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A Riesz decomposition theorem on harmonic spaces without positive potentials

Ibtesam BAJUNAID, Joel M. COHEN, Flavia COLONNA and David SINGMAN

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ABSTRACT. In this paper, we give a new definition of the flux of a superharmonic function defined outside a compact set in a Brelot space without positive potentials. We also give a new notion of potential in a BS space (that is, a harmonic space without positive potentials containing the constants) which leads to a Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. Furthermore, we give a characterization of the local axiom of proportionality in terms of a global condition on the space.

1. Introduction

The Riesz decomposition theorem for positive superharmonic functions states that any positive superharmonic function on a region of hyperbolic type in **C** can be expressed uniquely as the sum of a nonnegative potential and a harmonic function. This result is of no interest when the region is parabolic because there are no nonconstant positive superharmonic functions.

By a *Riesz decomposition theorem for a class of superharmonic functions* we mean a unique representation of each member of this class as a global harmonic function plus a member of the class of a special type. For a harmonic space with a Green function, this special class is the class of nonnegative potentials. In this case the most natural class to consider is the *admissible* superharmonic functions, which are the superharmonic functions that have a harmonic minorant outside a compact set. Indeed, the admissible superharmonic functions are precisely the superharmonic functions which can be written uniquely as sums of a nonnegative potential plus a harmonic function. This follows from the fact that every admissible superharmonic function in a harmonic space with a Green function possesses a global harmonic minorant and hence, the function minus its greatest harmonic minorant is a nonnegative potential.

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In this article, we give a Riesz decomposition theorem for the class of the admissible superharmonic functions on a harmonic space without positive potentials. This class was first studied by Anandam in [2]. See also [6] for a reference on the admissible subharmonic functions.

In this paper, we give a new definition of the flux of a function that is superharmonic outside a compact set which is equivalent to the various definitions of flux in the works of Anandam (see [2] through [8]). It also emcompasses the definitions of flux in the discrete setting given in [1], [9], and [15]. We then introduce a new class of potentials (called H-potentials) in the axiomatic setting that was first introduced in the recurrent tree and the complex plane settings in [15] and [16]. This new class allows us to obtain a global Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. We also give a characterization of the local axiom of proportionality in terms of a global condition on the space.

Many axiomatic treatments of potential theory have been formulated (for a survey and a historical context, see [13]). In this work we make use of the axiomatic theory of harmonic and superharmonic functions developed by Brelot (see [12]).

DEFINITION 1.1. A **Brelot space** is a connected, locally connected, locally compact but not compact Hausdorff space Ω together with a harmonic structure in the following sense. For each open set $U \subset \Omega$ there is an associated real vector space of real–valued continuous functions on U (which are called **harmonic functions on** U) satisfying the sheaf property, the regularity axiom, and the Harnack property.

The **harmonic support** of a superharmonic function s is the complement of the largest open set on which s is harmonic.

In a Brelot space, the Minimum Principle for superharmonic functions holds: *A nonnegative superharmonic function on a domain U in a Brelot space is either identically zero or positive everywhere on U* ([12], p. 71).

DEFINITION 1.2. A superharmonic function s on a Brelot space Ω is said to be **admissible** if there are a compact set K and a harmonic function h on $\Omega \backslash K$ such that $h(x) \leq s(x)$ for all $x \in \Omega \backslash K$.

Clearly, positive superharmonic functions and superharmonic functions of compact harmonic support are admissible. We shall see in Proposition 2.3 that K may be taken to be K_0, any fixed compact set independent of the superharmonic function.

Any nonnegative superharmonic function which has a harmonic minorant has a greatest harmonic minorant (see [12], p. 87).

Definition 1.3. A nonnegative superharmonic function on an open subset U of a Brelot space is called a **positive potential** (or briefly, a **potential**) if its greatest harmonic minorant on U is identically zero.

Definition 1.4. A **BH space** is a Brelot space whose sheaf of harmonic functions contains the constants. A **BP space** is a BH space on which there is a positive potential. A **BS space** (short for *espace Brelot sans potentiel positif*) is a BH space on which no positive potential exists.

