E}$ is preserved, i.e. $\langle f, g\rangle_{E}=\langle\tilde{f}, \tilde{g}\rangle_{E}$, and so in particular $\|f\|_{E}=\|\tilde{f}\|_{E}$. The Laplacian $\Delta_{E} \tilde{f}$ possibly introduces an additional nonzero value in coordinate $m+1$ of $\tilde{f}$, but since $f_{n+1}=0,\left\langle\Delta_{E} f, f\right\rangle_{E}$ is preserved, i.e. $\left\langle\Delta_{E} f, f\right\rangle_{E}=$ $\left\langle\Delta_{E} \tilde{f}, \tilde{f}\right\rangle_{E}$.

We first prove the result for the first column, i.e. that $x_{n, 1} \geq x_{n+1,1}$ for each $n$. Let $g \in \mathcal{H}_{n}$ such that $\|g\|_{E}=1$ and $\left\langle-\Delta_{E} g, g\right\rangle_{E}=x_{n, 1}$. Let $\tilde{g}$ be the element of $\mathcal{H}_{n+1}$ obtained from $g$ by setting the $(n+1)$ st coordinate equal to 0 . Then by (25)

$$
x_{n, 1}=\left\langle-\Delta_{E} g, g\right\rangle_{E}=\left\langle-\Delta_{E} \tilde{g}, \tilde{g}\right\rangle_{E} \geq \min \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n+1},\|f\|_{E}=1\right\}=x_{n+1,1}
$$

We now prove that $x_{n, k} \geq x_{n+1, k}, n \geq k$ for a fixed $k>1$. Let $f^{1}, f^{2}, \ldots, f^{k}$ be the eigenfunctions of norm 1 in $\mathcal{H}_{n}$ corresponding to $x_{n, 1}, \ldots, x_{n, k}$, and let $\tilde{f}^{1}, \ldots, \tilde{f}^{k}$ be the vectors in $\mathcal{H}_{n+1}$ obtained as above by inserting 0 as the last coordinate. Let $g^{1}, \ldots, g^{k-1}$ be the eigenfunctions in $\mathcal{H}_{n+1}$ corresponding to $x_{n+1,1}, \ldots, x_{n+1, k-1}$. We claim that there exist scalars $c_{1}, \ldots, c_{k}$ not all 0 such that $c_{1} \tilde{f}^{1}+\cdots+c_{k} \tilde{f}^{k}$ is orthogonal to each $g^{1}, \ldots, g^{k-1}$. If there were such scalars $c_{1}, \ldots, c_{k}$ then a necessary condition on them would be

$$
\begin{equation*}
\sum_{j=1}^{k} c_{j}\left\langle\tilde{f}^{j}, g_{m}\right\rangle_{E}=0, \quad m=1,2, \ldots, k-1 \tag{29}
\end{equation*}
$$

This just says that the vector $\left(c_{1}, \ldots, c_{k}\right)$ is in the null space of the $(k-1) \times k$ matrix whose entry in row $m$ column $j$ is $\left\langle\tilde{f}^{j}, g_{m}\right\rangle_{E}$. But the rank-nullity theorem for matrices guarantees this null space is nontrivial, so we can choose $c_{1}, \ldots, c_{k}$ not all 0 so that (29) is satisfied. Since $\tilde{f}^{1}, \ldots, \tilde{f}^{k}$ are linearly independent (recall that they are orthogonal with respect to $\langle\cdot, \cdot\rangle_{E}$ ) it follows that $c_{1} \tilde{f}^{1}+\cdots+c_{k} \tilde{f}^{k}$ is not the 0 vector. This completes the proof of the claim.

Let $f:=c_{1} f^{1}+\cdots+c_{k} f^{k}$ and $\tilde{f}:=c_{1} \tilde{f}^{1}+\cdots+c_{k} \tilde{f}^{k}$. By orthogonality, $\|f\|_{E}^{2}=\|\tilde{f}\|_{E}^{2}=c_{1}^{2}+\cdots+c_{k}^{2}$ and

$$
\begin{aligned}
\left\langle-\Delta_{E} \tilde{f}, \tilde{f}\right\rangle_{E}=\left\langle-\Delta_{E} f, f\right\rangle_{E} & =\sum_{r, s=1}^{k} c_{r} c_{s}\left\langle-\Delta_{E} f^{r}, f^{s}\right\rangle_{E} \\
& =\sum_{r, s=1}^{k} c_{r} c_{s} x_{n, r}\left\langle f^{r}, f^{s}\right\rangle_{E} \\
& =\sum_{j=1}^{k} c_{j}^{2} x_{n, j}\left\|f^{j}\right\|_{E}^{2} \\
& \leq x_{n, k} \sum_{j=1}^{k} c_{j}^{2}=x_{n, k}\|\tilde{f}\|_{E}^{2}
\end{aligned}
$$

so by (27) we get

$$
x_{n, k} \geq \frac{\left\langle-\Delta_{E} \tilde{f}, \tilde{f}\right\rangle_{E}}{\|\tilde{f}\|_{E}^{2}} \geq x_{n+1, k}
$$

(ii): We make use of (26) and (28) rather than (25) and (27) which we used in the proof of (i). First we prove $x_{n, n} \leq x_{n+1, n+1}$. Let $g \in \mathcal{H}_{n}$ such that $\|g\|_{E}=1$ and $\left\langle-\Delta_{E} g, g\right\rangle_{E}=x_{n, n}$. Let $\tilde{g}$ be the element of $\mathcal{H}_{n+1}$ obtained from $g$ by setting the $(n+1)$ st coordinate equal to 0 . Then

$$
x_{n, n}=\left\langle-\Delta_{E} g, g\right\rangle_{E}=\left\langle-\Delta_{E} \tilde{g}, \tilde{g}\right\rangle_{E} \leq \max \left\{\left\langle-\Delta_{E} f, f\right\rangle_{E}: f \in \mathcal{H}_{n+1},\|f\|_{E}=1\right\}=x_{n+1, n+1}
$$

We next prove that $x_{n, k} \leq x_{n+1, k+1}$ for $k<n$. Consider the $n-k+1$ unit eigenvectors $f^{k}, f^{k+1}, \ldots, f^{n}$ in $\mathcal{H}_{n}$ corresponding to the eigenvalues $x_{n, k}, x_{n, k+1}, \ldots, x_{n, n}$. Let $\tilde{f}^{k}, \ldots, \tilde{f}^{n}$ be the vectors in $\mathcal{H}_{n+1}$ obtained from the $f^{j}$ 's by inserting 0 as the last coordinate. Let $g^{k+2}, \ldots, g^{n+1}$ be the $n-k$ eigenfunctions in $\mathcal{H}_{n+1}$ corresponding to $x_{n+1, k+2}, \ldots, x_{n+1, n+1}$. As in the proof of the last part of (i), we can prove there exist scalars $c_{k}, c_{k+1}, \ldots, c_{n}$ not all 0 such that $\tilde{f}:=c_{k} \tilde{f}^{k}+c_{k+1} \tilde{f}^{k+1} \cdots+c_{n} \tilde{f}^{n}$ is orthogonal to each of $g^{k+2}, \ldots, g^{n}$. As before we can show that
so by (28)

$$
\left\langle-\Delta_{E} \tilde{f}, \tilde{f}\right\rangle=\sum_{j=k}^{n} c_{j}^{2} x_{n, j} \geq x_{n, k}\|\tilde{f}\|_{E}^{2}
$$

$$
x_{n, k} \leq \frac{\left\langle-\Delta_{E} \tilde{f}, \tilde{f}\right\rangle_{E}}{\|\tilde{f}\|_{E}^{2}} \leq x_{n+1, k+1}
$$

Finally, we prove the inequalities are strict. For each $x>0$, consider the radial function $f(x)$ on $T$ whose component in slot $n$ (starting with $n=0$ ) is $p_{n}(x)$. Then for each $n, \Delta_{E} f(x)_{n}=-x f(x)_{n}$. This says

$$
\frac{q}{q+1} p_{n+1}+\frac{1}{q+1} p_{n-1}=\left(1-x q^{-2 n}\right) p_{n}, \quad n \geq 1
$$

If it were true that $f(x)$ had two consecutive 0 's, i.e. $p_{n}(x)=0=p_{n+1}(x)$, then it would follow by induction that $p_{k}(x)=0$ for all $0 \leq k \leq n+1$. But this is impossible, since $p_{0}(x)=1$. It follows that $x_{n, k}<x_{n+1, k+1}$, for if they were equal then $p_{n}$ and $p_{n+1}$ would have a common root.
(iii): By parts (i) and (ii), we have $x_{n, k}<x_{n-1, k}<x_{n, k+1}$.

Remark 3.1. In section 4 below which deals with interlacing properties, we will deduce much stronger monotonicity properties of the roots $x_{n, k}$ of the $p_{n}$ 's.

### 3.3. Sum and product of the roots of $p_{n}$.

Theorem 3.4. The sum and product of the roots $x_{n, 1}, \ldots, x_{n, n}$ of $p_{n}$ are given by

$$
\begin{align*}
& \sum_{k=1}^{n} x_{n, k}=\frac{q^{2 n}-1}{q^{2}-1}  \tag{30}\\
& \prod_{k=1}^{n} x_{n, k}=\frac{q^{n^{2}-1}}{(q+1)^{n-1}}
\end{align*}
$$

Proof. As usual $x_{n, 1}, x_{n, 2}, \ldots, x_{n, n}$ denote the roots of $p_{n}$ listed in increasing order. Thus we can write $p_{n}(x)$ in two ways:

$$
\begin{equation*}
\sum_{k=0}^{n} p_{n, k} x^{k}=p_{n, n} \prod_{k=1}^{n}\left(x-x_{n, k}\right) \tag{31}
\end{equation*}
$$

If we equate the coefficents of $x^{n-1}$ on both sides of (31) and then make use of the formulas for $p_{n, n}$ and $p_{n, n-1}$ given in Theorem 2.1 we get

$$
\sum_{k=1}^{n} x_{n, k}=-\frac{p_{n, n-1}}{p_{n, n}}=\frac{q^{2 n}-1}{q^{2}-1}
$$

If we instead equate the constant term on both sides of (31), using that $p_{n, 0}=1$ gives

$$
\prod_{k=1}^{n} x_{n, k}=\frac{(-1)^{n}}{p_{n, n}}=\frac{q^{n^{2}-1}}{(q+1)^{n-1}}
$$

Corollary 3.1. We have the following bounds on the roots $x_{n, 1}<\cdots<x_{n, n}$ for $n \geq 2$ :

$$
x_{n, n}>q^{2 n-2} \text { and } x_{n, j}<\frac{q}{q^{2}-1} q^{2 j-1}=\frac{q^{2}}{q^{2}-1} q^{2 j-2}, \quad j=1, \ldots, n
$$

In particular the biggest root $x_{n, n}$ satisfies

$$
q^{2 n-2}<x_{n, n}<\frac{q^{2}}{q^{2}-1} q^{2 n-2}
$$

Proof. If we apply equation (30) with two successive values of n and subtract the results we obtain

$$
x_{n, n}+\sum_{k=1}^{n-1}\left(x_{n, k}-x_{n-1, k}\right)=\frac{q^{2 n}-1}{q^{2}-1}-\frac{q^{2 n-2}-1}{q^{2}-1}=q^{2 n-2}
$$

so, by this and part (i) of Theorem 3.3, we have

$$
x_{n, n}=q^{2 n-2}+\sum_{k=1}^{n-1}\left(x_{n-1, k}-x_{n, k}\right)>q^{2 n-2} .
$$

For $j=1, \ldots, n$, again applying part (i) of Theorem 3.3, we have

$$
x_{n, j} \leq x_{j, j}<\sum_{k=1}^{j} x_{j, k}=\frac{q^{2 j}-1}{q^{2}-1}<\frac{q^{2 j}}{q^{2}-1}=\frac{q}{q^{2}-1} q^{2 j-1} .
$$

## 4. Interlacing property of the roots of the sequence $p_{n}$

4.1. Stieltjes interlacing and strong Stieltjes interlacing of families of roots. Let $p$ and $r$ be two polynomials, each with real, simple, and disjoint roots, such that $\operatorname{deg}(p)>\operatorname{deg}(r)$. We say that the roots of $p$ and $r$ interlace if each root of $r$ lies between two adjacent roots of $p$, and there is at most one root of $r$ between any two adjacent roots of $p$. For example, it follows from Theorem 3.3 that the roots of $p_{n}$ and $p_{n+1}$ interlace for each $n \geq 0$.

Let $\left\{r_{n}\right\}_{n=0}^{\infty}$ be a sequence of polynomials such that for each $n$, the degree of $r_{n}$ is $n$. We say that the roots of the sequence satisfy Stieltjes interlacing provided for every $m$ and $n$, the roots of $r_{n}$ and $r_{m+n}$ interlace. Looking back at the table of roots provided in Subsection 2.1, we see that the roots of the sequence $\left\{p_{n}\right\}$ seem to show Stieltjes interlacing. In the next subsection we describe some work of Beardon [2] that gives a general framework in which one necessarily has Stieltjes interlacing for a sequence of polynomials satisfying a three-term recurrence formula. We will also show that after the appropriate change of variables, our polynomials fit that general framework.

However, Stieltjes interlacing is not strong enough for our purposes. For that reason, we introduce the following kind of interlacing.

Definition 4.1. Let $\left\{r_{n}\right\}_{n=0}^{\infty}$ be a sequence of polynomials such that for every $n, r_{n}$ has degree $n$, and roots which are real and simple. We say that the family of roots satisfies strong Stieltjes interlacing provided for every $m$ and $n$, the smallest $n+1$ roots of $r_{m+n}$ interlace with the $n$ roots of $r_{n}$.

The table of roots in Subsection 2.1 suggest that the roots of our polynomials $p_{n}$ satisfy strong Stieltjes interlacing. We will shortly prove that this is indeed the case.

Remark 4.1. Consider the table whose entry in row $n$ and column $k$ is $x_{n, k}$, i.e. the $k$ th smallest root of $p_{n}$. We have already proven in Theorem 3.3 that the entries of each column strictly decrease. Then to say that the roots of the $p_{n}$ 's satisfy strong Stieltjes interlacing just says that for each column $k$, the entry $x_{k, k}$ at the top of the column is strictly less than every entry of the next column. This property is highly nontrivial to prove.
4.2. Beardon results concerning a sequence of polynomials $P_{n}(x)$ satisfying a three-term recurrence relation. We will make use of some of the results in [2] which we describe here. In this section we'll use upper case letters to describe the polynomials in [2] and lower case to describe the analogous polynomials in our paper.

Theorem 4.1. ([2], Theorem 4) Let $\left\{\alpha_{n}\right\}_{n=0}^{\infty}$ and $\left\{\lambda_{n}\right\}_{n=0}^{\infty}$ be sequences of real numbers such that $\lambda_{n}>0$ for all $n$. Let $\left\{P_{n}\right\}_{n=0}^{\infty}$ be a sequence of monic polynomials satisfying the initial conditions $P_{0}(x) \equiv 1, P_{1}(x)=x-\alpha_{0}$ and the three-term recurrence relation

$$
P_{n+1}(x)=\left(x-\alpha_{n}\right) P_{n}(x)-\lambda_{n-1} P_{n-1}(x), \quad n \geq 1
$$

Then, given $n$, there are real polynomials $S_{2, n}, S_{3, n}, \ldots$, so that for each $m, S_{m, n}$ has degree $m-1$ and

$$
S_{m-1, n}(x) P_{n+m}(x)=S_{m, n}(x) P_{n+m-1}(x)+(-1)^{m} \lambda_{n} \cdots \lambda_{n+m-2} P_{n}(x) .
$$

Furthermore, if $P_{n+m}$ and $P_{n}$ do not have a common root, then the $m+n-1$ zeros of the product $S_{m, n}(x) P_{n}(x)$ interlace with the $n+m$ roots of $P_{n+m}$. In particular, if for each $i$ and $j$ it is the case that $P_{i}$ and $P_{j}$ have no root in common, then the family of roots of $\left\{P_{n}\right\}_{n=0}^{\infty}$ Stieltjes interlace.

Example 4.1. We give an example to show that the general theorem does not imply strong Stieltjes interlacing of the family of roots. Take $\alpha_{0}=2, \alpha_{n}=0$ for $n \geq 1$, and $\lambda_{n}=1$ for every $n$. The first nine of the resulting polynomials are as follows:

$$
\begin{aligned}
& P_{0}(x)=1, \\
& P_{1}(x)=-2+x, \\
& P_{2}(x)=-1-2 x+x^{2}, \\
& P_{3}(x)=2-2 x-2 x^{2}+x^{3}, \\
& P_{4}(x)=1+4 x-3 x^{2}-2 x^{3}+x^{4}, \\
& P_{5}(x)=-2+3 x+6 x^{2}-4 x^{3}-2 x^{4}+x^{5}, \\
& P_{6}(x)=-1-6 x+6 x^{2}+8 x^{3}-5 x^{4}-2 x^{5}+x^{6}, \\
& P_{7}(x)=2-4 x-12 x^{2}+10 x^{3}+10 x^{4}-6 x^{5}-2 x^{6}+x^{7}, \\
& P_{8}(x)=1+8 x-10 x^{2}-20 x^{3}+15 x^{4}+12 x^{5}-7 x^{6}-2 x^{7}+x^{8} .
\end{aligned}
$$

The estimates of the roots of $P_{1}$ through $P_{8}$, obtained using Mathematica, are as follows:

|  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P_{1}$ | 2 |  |  |  |  |  |  |  |  |
| $P_{2}$ | -.4142 | 2.4142 |  |  |  |  |  |  |  |
| $P_{3}$ | -1.17 | .6889 | 2.481 |  |  |  |  |  |  |
| $P_{4}$ | -1.4955 | -.2197 | 1.2197 | 2.4955 |  |  |  |  |  |
| $P_{5}$ | -1.6624 | -.7574 | .4249 | 1.49592 | 2.4989 |  |  |  |  |
| $P_{6}$ | -1.7587 | -1.0904 | -.1492 | .8459 | 1.6528 | 2.4997 |  |  |  |
| $P_{7}$ | -1.8192 | -1.30801 | -.55544 | .3072 | 1.127 | 1.748 | 2.49993 |  |  |
| $P_{8}$ | -1.859 | -1.45702 | -.8472 | -.1129 | .6452 | 1.321 | 1.8104 | 2.49998 |  |

Of course, roots of consecutive polynomials interlace. However, there are many places on the table which confirm that these polynomials do not have strong Stieltjes interlacing of the roots. For example, compare the roots of $P_{2}$ and the three smallest roots of $P_{8}$, or more simply, the smallest root of $P_{3}$ and the two smallest roots of $P_{8}$.
4.3. Our polynomials $p_{n}$ and the Beardon framework. We show in the following theorem how our polynomials $p_{n}(x)$ are related to polynomials $P_{n}(x)$ satisfying the Beardon framework.
Theorem 4.2. The sequence of polynomials defined by $P_{n}(x):=c_{n} p_{n}(x)$, where

$$
c_{n}= \begin{cases}1 & \text { if } n=0 \\ (-1)^{n} q^{n^{2}-1}(q+1)^{1-n} & \text { if } n \geq 1,\end{cases}
$$

satisfies the conditions of Theorem 4.1 with parameters $\alpha_{n}=q^{2 n}$, and $\lambda_{n-1}=\left\{\begin{array}{ll}\frac{q^{2}}{q+1} & \text { if } n=1 \\ \frac{q^{4 n-1}}{(q+1)^{2}} & \text { if } n \geq 2\end{array}\right.$. Consequently $P_{n}$ and $p_{n}$ have the same roots for all $n$.

Proof. We have $c_{0}=1$ and $c_{1}=-1$, and so $P_{0}=p_{0} \equiv 1$, and $P_{1}=-p_{1}=-(1-x)=x-1$. This gives the correct initial conditions with $\alpha_{0}=1$. It is easy to check that for $n \geq 1$,

$$
\frac{c_{n+1}}{c_{n}}=-\frac{q^{2 n+1}}{q+1} \quad \text { and } \quad \frac{c_{n+1}}{c_{n-1}}= \begin{cases}\frac{q^{3}}{q+1} & \text { if } n=1 \\ \frac{q^{4 n}}{(q+1)^{2}} & \text { if } n \geq 2\end{cases}
$$

It then follows that for $n \geq 1$

$$
c_{n+1} p_{n}=-\frac{q^{2 n+1}}{q+1} P_{n} \quad \text { and } \quad c_{n+1} p_{n-1}= \begin{cases}\frac{q^{3}}{q+1} P_{n-1} & \text { if } n=1 \\ \frac{q^{4 n}}{(q+1)^{2}} P_{n-1} & \text { if } n \geq 2\end{cases}
$$

Applying these formulas after multiplying by $c_{n+1}$ both sides of the recurrence relation

$$
p_{n+1}=\left(\frac{q+1}{q}\right)\left(1-x q^{-2 n}\right) p_{n}-\frac{1}{q} p_{n-1}
$$

we obtain $P_{n+1}=\left(x-\alpha_{n}\right) P_{n}-\lambda_{n-1} P_{n-1}$, as desired.
4.4. The polynomials $S_{m, n}$ and $s_{m, n}$. In order to show that the roots of our polynomials $p_{n}$ satisfy strong Stieltjes interlacing, we will look at Beardon's construction of the polynomials $S_{m, n}$ in Theorem 4.1. Fix $n$ and let $m>n$. We now show his construction, but instead with our polynomials $p_{n}$. We begin with our recurrence relation

$$
\begin{equation*}
p_{n+1}=\left(\frac{q+1}{q}\right)\left(1-x q^{-2 n}\right) p_{n}-\frac{1}{q} p_{n-1} . \tag{32}
\end{equation*}
$$

We wish to produce polynomials $s_{k, n}, t_{k, n}, u_{k, n}, v_{k, n}$ for each $k \geq 1$ such that

$$
\binom{p_{j+1}}{p_{j}}=\left(\begin{array}{cc}
s_{j-n+1, n} & t_{j-n+1, n} \\
u_{j-n+1, n} & v_{j-n+1, n}
\end{array}\right)\binom{p_{n+1}}{p_{n}}, j \geq n
$$