Any open subset of $\mathbf{R}^n$ for $n \geq 3$ is a BP space, while the complement of any subset of $\mathbf{R}^2$ of logarithmic capacity zero is a BS space.

Definition 1.5. A BP space is said to satisfy the **axiom of proportionality** if any two potentials with the same one–point harmonic support are proportional.

Theorem 1.1 ([17], p. 139). *In a Brelot space without potentials all positive superharmonic functions are harmonic and proportional. In particular, in a BS space, every positive superharmonic function must be constant.*

Thus, a Brelot space which possesses positive superharmonic functions which are not harmonic, has potentials.

The following result is the Riesz decomposition theorem for admissible superharmonic functions on a BP space.

Theorem 1.2 ([6], p. 66). *In a BP space a superharmonic function is admissible if and only if it can be written uniquely as the sum of a potential and a function harmonic on the whole space.*

Theorem 1.3 ([20], Theorem 16.1, [2], Theorem 3.6, and [14]). *If Ω is a Brelot space with positive potentials and a countable base of neighborhoods or if Ω is a BS space, then for any $x \in \Omega$, there exists a superharmonic function with harmonic support $\{x\}$.*

By Theorem 1.3 and Theorem 1.2, if Ω is a Brelot space with potentials and a countable base of neighborhoods, then for each $x \in \Omega$ there exists a potential with harmonic support $\{x\}$.

2. The flux of a superharmonic function

The concept of flux on a BS space was introduced for the purpose of associating a harmonic function on Ω to a function that is harmonic outside a compact set.

Nakai (see [22] or Theorem 1.20 of [2]) proved that if h is a function defined on a BP space Ω and harmonic on the complement of a compact set K, then $h = h_\Omega + b$ outside a compact set, where h_Ω is harmonic on Ω and b is bounded. This is not true if Ω is a BS space, but we shall describe an obstruction, called the *flux of h*, so that when the flux of h is zero, then h is the sum of a global harmonic function and a bounded function.

The concept of flux of a function superharmonic outside a compact subset of a BS space appeared in [8] and [19].

Proposition 2.1 ([3], Theorem 1.17). *Let Ω be a BS space, $K \subset \Omega$ compact not locally polar. Then there exists a function $H \geq 0$ unbounded and harmonic off K. If K is outer regular, then H may be chosen so that it tends continuously to* 0 *on ∂K.*

Definition 2.1. Let Ω be a BS space, $K \subset \Omega$ a nonempty compact set. A function H harmonic off K is called a **standard for** Ω associated with K if the following is true: given any function h which is harmonic off an arbitrary compact set, there exist a unique function h_Ω harmonic on the whole space and a unique real number α such that $b = h - \alpha H - h_\Omega$ is bounded off a compact set and $\liminf_{x \to \infty} b(x) = 0$, where the lim inf is taken with respect to the Alexandrov one-point compactification of Ω. This means that for any increasing exhaustion $\{C_n\}$ consisting of compact sets, and for any positive number ε there exists $N \in \mathbf{N}$ such that for all integers $n \geq N$ there is a point $x \in \Omega \backslash C_n$ such that $b(x) < \varepsilon$, and for all $y \in \Omega \backslash C_n$, $b(y) > -\varepsilon$. Note that by the uniqueness of α, H must be unbounded.

By Theorem 9.7 of [10], the function $H(z) = \log|z|$ is a standard for $\mathbf{C}$ associated with the closed unit disk. In this case, b has an actual limit of 0 at infinity because a function which is harmonic and bounded outside a compact set in $\mathbf{C}$ has a limit at infinity. On the other hand, for $\Omega = \mathbf{C} \backslash \{2\}$, H is still a standard for Ω associated with the closed unit disk, but a bounded harmonic function outside a compact set will have two limits as we approach the boundary of Ω in the extended plane (i.e. as $z \to 2$ and as $z \to \infty$).