We are only interested in the polynomials $s_{k, n}$ and $u_{k, n}$. If we take $j=n$, the resulting coefficient matrix is the identity matrix, and so $s_{1, n} \equiv 1$ and $u_{1, n} \equiv 0$. Next we make use of the recurrence relation with $n=1$ to write

$$
\binom{p_{n+2}}{p_{n+1}}=\left(\begin{array}{ll}
s_{2, n} & t_{2, n} \\
u_{2, n} & v_{2, n}
\end{array}\right)\binom{p_{n+1}}{p_{n}}=\left(\begin{array}{cc}
\frac{q+1}{q}\left(1-x q^{-2(n+1)}\right) & -1 / q \\
1 & 0
\end{array}\right)\binom{p_{n+1}}{p_{n}}
$$

then

$$
s_{2, n}=\left(\frac{q+1}{q}\right)\left(1-x q^{-2(n+1)}\right) \text { and } u_{2, n}=1 .
$$

If instead we take $j=n-1$, we get

$$
\binom{p_{n}}{p_{n-1}}=\left(\begin{array}{cc}
0 & 1 \\
-q & (q+1)\left(1-x q^{-2 n}\right)
\end{array}\right)\binom{p_{n+1}}{p_{n}}
$$

which says that $s_{0, n}=0$. Our next aim is to produce a recurrence relation for the $s_{k, n}$ 's. We have

$$
\begin{aligned}
\binom{p_{n+m+1}}{p_{n+m}} & =\left(\begin{array}{lc}
s_{m+1, n} & t_{m+1, n} \\
u_{m+1, n} & v_{m+1, n}
\end{array}\right)\binom{p_{n+1}}{p_{n}} \\
& =\left(\begin{array}{cc}
\frac{q+1}{q}\left(1-x q^{-2(n+m)}\right) & -1 / q \\
1 & 0
\end{array}\right)\binom{p_{n+m}}{p_{n+m-1}} \\
& =\left(\begin{array}{cc}
\frac{q+1}{q}\left(1-x q^{-2(n+m)}\right) & -1 / q \\
1 & 0
\end{array}\right)\left(\begin{array}{cc}
s_{m, n} & t_{m, n} \\
u_{m, n} & v_{m, n}
\end{array}\right)\binom{p_{n+1}}{p_{n}} .
\end{aligned}
$$

We deduce from this that $u_{m+1, n}=s_{m, n}$ for all $m$ (recall we have already shown that $u_{1, n}=s_{0, n}=0$ ), and so we obtain the recurrence relation

$$
\begin{equation*}
s_{m+1, n}=\left(\frac{q+1}{q}\right)\left(1-y q^{-2 m}\right) s_{m, n}-\frac{1}{q} s_{m-1, n}, s_{0, n}=0, s_{1, n}=1 \tag{33}
\end{equation*}
$$

where $y=x q^{-2 n}$ and we view each $s_{k, n}$ as a function of $y$.
The above construction is the same as the one in [2] where the author produced polynomials $S_{m, n}, T_{m, n}, U_{m, n}, V_{m, n}$ such that

$$
\binom{P_{m+n}}{P_{m+n-1}}=\left(\begin{array}{ll}
S_{m, n} & T_{m, n}  \tag{34}\\
U_{m, n} & V_{m, n}
\end{array}\right)\binom{P_{n+1}}{P_{n}}
$$

the only difference being that he used the recurrence relation for the $P_{k}$ 's given in Theorem 4.1 rather than the one in (32). We show next the relation between $S_{m, n}$ and $s_{m, n}$.

Theorem 4.3. For each $m$ and $n, S_{m, n}$ and $s_{m, n}$ have the same roots.
Proof. Recalling Theorem 4.2, we have $P_{k}=c_{k} p_{k}$, so substituting this in (34) we get

$$
\binom{c_{m+n} p_{m+n}}{c_{m+n-1} p_{m+n-1}}=\left(\begin{array}{cc}
S_{m, n} & T_{m, n} \\
U_{m, n} & V_{m, n}
\end{array}\right)\binom{c_{n+1} p_{n+1}}{c_{n} p_{n}} .
$$

This can be rewritten as

$$
\begin{aligned}
\binom{p_{m+n}}{p_{m+n-1}} & =\left(\begin{array}{cc}
1 / c_{m+n} & 0 \\
0 & 1 / c_{m+n-1}
\end{array}\right)\left(\begin{array}{ll}
S_{m, n} & T_{m, n} \\
U_{m, n} & V_{m, n}
\end{array}\right)\left(\begin{array}{cc}
c_{n+1} & 0 \\
0 & c_{n}
\end{array}\right)\binom{p_{n+1}}{p_{n}} \\
& =\left(\begin{array}{cc}
\left(c_{n+1} / c_{m+n}\right) S_{m, n} & \left(c_{n} / c_{m+n}\right) T_{m, n} \\
\left(c_{n+1} / c_{m+n-1}\right) U_{m, n} & \left(c_{n} / c_{m+n-1}\right) V_{m, n}
\end{array}\right)\binom{p_{n+1}}{p_{n}}
\end{aligned}
$$

from which we deduce that

$$
s_{m, n}=\frac{c_{n+1}}{c_{m+n}} S_{m, n}
$$

Since $c_{k} \neq 0$ for all $k$, it follows $s_{m, n}$ and $S_{m, n}$ have the same roots.
4.5. Strong Stieltjes interlacing and a characterization of the eigenvalues. One of the keys to identifying the eigenvalue counting function is to prove that the family of roots of the polynomials $p_{n}$ satisfy strong Stieltjes interlacing. Before proving this we have to prove a few preliminary results.

For $n$ fixed, consider the sequence $s_{m}:=s_{m+1, n}(y)$ defined in (33). Notice that $s_{m}$ and $p_{n}$ satisfy the same recurrence relation, but different initial conditions, namely $p_{0}=1$ and $p_{1}=1-x$ whereas $s_{0}=0$ and $s_{1}=1$. We show next that the smallest root of $s_{m}$ is an increasing function of $q$. The proof is similar to that of Theorem 3.2. The key is to modify the linear space $\mathcal{H}_{n}$ on which we work in order to reflect the change of initial conditions.
Lemma 4.1. The smallest and largest roots of $s_{m}$ are increasing functions of $q$.
Proof. Fix $m$. Consider the finite sector $S:=\left\{v \in T: v_{1} \leq v,|v| \leq m\right\}$, where $v_{1}$ is any fixed vertex of length 1. Then $S$ is a finite complete set of vertices whose boundary consists of the root of $T$ together with the vertices in $S$ of length $m$. Here we take our linear space to be the set of real-valued radial functions on $S$ which vanish on its boundary. This space can be identified with the set of sequences $\left(f_{0}, f_{1}, \ldots, f_{m}\right)$ where $f_{0}=f_{m}=0$. We view $\Delta_{E}$ as mapping the linear space to itself if we define $\Delta_{E} f$ to be 0 at the boundary vertices of $S$, and when needed make use of the 0 boundary values of $f$ to define $\Delta_{E} f$ in the usual way at interior vertices of $S$. We use the inner product $\langle f, g\rangle:=\sum_{0 \leq|v| \leq m} q^{-2|v|} f(v) g(v)$, and show next that with respect to this inner product, $-\Delta_{E}$ is positive and symmetric.

Let $f$ and $g$ be any functions in our linear space. By a special case of the general version of Green's theorem in ([5], Theorem 4.1, page 12) applied to the region $S$,

$$
\begin{aligned}
\left\langle-\Delta_{E} f, g\right\rangle & =\sum_{0 \leq|v| \leq m}-q^{-2|v|} \Delta_{E} f(v) g(v)=\sum_{0 \leq|v| \leq m}-\Delta_{H} f(v) g(v) \\
& =\frac{1}{q+1}\left[\sum_{2 \leq|v| \leq m-1}\left(f(v)-f\left(v^{-}\right)\right)\left(g(v)-g\left(v^{-}\right)\right)+\sum_{|v|=m, v \in S} f\left(v^{-}\right) g\left(v^{-}\right)+f\left(v_{1}\right) g\left(v_{1}\right)\right] .
\end{aligned}
$$

The last expression is clearly symmetric in $f$ and $g$, and is a sum of squares in case $f=g$. This shows $-\Delta_{E}$ is positive and symmetric. Because of the recurrence relation (33), the eigenvalues of $-\Delta_{E}$ on our linear space are the roots $y$ of $s_{m}$ with corresponding eigenfunction $\left(s_{0}(y), s_{1}(y), \ldots, s_{m}(y)\right)$. In particular, if $y$ is the smallest root of $s_{m}$,
then $y$ is the minimum of $\left\langle-\Delta_{E} f, f\right\rangle /\|f\|^{2}$, where the minimum is taken over the nonzero vectors in our linear space. If we replace minimum with maximum, we get the largest root of $s_{m}$. Using the fact that $f$ is radial, we get

$$
\begin{aligned}
& \left\langle-\Delta_{E} f, f\right\rangle=\frac{1}{q+1}\left[\sum_{k=2}^{m-1}\left(f_{k}-f_{k-1}\right)^{2} q^{k-1}+f_{m-1}^{2} q^{m-2}+f_{1}^{2}\right] \quad \text { and } \\
& \|f\|^{2}=\langle f, f\rangle=\sum_{v \in S} q^{-2|v|} f(v)^{2}=\sum_{k=1}^{m-1} q^{-2 k} q^{k-1} f_{k}^{2}=\sum_{k=1}^{m-1} q^{-(k+1)} f_{k}^{2} .
\end{aligned}
$$

We see that $\|f\|^{2}$ decreases as $q$ increases; looking at the three terms in the expression for $\left\langle-\Delta_{E} f, f\right\rangle$ the first two clearly increase as $q$ increases, and the third term $f_{1}^{2} /(q+1)$ doesn't increase with $q$ but it certainly does after division by $\|f\|^{2}$. This completes the proof.
Lemma 4.2. For each $m$ and $n$, all of the roots of $s_{m, n}$ are greater than all of the roots of $p_{n}$.
Proof. We make use of the recurrence relation and initial conditions satisfied by the sequence $s_{m, n}$ in (33) where we view $s_{m, n}$ as a function of $y$. Notice that this is the same as the recurrence relation which appears in (2), the only difference being in the initial conditions. They share some of the same properties, in particular, the smallest root of the $s_{m, n}$ 's decrease as $m$ increases.

We need a lower bound on the limiting value of the sequence of smallest roots of the $s_{m, n}$ 's. Taking $q=2$, for each specific choice of $y$ we can generate the sequence $\left\{s_{k, n}(y)\right\}_{k \geq 0}$. Starting from $y=0$ and gradually increasing $y$, we see using Excel that the limiting value of the sequence has its first transition from a positive to a negative value for $y$ between 2.2 and 2.25. By Lemma 4.1, this transition occurs at a larger value of $y$ when $q$ increases. Thus if $y^{\prime}$ is any root of $s_{m, n}$ for any $m$, then $y^{\prime}>2.2$. Replacing $y$ with $y=x q^{-2 n}$ and now viewing $s_{m, n}$ as being a function of $x$, it follows from Corollary 3.1 that

$$
x^{\prime}:=y^{\prime} q^{2 n}>(2.2) q^{2 n}>q^{2 n-1}>\frac{q}{q^{2}-1} q^{2 n-1}>x_{n, n} .
$$

Corollary 4.1. For $m \geq 1$, if it is the case that $p_{n}$ and $p_{m+n}$ have no root in common, then the first $n+1$ roots of $p_{m+n}$ interlace with the $n$ roots of $p_{n}$, i.e. for any $j$ from 1 to $n$, there is a unique root of $p_{n}$ between $x_{m+n, j}$ and $x_{m+n, j+1}$. Equivalently, for each $j, x_{n, j}<x_{m+n, j+1}$.

Proof. From Theorem 4.2 and Theorem 4.3, the roots of each $p_{k}$ and $P_{k}$ are the same, and the roots of each $s_{k}$ and $S_{k}$ are the same. The interlacing result is then immediate from Lemma 4.2 and Theorem 4.1. The final statement follows from Theorem 3.3(i).

We are now ready to prove the main result of this section.
Theorem 4.4. The roots of the polynomials in the sequence $\left\{p_{n}\right\}_{n=0}^{\infty}$ satisfy strong Stieltjes interlacing.
Proof. By Corollary 4.1, it remains only to prove that for each $m, n \geq 1, p_{n}$ and $p_{m+n}$ have no root in common.
We make the observation that by Theorem 3.3(i), if for $j<k$ the j roots of $p_{j}$ interlace with the first $j+1$ roots of $p_{k}$, then necessarily for any $k^{\prime}$ with $j<k^{\prime}<k$, the first $j+1$ roots of $p_{k^{\prime}}$ interlace with the roots of $p_{j}$ and the first $j$ roots of $p_{k^{\prime}}$ interlace with the first $j+1$ roots of $p_{k}$.

At the beginning of the proof of Theorem 3.3(i) we showed that consecutive polynomials $p_{j}$ and $p_{j+1}$ have no root in common. It also follows from Theorem 3.3 that the roots of $p_{j}$ and $p_{j+1}$ interlace.

We now argue by induction on $n$. For $n=1$, the only root of $p_{1}$ is $x_{1,1}=1$. If it were the case that $p_{m+1}$ had 1 as a root, then $p_{m+2}$ would not have 1 as a root, and so by Corollary 4.1, the first two roots of $p_{m+2}$ interlace with 1 , hence by the above observation so would the first two roots of $p_{m+1}$. Thus by Theorem 3.3, $x_{m+1,1}<1<x_{m+1,2}<x_{m+1, j}$, $j=3, \ldots, m+1$, contradicting that 1 is a root of $p_{m+1}$. Thus the result holds for $n=1$.

For the inductive step, let $n \geq 1$ and assume the result holds for $n$. Suppose that it is the case that $p_{n+1}$ has a root in common with $p_{m+n+1}$. We will be done if we are able to deduce a contradiction. By the inductive hypothesis $p_{n}$ has no root in common with $p_{m+n+1}$ so by Corollary 4.1, the n roots of $p_{n}$ interlace with the first $n+1$ roots of $p_{m+n+1}$. By the observation at the beginning of this proof, the first $n$ roots of $p_{n+1}$ interlace with the first $n+1$ roots of $p_{m+n+1}$, and so none of those roots of $p_{n+1}$ can be a root of $p_{m+n+1}$. Thus the only possible common root between $p_{n+1}$ and $p_{m+n+1}$ is $x_{n+1, n+1}$. The same argument shows that if there is a common root between $p_{n+1}$ and $p_{m+n+2}$ it must be $x_{n+1, n+1}$. However, $p_{m+n+1}$ and $p_{m+n+2}$ have no root in common, so it follows $p_{n+1}$ and $p_{m+n+2}$ have no root in common. Thus by Corollary 4.1, the $n+1$ roots of $p_{n+1}$ interlace with the first $n+2$ roots of $p_{m+n+2}$, so again using the observation at the beginning of this proof, the $n+1$ roots of $p_{n+1}$ interlace with the first $n+2$ roots of $p_{m+n+1}$. In particular $x_{m+n+1, n+1}<x_{n+1, n+1}<x_{m+n+1, n+2}$, and so it is impossible for any of the roots of $x_{m+n+1}$ to equal $x_{n+1, n+1}$. This contradiction completes the proof.

Corollary 4.2. Let $x_{k}:=\lim _{n \rightarrow \infty} x_{n, k}$. Then the eigenvalues for the problem (2), (3), (4), i.e. the roots of the entire function $p_{\infty}$, are precisely $\left\{x_{k}\right\}_{k=1}^{\infty}$.

Proof. Since the polynomials $p_{n}$ converge locally uniformly to $p_{\infty}$, it is an easy exercise to show that the sequence of roots $\left\{x_{n, k}\right\}_{n \geq 1}$ converges to a root of $p_{\infty}$ (see [8], Chapter VII, Section 1, Exercise 5). It remains only to prove that $p_{\infty}$ doesn't have any additional roots.

By Theorem 4.4, we have the inequalities $x_{1}<x_{2}<\cdots<x_{n-1}<x_{n}<\ldots$.
For each $n$, since $p_{n}$ has only simple roots, the graph of $\left|p_{n}\right|$ on the interval joining successive roots is concave down and 0 at the two ends. Thus on any proper subinterval, $\left|p_{n}\right|$ takes its minimum at one of the two endpoints.

Now suppose that $p_{\infty}$ has a root at a point $z$ between $x_{k}$ and $x_{k+1}$. Then again by Theorem 4.4, there exists an interval $I$ centered at $z$ such that for all sufficiently large $n$ none of the $p_{n}$ 's has a root on $I$. But $I$ is an interval lying between two successive roots of $p_{n}$. By definition we have that $\left|p_{n}(z)\right|$ converges to 0 as $n \rightarrow \infty$. Now take a sequence of intervals $I_{j}$ centered at $z$ which shrink down to $z$. On each one of those intervals for each $n$, by the reasoning above, the value of $\left|p_{n}(z)\right|$ is greater or equal to the value of $\left|p_{n}\right|$ at one of the two endpoints of $I_{j}$, so that implies there is a subsequence of $\left|p_{n}\right|$ evaluated at either the left or right endpoint of $I_{j}$ which converges to 0 . Thus $p_{\infty}$ must be 0 at one of the endpoints of $I_{j}$. But that means $p_{\infty}$ has a set of roots with a limit point, something which cannot happen since $p_{\infty}$ is entire.

## 5. The Discrete Version of Weyl's Law

Define the sequences $\left\{a_{n}\right\}_{n \geq 1}$ and $\left\{b_{n}\right\}_{n \geq 0}$ by

$$
a_{n}=q^{2 n-2}, \quad b_{n}= \begin{cases}0.22 & \text { if } n=0 \\ \frac{q^{2}}{q^{2}-1} q^{2 n-2} & \text { if } n \geq 1\end{cases}
$$

Then $b_{0}<a_{1}<b_{1}<a_{2}<b_{2}<\ldots$, so to these sequences we can associate a partition of $[0, \infty)$.
Theorem 5.1. Let $\beta$ denote the eigenvalue counting function as defined in (5). Then

$$
\beta(x)= \begin{cases}0 & \text { if } x<b_{0}, \\ 0 \text { or } 1 & \text { if } b_{0} \leq x \leq a_{1}=1, \\ 1 \text { or } 2 & \text { if } a_{1}=1<x \leq a_{2}, \\ n-1, n, \text { or } n+1 & \text { if } a_{n}<x \leq b_{n}, n \geq 2 \\ n \text { or } n+1 & \text { if } b_{n}<x \leq a_{n+1}, n \geq 2 .\end{cases}
$$

Consequently.

$$
\begin{equation*}
\left|\beta(x)-\log _{q} \sqrt{x}\right|<2 . \tag{35}
\end{equation*}
$$

Proof. Let $\left\{x_{k}\right\}_{k \geq 1}$ denote the eigenvalues in increasing order. In the proof of Lemma 4.2 we showed that $x_{1}$ lies between $b_{0}$ and 1 , so that gives the first two equalities of the theorem. By Corollary 3.1 we have $a_{n} \leq x_{n, n} \leq b_{n}$, $n \geq 2$. By Theorems 4.4, $x_{n-1, n-1} \leq x_{n} \leq x_{n, n}$, and so we deduce

$$
a_{n-1} \leq x_{n} \leq b_{n} \text { for } n \geq 2
$$

Suppose that $1<x \leq a_{2}$. Since $1<x_{2}<b_{2}$, either $x<x_{2}$ in which case $\beta(x)=1$, or $x \geq x_{2}$ in which case $\beta(x)$ is at least 2. But $x_{3}>a_{2}$, so $\beta(x)$ must equal 2 . Thus $\beta(x)=1$ or 2 .

Suppose that $a_{2}<x \leq b_{2}$. Since $x_{1}<1<a_{2}, \beta(x)$ is at least 1 . Since $1<x_{2}<b_{2}$, either $x<x_{2}$ in which case $\beta(x)=1$ or $x \geq x_{2}$ in which case $\beta(x)$ is at least 2. Since $a_{2}<x_{3}<b_{3}$, either $x<x_{3}$ in which case $\beta(x)=2$ or $x \geq x_{3}$ in which case $\beta(x)$ is at least 3 . But $x_{4}>a_{3}>b_{2}$, so $\beta(x)$ is less than 4 . Thus in any case $\beta(x)$ is 1,2 , or 3 .

Suppose that $b_{2}<x<a_{3}$. Since $1<x_{2}<b_{2}$, it follows that $\beta(x)$ is at least 2. Since $a_{2}<x_{3}<b_{3}$, either $x<x_{3}$ in which case $\beta(x)=2$, or $x \geq x_{3}$ in which case $\beta(x)$ is at least 3. But $x_{4}>a_{4}>b_{3}$, so $\beta(x)$ is less than 4 . Thus in any case $\beta(x)$ is 2 or 3 .

The general case follows with similar reasoning.
To prove (35), suppose first that $a_{n}<x \leq b_{n}$. Then $q^{2 n-2}<x \leq \frac{q^{2}}{q^{2}-1} q^{2 n-2}$, so taking logs base $q$ and rearranging gives

$$
\log _{q} \sqrt{q^{2}-1}<n-\log _{q} \sqrt{x} \leq 1
$$

Note that the quantity $\log _{q} \sqrt{q^{2}-1}$ increases from about 0.792 to 1 as $q$ increases from 2 to $\infty$. We also have

$$
1+\log _{q} \sqrt{q^{2}-1}<n+1-\log _{q} \sqrt{x} \leq 2 \text { and }-1+\log _{q} \sqrt{q^{2}-1}<n-1-\log _{q} \sqrt{x} \leq 0
$$

Since $\beta(x)$ is either $n-1, n$, or $n+1$, we have proven (35) in this case. A similar argument shows that if $b_{n}<x \leq a_{n+1}$ then

$$
0 \leq n-\log _{q} \sqrt{x}<\log _{q} \sqrt{q^{2}-1} \text { and } 1 \leq n+1-\log _{q} \sqrt{x}<1+\log _{q} \sqrt{q^{2}-1}
$$

In this case $\beta(x)$ is either $n$ or $n+1$, so once again we obtain (35).

## 6. Orthogonal polynomials and Favard's Theorem

In this section $\left\{P_{n}\right\}_{n \geq 0}$ denotes the sequence satisfying all of the conditions of the hypotheses of Theorem 4.1, namely each $P_{n}$ is a polynomial of degree $n, P_{0} \equiv 1, P_{1}(x)=x-\alpha_{0}$, and $P_{n+1}(x)=\left(x-\alpha_{n}\right) P_{n}(x)-\lambda_{n-1} P_{n-1}(x)$ for $n \geq 1$, where $\left\{\alpha_{n}\right\}_{n \geq 0}$ and $\left\{\lambda_{n}\right\}_{n \geq 0}$ are real sequences such that $\lambda_{n}>0$ for each $n$. A theorem of Favard [10], stated below as Theorem 6.3, asserts that the functions of any such sequence are orthogonal with respect to a certain Borel probability measure $\mu$ on $\mathbb{R}$. Thus by Theorem 4.2 our polynomials $\left\{p_{n}\right\}_{n \geq 0}$ also satisfy orthogonality relations with respect to the same $\mu$.