In [8] (see note following Lemma 2) the following result is shown.

Theorem 2.1. *Let Ω be a BS space and let H be a nonnegative harmonic function defined outside an outer regular compact set K, not identically* 0, *and tending to* 0 *at the boundary of K. Then H is a standard for Ω.*

In [11] (Theorem 4.2), we proved that the outer regularity of the compact set is unnecessary.

In the remainder of this section and in the next section, we shall fix a BS space Ω, a nonempty compact subset K_0 which is not locally polar, and a standard H associated with K_0 tending to 0 on ∂K_0, which for convenience we extend to be identically 0 in the interior of K_0. Observe that this extension is subharmonic on all of Ω.

The following is the key result which we shall use to calculate the flux of a superharmonic function outside a compact set.

Proposition 2.2 ([8], Proposition 1). *Let s be superharmonic outside some compact set. Then there exist a superharmonic function s_Ω on Ω and a constant β such that $s = s_\Omega + \beta H$ outside a compact set.*

Remark 2.1. Let s be a function superharmonic outside a compact set. When it is clear from the context that the values of s on any particular compact set are unimportant, we shall use Proposition 2.2 to change s on a compact set so that it is defined globally and it is superharmonic outside K_0. In particular, whenever it is useful, we shall assume that s is defined globally, is superharmonic outside K_0 and is lower bounded on K_0.

The decomposition in Proposition 2.2 is not unique because we may increase β by subtracting an appropriate multiple of $-H$ from s_Ω, since $-H$ is globally superharmonic. Thus, it would be interesting to know the following: let $\beta_0 = \inf \beta$ over all β such that $s = s_\Omega + \beta H$ for some s_Ω superharmonic on Ω. When is $\beta_0 > -\infty$? We shall discuss this question at the end of this section.

Remark 2.2. Let s be an admissible superharmonic function, and let h_1 be a function which is harmonic outside a compact set K and such that $h_1 \le s$ outside K. By taking the restriction of h_1 to an outer regular compact set containing K, we may assume that K is outer regular. We may extend h_1 to a continuous function on Ω. Let b_1 be a lower bound of s on K and let b_2 be an upper bound of h_1 on K. Then $h = h_1 - |b_2 - b_1| \le s$ on Ω and h is harmonic outside K. Thus, when we say that s has a harmonic minorant h outside a compact set, we mean that $h \le s$ globally and h is harmonic outside a compact set.

Remark 2.3 ([2], p. 133). The difference of the greatest harmonic minorants h_i of a superharmonic function outside a compact set K_i $(i = 1, 2)$ is bounded.

Definition 2.2. Let s be superharmonic outside a compact set. Define

$$A_s = \{\alpha \in \mathbf{R}\colon \text{ there exists } h_\Omega \text{ harmonic on } \Omega \text{ such that } s - \alpha H \ge h_\Omega\}.$$

Lemma 2.1. *If s is superharmonic outside a compact set, then A_s is bounded above. Furthermore, $A_s \neq \emptyset$ if and only if s has a harmonic minorant outside a compact set.*

Proof. By Proposition 2.2, there exist $\beta \in \mathbf{R}$ and s_Ω superharmonic on Ω such that $s = \beta H + s_\Omega$ outside a compact set. For $\alpha \in A_s$, let h_Ω be harmonic on Ω such that $s - \alpha H \geq h_\Omega$. If $\alpha > \beta$, then $s_\Omega - h_\Omega$ is a lower bounded superharmonic function (since it is necessarily bounded below on any compact set), hence is constant. But by Proposition 2.1, a constant cannot be bounded below by a positive multiple of H. Thus, $\alpha \leq \beta$ and so A_s is bounded above by β.