The proof of Favard's theorem which we find the most accessible, is the one in [2]. We highly recommend reading that paper. See also ([15], Chapter 4.1). Our intention here is to offer some alternative details to the proof of Favard's theorem in [2], specifically in order to avoid the use of the Riemann-Stieltjes integral in that proof. And so what we write here will only be of use to readers who go through the details of Theorem 7 in [2]. We begin with some background material taken from [2].

Let $\mathcal{P}$ denote the set of polynomials with real coefficients. Since $P_{n}$ is a real polynomial of degree $n$, it follows that the set of these polynomials forms a basis for $\mathcal{P}$. Thus any polynomial $p$ can be written uniquely as a linear combination $\sum_{k=0}^{n} c_{k} P_{k}$, where $n$ is the degree of $p$. Define the linear functional $\Phi_{0}$ on $\mathcal{P}$ by $\Phi_{0}(p):=c_{0}$. Note that in particular, $\Phi_{0}(1)=\Phi_{0}\left(P_{0}\right)=1$ and $\Phi_{0}\left(P_{k}\right)=0$ for $k \geq 1$.

Theorem 6.1. ([2],Theorem 5) For polynomials $p, q$, define $[p, q]:=\Phi_{0}(p q)$. Then the following hold:
(i) $[\cdot, \cdot]$ is an inner product on $\mathcal{P}$;
(ii) $\left[P_{m}, P_{n}\right]= \begin{cases}1 & \text { if } m=n=0, \\ \lambda_{0} \lambda_{1} \ldots \lambda_{n-1} & \text { if } m=n \geq 1, \\ 0 & \text { if } m \neq n .\end{cases}$
(iii) $\Phi_{0}$ is a positive linear functional on $\mathcal{P}$, i.e. if $p \in \mathcal{P}$ with $p(x) \geq 0$ for all $x \in \mathbb{R}$ and $p$ is not identically 0 , then $\Phi_{0}(p)>0$.
The next step is to associate a discrete measure $\mu_{n}$ to $P_{n}$ for each $n \geq 1$. Let $Z_{n}:=\left\{x_{n, k}, k=1, \ldots, n\right\}$ denote the set of roots of $P_{n}$. For each $i$ from 1 to $n$ let

$$
L_{n, i}(x):=\prod_{j \neq i} \frac{x-x_{n, j}}{x_{n, i}-x_{n, j}}
$$

namely the unique polynomial of degree $n-1$ which is 1 at $x_{n, i}$ and 0 at each of the other $n-1$ roots of $P_{n}$. Note that $L_{n, 1}(x)+\cdots+L_{n, n}(x) \equiv 1$.

Theorem 6.2. ([2],Theorem 6) The linear functional $\Phi_{0}$ satisfies the following:
(i) $\Phi_{0}\left(L_{n, i}\right)>0$ for each $n$ and $1 \leq i \leq n$.
(ii) For each $n$ and polynomial $p$ of degree at most $2 n-1$,

$$
\Phi_{0}(p)=\sum_{j=1}^{n} p\left(x_{n, i}\right) \Phi_{0}\left(L_{n, i}\right)
$$

Consequently if $\mu_{n}$ is the discrete measure with support equal to $Z_{n}$ such that $\mu_{n}\left\{x_{n, i}\right\}=\Phi_{0}\left(L_{n, i}\right)$, then $\mu_{n}$ is a probability measure and for any polynomials $p, q$ the sum of whose degrees is at most $2 n-1$,

$$
\int_{\mathbb{R}} p q d \mu_{n}=\sum_{i=1}^{n} p\left(x_{n, i}\right) q\left(x_{n, i}\right) \mu\left(x_{n, i}\right)=\Phi_{0}(p q)=[p, q] .
$$

We now state the theorem of Favard as given in [2].
Theorem 6.3. Let $\left\{P_{n}\right\}_{n \geq 0}$ satisfy the conditions stated in Theorem 4.1. Then there is a Borel probability measure $\mu$ on $\mathbb{R}$ such that each real polynomial $p$ is $\mu$-integrable, and for each $p, q \in \mathcal{P}$,

$$
\int_{\mathbb{R}} p q d \mu=[p, q]
$$

In particular $\int_{\mathbb{R}} P_{m} P_{n} d \mu=0$ for each $m \neq n$.
Next some background material on measure theory taken from [4]. A Borel measure $\mu$ on $\mathbb{R}$ will called a subprobability measure (s.p.m.) if $\mu(\mathbb{R}) \leq 1$. To each such measure we associate its subdistribution function (s.d.f.) $F: \mathbb{R} \rightarrow[0,1], \quad x \mapsto \mu(-\infty, x]$. As a result we have for each left open and right closed interval $(a, b]$,

$$
\begin{equation*}
\mu(a, b]=F(b)-F(a) . \tag{36}
\end{equation*}
$$

The s.d.f. $F$ has the properties that $F$ is increasing, $\lim _{x \rightarrow-\infty} F(x)=0, \lim _{x \rightarrow \infty} F(x) \leq 1$, and $F$ is right continuous. Conversely, to any function $F$ satisfying these properties, if we define $\mu$ on left open right closed intervals by (36),
then $\mu$ extends uniquely to a s.p.m. on $\mathbb{R}$, known as a Lebesgue-Stieltjes measure. Since $F$ is increasing, there are at most countably many points of discontinuity. An interval ( $a, b$ ] is called a continuity interval of $F$ (or for $\mu$ ) if both $a$ and $b$ are points of continuity of $F$. Thus an interval $(a, b]$ is a continuity interval if and only if both $a$ and $b$ are not atoms of $\mu$, i.e. $\mu\{a\}=\mu\{b\}=0$.

A sequence of s.p.m.'s $\mu_{n}$ is said to converge vaguely to the s.p.m. $\mu$ provided $\mu_{n}(a, b] \rightarrow \mu(a, b]$ for all $a$ and $b$ in a dense set of $\mathbb{R}$. The fundamental results concerning vague convergence of which we make use are given in

Theorem 6.4. (i) ([4], Theorem 4.3.1) A sequence $\mu_{n}$ converges vaguely to $\mu$ if and only if $\mu_{n}(a, b] \rightarrow \mu(a, b]$ for every continuity interval ( $a, b]$.
(ii) ([4],Theorem 4.4.2.) A sequence $\mu_{n}$ converges vaguely to $\mu$ if and only if for every bounded continuous function $f: \mathbb{R} \rightarrow \mathbb{R}, \int_{\mathbb{R}} f(x) d \mu_{n}(x) \rightarrow \int_{\mathbb{R}} f(x) d \mu(x)$.
(iii) ([4], Exercise 2 of section 4.4) If $\mu_{n}$ converges vaguely to $\mu$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{[a, b]} f d \mu_{n}=\int_{[a, b]} f d \mu \tag{37}
\end{equation*}
$$

for every continuous function $f$ provided ( $a, b]$ is a bounded continuity interval.
(iv) ([4], Theorem 4.3.3) Given any sequence $\mu_{n}$ of s.p.m.'s, there is a subsequence that converges vaguely to an s.p.m. $\mu$.

In the proof of Favard's theorem in [2], rather than applying Theorem 6.4, the author uses
Theorem 6.5. Suppose that $g_{1}, g_{2}, \ldots$ are increasing functions from $\mathbb{R}$ to $[0,1]$. Then there is a subsequence $g_{n_{1}}, g_{n_{2}}, \ldots$ which converges pointwise everywhere in $\mathbb{R}$ to an increasing function $g: R \rightarrow[0,1]$, and which is such that, for every polynomial $p$, and every compact interval $[a, b]$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{a}^{b} p d g_{n}=\int_{a}^{b} p d g \tag{38}
\end{equation*}
$$

where these integrals are to be interpreted as Riemann-Stieltjes integrals.
To each of the measures $\mu_{n}$ of Theorem 6.2, one associates the distribution function $g_{n}$. The author then applies Theorem 6.5 to the resulting sequence to obtain an increasing function $g$ satisfying the conclusions of that theorem. In order to complete the proof of Favard's theorem using the tools of measure theory, the integrals need to be Lebesgue-Stieltjes integrals rather than Riemann-Stieltjes integrals, and so the author replaces $g$ with an appropriate right continuous function $g^{*}$.

The approach we suggest here is to make use of Theorem 6.4 including (37) rather than Theorem 6.5 and (38). Thus we obtain the Lebesgue-Stieltjes measure $\mu$ as the vague limit of a subsequence of the $\mu_{n}$, with $g$ the associated right continuous distribution function of $\mu$. The proof as presented in [2] then works essentially as written without any need to modify $g$. One change that needs to be made is that the intervals $[-k, k], k \in \mathbb{Z}^{+}$, which the author applies to the intervals in (38) should be replaced with intervals $\left[-a_{k}, a_{k}\right]$ to which we apply (37), where $a_{k}$ is chosen with the properties that $a_{k} \rightarrow \infty$ and $a_{k}$ and $-a_{k}$ are points of continuity of the s.d.f. associated with $\mu$. This completes our discussion of the proof of Favard's theorem.

Finally, we observe
Theorem 6.6. The measure $\mu$ associated with the original sequence $\left\{p_{n}\right\}_{n \geq 0}$, whose existence is guaranteed by Favard's theorem, has support equal to the set of roots of $p_{\infty}$.

This theorem follows from the fact that $\mu$ is the vague limit of a subsequence of $\mu_{n}$ given in Theorem 6.2, the roots of $p_{\infty}$ are discrete, and from Corollary 4.2 as well as part (iii) of Theorem 6.4.

## 7. A FEW CONJECTURES AND THEIR CONSEQUENCES

One of the frustrating aspects of our work here is our inability to come up with proofs of certain properties of the $p_{n}$ 's and $p_{\infty}$ which appear to be obviously true based on their graphs or simple numerical calculations. We discuss several of these in this section. This gives rise to a number of conjectures and a few theorems describing the relations between some of those conjectures.
7.1. A modified eigenvalue problem. A more general eigenvalue problem than we have so far considered is to find the eigenvalues of the Euclidean Laplacian on $T$ corresponding to radial eigenfunctions having boundary values equal to a fixed constant $M$, and then to study the associated eigenvalue counting function $\beta_{M}(x)$, defined to be the number of such eigenvalues less or equal to $x$. Our results so far have concerned $\beta_{0}(x)=\beta(x)$. We will refer to this problem as the modified eigenvalue problem corresponding to $M$.

Recall that $p_{\infty}$ denotes the local uniform limit of our sequence of polynomials $p_{n}$, and the eigenvalues for the problem with $M=0$ are precisely the roots of $p_{\infty}$. We have denoted these roots written in increasing order by $x_{n}$,
$n \geq 1$. Each expression $\left[p_{0}\left(x_{n}\right), p_{1}\left(x_{n}\right), p_{2}\left(x_{n}\right), \ldots\right]$ determines the sequence of values of the corresponding radial eigenfunction on $T$.

Calculations suggest certain features of $p_{\infty}(x)$ for $x \in[0, \infty)$, namely that for large enough $x$ the roots go up by factors of approximately $q^{2}, p_{\infty}(x)$ switches sign between every other pair of roots, and the maximum absolute value on each interval between consecutive roots increases to $\infty$. In short, for large $x$ the graph of $p_{\infty}(x)$ superficially resembles the graph of $f(x)=x \sin \left(\pi \log _{q} \sqrt{x}\right)$. This suggests that $\beta_{M}(x)-\beta(x)$ is bounded. We formulate as conjectures each of the above properties which we are unable to prove. The first of these was noted from the calculation of the first few roots of $p_{\infty}$.
Conjecture 1. $\lim _{n \rightarrow \infty} \frac{x_{n+1}}{x_{n}}=q^{2}$.
Conjecture 2. $\max \left\{\left|p_{\infty}(x)\right|: x_{n} \leq x \leq x_{n+1}\right\} \rightarrow \infty$ as $n \nearrow \infty$.
Conjecture 3. For all $M \in \mathbb{R}, \beta_{M}(x)-\beta(x)=O(1)$.
Our primary interest in this subsection as well as in two later ones is Conjecture 3. The next theorem shows that Conjecture 2 gives a sufficient condition for it.

Theorem 7.1. Conjecture 2 $\Longrightarrow$ Conjecture 3.
Proof. Since $p_{n}(0)=1$ for each $n$ and $p_{\infty}(0)=1$, and each of the roots is simple, it follows without any assumption of Conjecture 2, that

$$
\begin{cases}p_{\infty}(x)<0 & \text { for } x \in\left(x_{n}, x_{n+1}\right), n \text { odd } \\ p_{\infty}(x)>0 & \text { for } x \in\left(x_{n}, x_{n+1}\right), n \text { even }\end{cases}
$$

and for each $k \geq 1$,

$$
\begin{cases}p_{n+k}(x)>0 & \text { for } x \in\left(x_{n}, x_{n+k, n}\right), n \text { odd } \\ p_{n+k}(x)<0 & \text { for } x \in\left(x_{n+k, n}, x_{n+1}\right), n \text { odd } \\ p_{n+k}(x)<0 & \text { for } x \in\left(x_{n}, x_{n+k, n}\right), n \text { even } \\ p_{n+k}(x)>0 & \text { for } x \in\left(x_{n+k, n}, x_{n+1}\right), n \text { even }\end{cases}
$$

Recalling that each $p_{n}$ has $n$ roots that are real and simple, we also have that for each $n$ and each interval $\left(x_{n, j}, x_{n, j+1}\right)$ for $1 \leq j \leq n, p_{n}$ has a unique absolute max or absolute min and the concavity is constant. By Rolle's theorem, since $p_{n}^{\prime}$ has degree $n-1, p_{n}^{\prime}$ has a unique root on each such interval. Since $\left\{p_{n}\right\}$ converges locally uniformly to $p_{\infty}$, it follows from Cauchy's integral formula that $\left\{p_{n}^{\prime}\right\}$ and $\left\{p_{n}^{\prime \prime}\right\}$ converge locally uniformly to $p_{\infty}^{\prime}$ and $p_{\infty}^{\prime \prime}$, respectively. Thus on each interval $\left(x_{n}, x_{n+1}\right)$, if $n$ is even, then $p_{\infty}$ has a unique absolute maximum and $p_{\infty}^{\prime \prime}<0$, and if $n$ is odd, then $p_{\infty}$ has a unique absolute minimum and $p_{\infty}^{\prime \prime}>0$.

If we assume that Conjecture 2 holds, then for a fixed $M>0, p_{\infty}$ takes the value $M$ exactly twice on $\left(x_{n}, x_{n+1}\right)$ for $n$ even, and so each such interval contributes exactly two eigenvalues to the eigenvalue problem corresponding to $M$. Thus in this case $\beta_{M}(x)-\beta_{0}(x)=O(1)$. The proof in case $M<0$ is similar.
7.2. Conjecture 4 and a more elegant form for $\beta(x)$. As we observed in formulating Conjecture 1 , after a while the roots of $p_{\infty}$ seem to go up by a factor of $q^{2}$. Rather than use this, we will instead formulate some conjectures about the interlacing properties of the roots of $p_{\infty}$ and these will lead more naturally to an elegant formula for $\beta(x)$.

We begin with some examples of properties of the $p_{n}$ 's which appear to be obviously true based on their graphs or simple numerical calculations but which we are unable to prove.

Example 7.1. (i) The roots $\left\{x_{n, 1}, \ldots, x_{n, n}\right\}$ of $p_{n}$ have the following interlacing properties for $n \geq 2$ :

$$
\begin{aligned}
& x_{n, 1}<q<x_{n, 2}<q^{3}<x_{n, 3}<\cdots<x_{n, n-1}<q^{2 n-3}<x_{n, n}<q^{2 n-1} \\
& x_{n, 1}<q^{0}<x_{n, 2}<q^{2}<x_{n, 3}<q^{4}<\cdots<x_{n, n-1}<q^{2 n-4}<x_{n, n} .
\end{aligned}
$$

(ii) The roots of each $p_{n}$ written in increasing order are successively farther apart. Consequently the same holds for $p_{\infty}$.
(iii) The maximum of each $\left|p_{n}\right|$ between any of its roots and its successor root is an increasing function of the roots. Consequently the same holds for $p_{\infty}$.

Since the $p_{n}$ 's are defined by means of a homogenous linear recurrence relation, one might expect there to be simple inductive proofs for at least some of these. The problem is that between successive roots of $p_{n}$ the derivative $p_{n}^{\prime}$ can vary enormously, and so very small changes in $x$ can produce great changes in $p_{n}(x)$. Thus inductive proofs using rough estimates on the terms of the recurrence relation (2) don't seem to give what we need.

The facts that (ii) holds for $p_{\infty}$ if it holds for all $p_{n}$ and (iii) holds for $p_{\infty}$ if it holds for all $p_{n}$ both follow by a simple continuity argument.

At the moment we don't have proofs for any of (i)-(iii), however, we do show below that (ii) and (iii) are both consequences of (i). Thus, we formulate (i) as a conjecture which we obtain by combining the two sets of inequalities in (i):

Conjecture 4. The roots $\left\{x_{n, 1}, \ldots, x_{n, n}\right\}$ of $p_{n}$ have the following interlacing property:

$$
\begin{align*}
x_{n, 1}<1<q<x_{n, 2}<q^{2}<q^{3}<x_{n, 3} & <q^{4}<q^{5}<x_{n, 4}  \tag{39}\\
& <\cdots<q^{2 n-5}<x_{n, n-1}<q^{2 n-4}<q^{2 n-3}<x_{n, n} .
\end{align*}
$$

Theorem 7.2. Among the conditions in Example 7.1 the following implications hold: $(i) \Longrightarrow(i i) \Longrightarrow$ (iii). In particular, under the assumption of Conjecture 4, both (ii) and (iii) hold.

Proof. $(i) \Longrightarrow(i i)$ : Suppose that (i) is true. Then we get the following upon combining (39) with the result of Corollary (3.1) that $x_{n, n}>q^{2 n-2}$ :

$$
\begin{align*}
x_{n, 1}<1<q<x_{n, 2}<q^{2}<q^{3}<x_{n, 3} & <q^{4}<q^{5}<x_{n, 4}  \tag{40}\\
& <\cdots<q^{2 n-5}<x_{n, n-1}<q^{2 n-4}<q^{2 n-3}<q^{2 n-2}<x_{n, n} .
\end{align*}
$$

In particular

$$
\begin{equation*}
q^{2 j-3}<x_{n, j}<q^{2 j-2}<q^{2 j-1}<x_{n, j+1}<q^{2 j}<q^{2 j+1}<x_{n, j+2}, \quad \text { for } 2 \leq j \leq n-2 . \tag{41}
\end{equation*}
$$

We have $q^{2 j+1}+q^{2 j-3}>q^{2 j+1} \geq 2 q^{2 j}$, and so $q^{2 j+1}-q^{2 j}>q^{2 j}-q^{2 j-3}$. It then follows that

$$
x_{n, j+2}-x_{n, j+1} \geq q^{2 j+1}-q^{2 j}>q^{2 j}-q^{2 j-3} \geq x_{n, j+1}-x_{n, j}
$$

which proves that the roots $x_{n, 2}, \ldots, x_{n, n}$ are successively farther apart. Finally, $x_{n, 1}+x_{n, 3}>x_{n, 3}>q^{3} \geq 2 q^{2}>$ $2 x_{n, 2}$, so $x_{n, 2}-x_{n, 1}<x_{n, 3}-x_{n, 2}$. Thus (ii) holds.
$($ ii $) \Longrightarrow(i i i)$ : Suppose that (ii) is true. Let $p(x)=\left(x-r_{1}\right)\left(x-r_{2}\right) \ldots\left(x-r_{n}\right), n \geq 2$, be any monic polynomial in which the roots are real, distinct, and successively farther apart, i.e. $r_{1}<r_{2}<\cdots<r_{n}$ and $0<r_{2}-r_{1} \leq r_{3}-r_{2}<$ $\ldots \leq r_{n}-r_{n-1}$. We shall prove by induction on $n$ that $\max \left\{|p(x)|: r_{j} \leq x \leq r_{j+1}\right\}$ is an increasing function of $j$ for $1 \leq j \leq n-1$. Since $p_{n}$ is such a polynomial, (iii) will follow.

As there is nothing to do in case $n=2$, we proceed to the inductive step. Suppose $n \geq 2$ and the result holds for $n$. Let $p(x)=\left(x-r_{1}\right) \ldots\left(x-r_{n+1}\right)$ where the roots satisfy the above assumptions. Let $x$ be the unique point in $\left(r_{n-1}, r_{n}\right)$ such that $p^{\prime}(x)=0$. Then by the inductive hypothesis, it follows that

$$
\prod_{i=1}^{n}\left|y-r_{i}\right| \leq \prod_{i=1}^{n}\left|x-r_{i}\right| \text { for every } y \text { with } r_{1} \leq y \leq x
$$

Choose $z \in\left(r_{n}, r_{n+1}\right)$ such that $r_{n+1}-z=x-r_{n-1}$. Then we have

$$
\left|x-r_{n-1}\right|=\left|z-r_{n+1}\right|,\left|x-r_{n+1}\right|=\left|z-r_{n-1}\right|, \text { and }\left|x-r_{n}\right| \leq\left|z-r_{n}\right|
$$

Since $\left|x-r_{j}\right|<\left|z-r_{j}\right|$ for $j=1, \ldots, n-2$, we obtain

$$
\prod_{j=1}^{n+1}\left|x-r_{j}\right|<\prod_{j=1}^{n+1}\left|z-r_{j}\right|
$$

and the latter expression is less or equal to the same expression if we replace $z$ with the root of $p^{\prime}$ in $\left(r_{n}, r_{n+1}\right)$. This completes the induction and so the proof of $(i i) \Longrightarrow$ (iii).