Assume $\alpha \in A_s$. Then there exists h_Ω harmonic on Ω such that $s \geq \alpha H + h_\Omega$. Thus s has a harmonic minorant outside a compact set. Conversely, if s has a harmonic minorant h outside a compact set, then $h = \alpha H + h_\Omega + b$ outside a compact set, for some $\alpha \in \mathbf{R}$, h_Ω harmonic on Ω, and b bounded. Then outside a compact set $s - \alpha H \geq h_\Omega + \inf b$, a global harmonic function. Thus $\alpha \in A_s$. $\square$

Definition 2.3. Let s be a function on Ω superharmonic outside a compact set. Define the **flux of** s **at infinity** (or simply the **flux of** s) with respect to H by

$$\mathrm{flux}(s) = \sup A_s.$$

By convention, $\mathrm{flux}(s) = -\infty$ if $A_s = \varnothing$. By Lemma 2.1, the flux of s is finite if A_s is nonempty.

The following result ties together our definition of flux with various earlier definitions, as well as compiling many useful properties of flux.

Theorem 2.2. (a) *If h is harmonic outside a compact set, then the flux of h is the unique constant α of Definition* 2.1 *such that $h = \alpha H + h_\Omega + b$ outside a compact set with h_Ω harmonic on Ω and b bounded. In particular, $A_h = (-\infty, \alpha]$.*

(b) *If h is bounded harmonic outside a compact set or harmonic everywhere, then the flux of h is zero. If s is superharmonic everywhere, then* $\mathrm{flux}(s) \leq 0$.

(c) *Let s be an admissible superharmonic function on Ω. If h_1 and h_2 are the greatest harmonic minorants of s outside compact sets K_1 and K_2, respectively, then the flux of h_1 and the flux of h_2 are equal.*

(d) *If s is a function superharmonic outside a compact set K and has a subharmonic minorant on $\Omega \backslash K$ (in particular, if s is admissible), then its flux is equal to the flux of its greatest harmonic minorant on $\Omega \backslash K$. Consequently, admissible superharmonic functions have finite flux.*

(e) *The flux of a nonadmissible superharmonic function s is equal to $-\infty$.*

(f) *Let s be superharmonic outside a compact set, and write $s = s_\Omega + \beta H$ as in Proposition* 2.2. *Then* $\mathrm{flux}(s) = \mathrm{flux}(s_\Omega) + \beta$.

(g) *The set of functions which are superharmonic outside a compact set is closed under addition and scalar multiplication by a positive number and the flux is linear on that set.*

(h) *If s is superharmonic outside a compact set and $A_s \neq \emptyset$, then* $\operatorname{flux}(s) \in A_s$, *so that* $A_s = (-\infty, \operatorname{flux}(s)]$.

(j) *If s is superharmonic on Ω and* $\operatorname{flux}(s) = 0$, *then s is harmonic on Ω.*

Proof. We use Proposition 2.2, Lemma 2.1, Remark 2.3, and the following immediate facts:

(i) There are no nonconstant positive superharmonic functions, hence no nonharmonic superharmonic function can be bounded below on Ω by a harmonic function.

(ii) If s is superharmonic outside a compact set, then for all $\alpha \in \mathbf{R}$ and $c > 0$, $A_{s+\alpha H} = A_s + \alpha$, and $A_{cs} = cA_s$.

(iii) (Monotonicity of the flux) If $s_1 \leq s_2$, with s_1, s_2 superharmonic outside a compact set, then $A_{s_1} \subset A_{s_2}$, so that $\operatorname{flux}(s_1) \leq \operatorname{flux}(s_2)$.

(iv) If h_Ω is harmonic on Ω and $h_\Omega + \gamma H$ is bounded below, then $\gamma \geq 0$.

(v) If s_1 and s_2 are superharmonic outside a compact set K, then the greatest harmonic minorant of $s_1 + s_2$ is the sum of the greatest harmonic minorants of s_1 and s_2 on K.

Part (a) follows from (iv). Part (b) follows from (a) and Lemma 2.1. Part (c) holds since Remark 2.3 implies that $A_{h_1} = A_{h_2}$.

To prove (d), let h be the greatest harmonic minorant of s on $\Omega \backslash K$. By (c), without loss of generality we may assume that K contains K_0. By (a), $A_h \neq \emptyset$, and by (iii), $A_h \subset A_s$, so $A_s \neq \emptyset$. Let $\alpha \in A_s$ so that $s - \alpha H \geq h_\Omega$, a function harmonic on Ω. Thus $s \geq \alpha H + h_\Omega$, which is harmonic off K_0. Then on $\Omega \backslash K$, $s \geq h \geq \alpha H + h_\Omega$, so by (iii) and (a), $\operatorname{flux}(s) \geq \operatorname{flux}(h) \geq \alpha$. Since this is true for all $\alpha \in A_s$, it follows that $\operatorname{flux}(s) = \operatorname{flux}(h)$.

To prove (e), assume there exists $\alpha \in A_s$. Then $s \geq \alpha H + h_\Omega$, where h_Ω is a function harmonic on Ω. Since $\alpha H + h_\Omega$ is harmonic outside K_0, s is admissible.

Part (f) follows from (ii). To prove (g), let s_1, s_2 be superharmonic outside the same compact set K, and let h_1 and h_2 be their respective greatest harmonic minorants outside K. Then by (v), $h_1 + h_2$ is the greatest harmonic minorant of $s_1 + s_2$ outside K. Thus by (d) and (a), $\operatorname{flux}(s_1 + s_2) = \operatorname{flux}(h_1 + h_2) = \operatorname{flux}(h_1) + \operatorname{flux}(h_2) = \operatorname{flux}(s_1) + \operatorname{flux}(s_2)$. Linearity with respect to multiplication by a positive constant follows from (ii).

Part (h) follows from (d) and (a). Part (j) follows from (h) and (i). $\square$

Remark 2.4. The original definitions of flux given by Anandam [8] separately first for harmonic functions outside a compact set, then for global superharmonic functions, and finally for functions superharmonic outside a compact set, are equivalent to ours, by Theorem 2.2, parts (a), (d), (e) and (f).

Proposition 2.3. *A superharmonic function on a BS space Ω is admissible if and only if it has a minorant which is harmonic outside K_0.*

Proof. Let s be an admissible superharmonic function on Ω with flux α. Then by part (h) of Theorem 2.2, there exists a harmonic function h_Ω on Ω such that $s \geq \alpha H + h_\Omega$, which is harmonic outside K_0. The converse is obvious. $\square$

We now respond to the question raised after Remark 2.1.

Proposition 2.4. *Let s be superharmonic outside a compact set in a BS space Ω. Let B be the set consisting of all $\beta \in \mathbf{R}$ such that $s = s_\Omega + \beta H$ outside a compact set as in Proposition 2.2. Then*

(a) *The set B is an interval unbounded above and* $\inf B \geq \operatorname{flux}(s)$. *In particular, if s has finite flux, then B is bounded below.*

(b) *When $\Omega = \mathbf{R}$ with the harmonic structure inherited from the Laplace operator, s has finite flux if and only if B is bounded below.*

Proof. To prove (a) assume $\beta \in B$ and $\gamma > \beta$. If $s = s_\Omega + \beta H$ outside a compact set K, then $s = s'_\Omega + \gamma H$ outside K, where $s'_\Omega = s_\Omega - (\gamma - \beta)H$ which is globally superharmonic. Thus $\gamma \in B$, proving that B is an interval unbounded above. On the other hand, since the flux of a superharmonic function on Ω is less than or equal to 0, if $s = s_\Omega + \beta H$ outside a compact set then $\beta \geq \operatorname{flux}(s)$. Thus, if $\operatorname{flux}(s)$ is finite, B is bounded below.