One consequence of Conjecture 4 is that we get a more elegant description of the eigenvalue counting function $\beta(x)$ if we make use of the partition of $[0, \infty)$ given by $\left\{1, q, q^{2}, q^{3}, q^{4}, \ldots\right\}$.
Theorem 7.3. If Conjecture 4 holds, then $\beta(x)=\left\lfloor\frac{1}{2}+\log _{q} \sqrt{x}\right\rfloor,\left\lfloor 1+\log _{q} \sqrt{x}\right\rfloor$, or $\left\lfloor\frac{3}{2}+\log _{q} \sqrt{x}\right\rfloor$ where $\lfloor\cdot\rfloor$ denotes the floor function. Specifically,

$$
\beta(x)= \begin{cases}\left\lfloor 1+\log _{q} \sqrt{x}\right\rfloor & \text { if } q^{2 j-2} \leq x<q^{2 j-1} \text { for some } j, \\ \left\lfloor\frac{1}{2}+\log _{q} \sqrt{x}\right\rfloor \text { or }\left\lfloor\frac{3}{2}+\log _{q} \sqrt{x}\right\rfloor & \text { if } q^{2 j-1} \leq x<q^{2 j} \quad \text { for some } j\end{cases}
$$

Proof. Letting $n$ go to $\infty$ in (41) and recalling that $x_{n, j}$ decreases with $n$, we get

$$
q^{2 j-3} \leq x_{j}<q^{2 j-2}<q^{2 j-1} \leq x_{j+1}<q^{2 j}<q^{2 j+1}
$$

where $x_{j}$ denotes the $j$ th smallest root of $p_{\infty}$. If $q^{2 j-2} \leq x<q^{2 j-1}$ then $\beta(x)=j$. In this case $\frac{1}{2}+\log _{q} \sqrt{x}<j \leq$ $1+\log _{q} \sqrt{x}$, so since $j$ is an integer, it must be the floor of $1+\log _{q} \sqrt{x}$. On the other hand if $q^{2 j-1} \leq x<q^{2 j}$, then $\beta(x)$ is either $j$ or $j+1$. In this case, $\log _{q} \sqrt{x}<j \leq \frac{1}{2}+\log _{q} \sqrt{x}$, so $j$ is the floor of $\frac{1}{2}+\log _{q} \sqrt{x}$ and $j+1$ is the floor of $\frac{3}{2}+\log _{q} \sqrt{x}$.
7.3. Idea for a proof of Conjecture 4. Though we aren't able to prove Conjecture 4, we have a natural approach to a proof which we think is worth describing. It makes use of the following theorem. Both (i) and (ii) of the theorem are variations on Theorem 1 in [2] and (i) is Theorem 1.20 on page 13 of [11]. As we don't find a reference for (ii), for convenience we include a proof of it.

Theorem 7.4. Let $f$ and $g$ be monic polynomials both with all roots real, distinct, and having no roots in common. Denote the roots of $f$ by $a_{1}<a_{2}<\ldots$ and the roots of $g$ by $b_{1}<b_{2}<\ldots$.
(i) Let $\operatorname{deg}(f)=\operatorname{deg}(g)=n$, and the partial fraction decomposition of $f / g$ given by

$$
\frac{f(x)}{g(x)}=\frac{\left(x-a_{1}\right) \ldots\left(x-a_{n}\right)}{\left(x-b_{1}\right) \ldots\left(x-b_{n}\right)}=1+\frac{c_{1}}{x-b_{1}}+\frac{c_{2}}{x-b_{2}}+\cdots+\frac{c_{n}}{x-b_{n}} .
$$

Then the roots interlace in the sense that

$$
a_{1}<b_{1}<a_{2}<b_{2}<\cdots<a_{n}<b_{n}
$$

if and only if the coefficients $c_{i}$ are all positive.
(ii) Let $\operatorname{deg}(f)=n$ and $\operatorname{deg}(g)=n-1$ and the partial fraction decomposition of $f / g$ given by

$$
\frac{f(x)}{g(x)}=\frac{\left(x-a_{1}\right) \ldots\left(x-a_{n}\right)}{\left(x-b_{1}\right) \ldots\left(x-b_{n-1}\right)}=x-c+\frac{c_{1}}{x-b_{1}}+\frac{c_{2}}{x-b_{2}}+\cdots+\frac{c_{n-1}}{x-b_{n-1}} .
$$

Then the roots interlace in the sense that

$$
a_{1}<b_{1}<a_{2}<b_{2}<\cdots<b_{n-1}<a_{n}
$$

if and only if the coefficients $c_{i}$ are all negative.
Proof of (ii): Suppose first that the roots interlace in the sense described in (ii). Then for each $i$ from 1 to $n-1$ if $x$ approaches $b_{i}$ on the right, the sign of $f(x) / g(x)$ is $(-1)^{n-i} /(-1)^{n-1-i}=-1$. Thus $\lim _{x \rightarrow b_{i}^{+}} f(x) / g(x)=-\infty$, and so since the term $c_{i} /\left(x-b_{i}\right)$ dominates it must be the case that $c_{i}<0$.

Conversely, suppose that $c_{1}, \ldots, c_{n-1}$ are all negative. For each $i$ from 1 to $n-2$ consider the interval $\left[b_{i}, b_{i+1}\right]$. Since $c_{i}<0$ it follows that $f(x) / g(x)$ converges to $-\infty$ as $x \rightarrow b_{i}^{+}$and to $\infty$ as $x \rightarrow b_{i+1}^{-}$. Thus $f$ has at least one root in that interval. This then accounts for $n-2$ of the roots of $f$. Because of the $x-c$ term in $f / g, f(x) / g(x)$ has a limit of $\infty$ as $x \rightarrow \infty$ and $-\infty$ as $x \rightarrow-\infty$. Since $c_{n-1}<0$, the limit of $c_{n-1} /\left(x-b_{n-1}\right)$ as $x \rightarrow b_{n-1}^{+}$is $-\infty$. It follows that $f$ has a root which is greater than $b_{n-1}$. A similar argument shows that $f$ has a root which is less than $b_{1}$. Since we have accounted for all $n$ roots of $f$, we have shown that the roots interlace as claimed.

To try to prove the first set of inequalities in (i) of Example 7.1 we make use of part (i) of Theorem 7.4. Take $f=P_{n}$, with $P_{n}$ the polynomial defined in Theorems 4.1 and 4.2 , and take $g=V_{n}(x)=\prod_{j=1}^{n}\left(x-q^{2 j-1}\right)$. The partial fraction decomposition of $P_{n} / V_{n}$ is given by

$$
\frac{P_{n}(x)}{V_{n}(x)}=1+\frac{c_{n, 1}}{x-q}+\frac{c_{n, 2}}{x-q^{3}}+\ldots \frac{c_{n, n}}{x-q^{2 n-1}}
$$

The aim is to prove using induction on $n \geq 1$ that $c_{n, 1}, \ldots, c_{n, n}$ are all greater than 0 . Recall that $P_{0}=1, P_{1}=x-1$, $P_{2}=\left(x-q^{2}\right)(x-1)-\frac{q^{2}}{q+1}$ and $P_{n+1}=\left(x-q^{2 n}\right) P_{n}-\frac{q^{4 n-1}}{(q+1)^{2}} P_{n-1}$. We have $\frac{P_{1}(x)}{V_{1}(x)}=\frac{x-1}{x-q}=1+\frac{q-1}{x-q}$ and

$$
\begin{aligned}
\frac{P_{2}(x)}{V_{2}(x)} & =\frac{\left(x-q^{2}\right)(x-1)-\frac{q^{2}}{q+1}}{(x-q)\left(x-q^{3}\right)} \\
& =1+\frac{q^{3}-q^{2}+1}{(q-1)(q+1)^{2}} \cdot \frac{1}{x-q}+\frac{q^{6}-q^{3}-q^{4}}{(q-1)(q+1)^{2}} \cdot \frac{1}{x-q^{3}}
\end{aligned}
$$

so it is true for $n=1$ and $n=2$. For the inductive step we assume the result holds for $n \geq 2$, and try to show it for $n+1$ :

$$
\begin{aligned}
1+\frac{c_{n+1,1}}{x-q}+\cdots+\frac{c_{n+1, n+1}}{x-q^{2 n+1}} & =\frac{P_{n+1}(x)}{V_{n+1}(x)} \\
& =\frac{\left(x-q^{2 n}\right) P_{n}(x)}{\left(x-q^{2 n+1}\right) V_{n}(x)}-\frac{\frac{q^{4 n-1}}{(q+1)^{2}} P_{n-1}(x)}{\left(x-q^{2 n+1}\right)\left(x-q^{2 n-1}\right) V_{n-1}(x)} \\
= & \frac{x-q^{2 n}}{x-q^{2 n+1}}\left[1+\frac{c_{n, 1}}{x-q}+\cdots+\frac{c_{n, n}}{x-q^{2 n-1}}\right] \\
& \quad-\frac{q^{4 n-1}}{(q+1)^{2}\left(x-q^{2 n+1}\right)\left(x-q^{2 n-1}\right)}\left[1+\frac{c_{n-1,1}}{x-q}+\cdots+\frac{c_{n-1, n-1}}{x-q^{2 n-3}}\right] .
\end{aligned}
$$

We need to equate coefficients of like terms $1 /\left(x-q^{2 j-1}\right)$ on both sides. To do so we make use of the following partial fraction decompositions:

$$
\begin{aligned}
& \frac{x-q^{2 n}}{x-q^{2 n+1}}=1+\frac{q^{2 n+1}-q^{2 n}}{x-q^{2 n+1}}, \\
& \frac{x-q^{2 n}}{\left(x-q^{2 n+1}\right)\left(x-q^{2 j-1}\right)}=\frac{\left(q^{2 j-1}-q^{2 n}\right)\left(q^{2 j-1}-q^{2 n+1}\right)^{-1}}{x-q^{2 j-1}}+\frac{\left(q^{2 n+1}-q^{2 n}\right)\left(q^{2 n+1}-q^{2 j-1}\right)^{-1}}{x-q^{2 n+1}}, 1 \leq j \leq n, \\
& \frac{1}{\left(x-q^{2 n+1}\right)\left(x-q^{2 n-1}\right)}=\frac{\left(q^{2 n-1}-q^{2 n+1}\right)^{-1}}{x-q^{2 n-1}}+\frac{\left(q^{2 n+1}-q^{2 n-1}\right)^{-1}}{x-q^{2 n+1}}, \\
& \frac{1}{\left(x-q^{2 n+1}\right)\left(x-q^{2 n-1}\right)\left(x-q^{2 j-1}\right)}=\frac{\left(q^{2 j-1}-q^{2 n+1}\right)^{-1}\left(q^{2 j-1}-q^{2 n-1}\right)^{-1}}{x-q^{2 j-1}}+\frac{\left(q^{2 n-1}-q^{2 n+1}\right)^{-1}\left(q^{2 n-1}-q^{2 j-1}\right)^{-1}}{x-q^{2 n-1}} \\
& \quad+\frac{\left(q^{2 n+1}-q^{2 n-1}\right)^{-1}\left(q^{2 n+1}-q^{2 j-1}\right)^{-1}}{x-q^{2 n+1}}, 1 \leq j \leq n-1 .
\end{aligned}
$$

We then get the following relations:

$$
\begin{aligned}
c_{n+1, j} & =\frac{q^{2 n}-q^{2 j-1}}{q^{2 n+1}-q^{2 j-1}} c_{n, j}-\frac{q^{4 n-1}}{(q+1)^{2}\left(q^{2 n+1}-q^{2 j-1}\right)\left(q^{2 n-1}-q^{2 j-1}\right)} c_{n-1, j}, 1 \leq j \leq n-1, \\
c_{n+1, n} & =\frac{q^{2 n}-q^{2 n-1}}{q^{2 n+1}-q^{2 n-1}} c_{n, n}+\frac{q^{4 n-1}}{(q+1)^{2}}\left[\frac{1}{q^{2 n+1}-q^{2 n-1}}+\sum_{j=1}^{n-1} \frac{c_{n-1, j}}{\left(q^{2 n+1}-q^{2 n-1}\right)\left(q^{2 n-1}-q^{2 j-1}\right)}\right], \\
c_{n+1, n+1} & =q^{2 n+1}-q^{2 n}+\sum_{j=1}^{n} \frac{q^{2 n+1}-q^{2 n}}{q^{2 n+1}-q^{2 j-1}} c_{n, j}-\frac{q^{4 n-1}}{(q+1)^{2}}\left[\frac{1}{q^{2 n+1}-q^{2 n-1}}+\sum_{j=1}^{n-1} \frac{c_{n-1, j}}{\left(q^{2 n+1}-q^{2 n-1}\right)\left(q^{2 n+1}-q^{2 j-1}\right)}\right] .
\end{aligned}
$$

The inductive hypothesis makes it clear that $c_{n+1, n}>0$, but $c_{n+1, j}$ for $j=1, \ldots, n-1$ and $j=n+1$ are differences of positive quantities, so it is not clear to us how to use the inductive hypothesis to deduce that they are positive.
7.4. Conjecture 5 and another look at the modified eigenvalue problem. We return to the modified eigenvalue problem and ask if it is possible to deduce Conjecture 3 from Conjecture 4 instead of from Conjecture 2 as we did in Theorem 7.1. Thus we are asking if Conjecture 4 is more fundamental than Conjecture 2. Since we don't have the answer, we formulate it as another conjecture.
Conjecture 5. Conjecture $4 \Longrightarrow$ Conjecture 2.
From Theorem 7.2, it follows that Conjecture 5 holds if with the assumption of Conjecture 4 we can deduce the following:

$$
\begin{equation*}
(\forall M>0)\left(\exists m \in \mathbb{Z}^{+}\right)\left(\exists N \in \mathbb{Z}^{+}\right)\left[n \geq N \Longrightarrow \max \left\{\left|p_{n}(x)\right|: x_{m} \leq x \leq x_{m+1}\right\}>M\right] \tag{42}
\end{equation*}
$$

A related but much simpler result which doesn't require any conjecture is the following one.
Theorem 7.5. The maximum of $\left|p_{n}\right|$ over the interval between its two largest roots goes to $\infty$ as $n \rightarrow \infty$.
Proof. From Corollary 3.1 we have

$$
x_{n, n}-x_{n, n-1}>q^{2 n-2}-\frac{q^{2}}{q^{2}-1} q^{2 n-4}=\left(\frac{q^{2}-2}{q^{2}-1}\right) q^{2 n-2}>\left(\frac{q^{2}-2}{q^{2}-1}\right)\left(\frac{q^{2}-1}{q^{2}}\right) x_{n, n}=\frac{q^{2}-2}{q^{2}} x_{n, n} \geq \frac{x_{n, n}}{2}
$$

and so $x_{n, n-1}<\frac{1}{2} x_{n, n}$. Then taking $y=\frac{3}{4} x_{n, n}$, it follows that $y-x_{n, n-1}>y-\frac{1}{2} x_{n, n}=\frac{1}{4} x_{n, n}$, and obviously $y-x_{n, n}=\frac{1}{4} x_{n, n}$. Since $x_{n, n}>q^{2 n-2}$, we deduce that the distance of $y$ to each of the roots of $p_{n}$ is at least $q^{2 n-4}$. Thus from Theorem 2.1,

$$
\max \left\{\left|p_{n}(x)\right|: x_{n, n-1} \leq x \leq x_{n, n}\right\} \geq\left|p_{n}(y)\right|=\frac{(q+1)^{n}}{q^{n^{2}-1}} \prod_{j=1}^{n}\left|y-x_{n, j}\right|>\frac{(q+1)^{n}}{q^{n^{2}-1}} q^{(2 n-4) n}>q^{n^{2}-3 n+1}
$$

and this last term goes to $\infty$ as $n$ goes to $\infty$.
We had hoped that with the assumption of Conjecture 4 we could deduce (42) (and hence prove Conjecture 5 is true) with a proof similar to that of Theorem 7.5 but using estimates inspired by (40). The idea is that for the given $M$ and for $m$ to be determined, we consider $p_{n}$ for $n>m$ and we fix $x$ in one of the gaps between the roots of $p_{n}$ (we used $x$ equal to the midpoint of the interval $\left[q^{2 m}, q^{2 m+1}\right]$ ). Then $\left|p_{n}(x)\right|$ is obtained by multiplying together the distances of $x$ to each of the roots of $p_{n}$ and then in turn multiplying by the factor $\left|p_{n, n}\right|=(q+1)^{n-1} / q^{n^{2}-1}$. In order to deduce $\left|p_{n}(x)\right|>M$, the products of $\left|p_{n, n}\right|$ together with the distances of $x$ to the roots of $p_{n}$ greater than $x$ would have to be bounded away from 0 independent of $n$, say by a quantity we call $\varepsilon$. Then it would be a simple matter to choose $m$ large enough that the product of the distances of $x$ to the roots of $p_{n}$ smaller than $x$ is larger
than $M / \varepsilon$ and we would have (42). The lower bounds on the distances of $x$ to each of the roots of $p_{n}$ are obtained in the obvious way using (40). Unfortunately the lower bounds aren't quite strong enough and the proof breaks down. To give an idea of what goes wrong, consider the following two calculations:

$$
\begin{aligned}
1 \cdot q^{2} \cdot q^{4} \cdots \cdot q^{2 n-2} & =q^{n^{2}-n} \\
q \cdot q^{3} \cdot q^{5} \cdot \ldots q^{2 n-5} \cdot q^{2 n-2} & =q^{n^{2}-2 n+2} .
\end{aligned}
$$

It is the second of these which relates to our proof. After multiplication by $(q+1)^{n-1} / q^{n^{2}-1}$ the first of these remains bounded away from 0 , whereas the second is $(q+1)^{n-1} q^{-2 n+3}$, which is not bounded away from 0 .
7.5. Conjecture 6 and yet another look at the modified eigenvalue problem. Let us continue to denote by $p_{n}$ the polynomials we have previously studied associated with the original eigenvalue problem. Recalling the notation $x_{n}, n \geq 1$, for the roots of $p_{\infty}$ and $x_{n, k}, 1 \leq k \leq n$, for the roots of $p_{n}$, by the strong Stieltjes interlacing, for each $n$,

$$
x_{n}<\cdots<x_{n+4, n}<x_{n+3, n}<x_{n+2, n}<x_{n+1, n}<x_{n, n}<x_{n+1} \text { for } j>n .
$$

It follows that for each $n$ there exists $\bar{x}_{n}$ in the interval $\left[x_{n, n}, x_{n+1}\right]$ such that $p_{n}\left(\bar{x}_{n}\right)=p_{\infty}\left(\bar{x}_{n}\right)$. We are interested in obtaining a lower bound on the quantity $\bar{x}_{n}-x_{n, n}$. Numerical calculation suggest that it increases with $n$. We did the calculations with $q=2$ and with $p_{14}$ replacing $p_{\infty}$. The results are

| $n$ | $x_{n, n}$ | $\approx \bar{x}_{n}$ |
| :---: | :---: | :---: |
| 1 | 1 | 1.9 |
| 2 | 4.39 | 7.16 |
| 3 | 17.09 | 28.5 |
| 4 | 68.4 | 114.049 |
| 5 | 274 | 456.343 |

For our purposes we merely require $\bar{x}_{n}-x_{n, n}$ to be bounded away from 0 , and so we propose
Conjecture 6. With $\bar{x}_{n}$ defined as above, $\inf \left\{\bar{x}_{n}-x_{n, n}: n \geq 1\right\}>0$.
We show in the following theorem that Conjecture 6 is more fundamental than Conjecture 2 and hence by Theorem 7.1 gives another condition which implies the formula $\beta_{M}(x)-\beta(x)=O(1)$.

Theorem 7.6. Conjecture $6 \Longrightarrow$ Conjecture 2.
Proof. We first prove

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|p_{n}^{\prime}\left(x_{n, n}\right)\right|=\infty \tag{43}
\end{equation*}
$$

The polynomials $p_{n}$ are given by

$$
p_{n}(x)=(-1)^{n} \frac{(q+1)^{n-1}}{q^{n^{2}-1}}\left(x-x_{n, 1}\right)\left(x-x_{n, 2}\right) \ldots\left(x-x_{n, n}\right) .
$$

Using logarithmic differentiation we get

$$
p_{n}^{\prime}(x)=(-1)^{n} \frac{(q+1)^{n-1}}{q^{n^{2}-1}}\left(x-x_{n, 1}\right) \ldots\left(x-x_{n, n}\right)\left(\frac{1}{x-x_{n, 1}}+\cdots+\frac{1}{x-x_{n, n}}\right) .
$$

If we distribute the multiplication over the sum and replace $x$ with $x_{n, n}$, all but one term vanish, and the result is

$$
p_{n}^{\prime}\left(x_{n, n}\right)=(-1)^{n} \frac{(q+1)^{n-1}}{q^{n^{2}-1}}\left(x_{n, n}-x_{n, 1}\right)\left(x_{n, n}-x_{n, 2}\right) \ldots\left(x_{n, n}-x_{n, n-1}\right) .
$$

By Corollary 3.1, $0<\frac{x_{n, n-1}}{x_{n, n}}<\frac{q^{2 n-3}}{q^{2 n-2}}=\frac{1}{q}$, so after taking out $x_{n, n}$ from each of the factors we deduce

$$
\left|p_{n}^{\prime}\left(x_{n, n}\right)\right|>\frac{(q+1)^{n-1}}{q^{n^{2}-1}}\left|x_{n, n}\right|^{n-1}\left(1-\frac{x_{n, n-1}}{x_{n, n}}\right)^{n-1}>\frac{q^{n-1}}{q^{n^{2}-1}} q^{(2 n-2)(n-1)}\left(1-\frac{1}{q}\right)^{n-1}=q^{n^{2}-4 n+3}(q-1)^{n-1} \rightarrow \infty
$$ as $n \rightarrow \infty$. This proves (43).

We need to show that for all sufficiently large even values of $n$, the maximum value of $p_{\infty}$ on $\left[x_{n}, x_{n+1}\right]$ is greater than $M$. By the assumption of Conjecture 6 , there exists $\varepsilon>0$ such that $x_{n, n}+\varepsilon<\bar{x}_{n}$ for all $n$, so $p_{n}(x)<p_{\infty}(x)$ for all $x$ in $\left[x_{n, n}, x_{n, n}+\varepsilon\right]$. Since $x_{n, n}$ is the biggest root of $p_{n}$, the graph of $p_{n}$ to the right of $x_{n, n}$ lies above its tangent line, so in particular

$$
p_{\infty}\left(x_{n, n}+\varepsilon\right)>p_{n}\left(x_{n, n}+\varepsilon\right)>\varepsilon p_{n}^{\prime}\left(x_{n, n}\right)
$$

The result then follows from (43).

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