By part (a), to prove (b) we need to show that if B is bounded below then s has finite flux. Let $K_0 = \{0\}$. The function $H(x) = |x|$ is a standard for K_0. Let us consider $s(x) = |x| - x^2$ for $x \in \mathbf{R}$, which has flux $-\infty$ since it does not have a harmonic minorant ouside a compact set. Since s is smooth on $\mathbf{R}\backslash\{0\}$ and its Laplacian is -2 there, s is superharmonic on $\mathbf{R}\backslash\{0\}$. For $n \in \mathbf{N}$, let $\tilde{s}_n(x) = s(x) + (2n-1)H(x)$ and observe that for $x \geq n$, $s'(x) \leq 0$ and for $x \leq -n$, $s'(x) \geq 0$, where s' denotes the derivative of s. Thus the function

$$s_n(x) = \begin{cases} \tilde{s}_n(n) & \text{for } |x| \leq n \\ \tilde{s}_n(x) & \text{for } |x| \geq n \end{cases}$$

is globally superharmonic and $s(x) = s_n(x) + \beta_n H(x)$ outside $[-n, n]$, where $\beta_n = -(2n-1)$. Thus, B is unbounded below. $\square$

3. Potentials in a BS space and Riesz decomposition of admissible superharmonic functions

In this section we shall present several classes of admissible superharmonic functions which in a BS space play the role analogous to that of positive potentials in a BP space. We first introduce two operators which we use to define these classes.

Definition 3.1. Let s be superharmonic on $\Omega \backslash K_0$. Let $\mathscr{E} = \{U_n\}$ be an increasing exhaustion consisting of relatively compact regular sets containing K_0 (which exists by [21]). Let $h_n = h_s^{U_n}$, the solution of the Dirichlet problem with boundary values s on ∂U_n. Define

$$D_{\mathscr{E}} s(x) = \lim_{n \to \infty} h_n(x)$$

if this limit exists locally uniformly, in which case $D_{\mathscr{E}} s$ is harmonic on Ω.

Definition 3.2. Let K_0 be outer regular and let s be superharmonic on $\Omega \backslash K_0$. Define Ds to be the greatest harmonic minorant of s on each component of the complement of K_0 if such minorant exists and $-\infty$ if it does not exist on that component.

We now present different classes of potentials introduced by Anandam in [6] and [7].

Definition 3.3. An admissible superharmonic function s is said to be in the **class** $\mathscr{P}$ if there exists an exhaustion $\mathscr{E}$ such that $D_{\mathscr{E}}(s - \alpha H)$ exists and is constant, where α is the flux of s. If, furthermore, that constant is 0 for some exhaustion $\mathscr{E}$, s is called a **BS potential**. Define the **class** $\mathscr{Q}$ as the collection of all admissible superharmonic functions s satisfying the property: there exists $s' \in \mathscr{P}$ such that the difference of the greatest harmonic minorants of s and s' outside a compact set is bounded. This class is independent of the choice of the compact set.

Observation 3.1. *Suppose s is in class $\mathscr{P}$ and has flux α, so that for some exhaustion $\mathscr{E}$, $D_{\mathscr{E}}(s - \alpha H)$ exists and is constant. By Lemma 2, p. 235 in [4], $s - \alpha H$ is lower bounded. If $D_{\mathscr{E}'}(s - \alpha H)$ exists for some other exhaustion $\mathscr{E}'$, then $D_{\mathscr{E}'}(s - \alpha H)$ is a lower bounded harmonic function, hence it is also constant.*

Anandam proved the following partial Riesz decomposition theorem for admissible superharmonic functions on a BS space.

Proposition 3.1 ([5], Lemmas 2 and 3). *Any admissible superharmonic function s on a BS space is a sum of a function in the class $\mathscr{Q}$ and a harmonic function. This decomposition is unique up to an additive constant. If s has compact harmonic support, then the element of $\mathscr{Q}$ can be chosen uniquely to be a BS potential.*

One difficulty in working with these classes of potentials is that given an admissible superharmonic function s, there is no procedure for determining whether s is in such classes. To overcome this difficulty and the lack of uniqueness in the decomposition of Proposition 3.1, we introduce a new class of potentials.

Definition 3.4. An admissible superharmonic function s is an H-**potential** if

$$\liminf_{x\to\infty}\{Ds(x)-\alpha H(x)\}=0,$$

where D is the operator of Definition 3.2, α is the flux of s, and the lim inf is taken with respect to the Alexandrov one-point compactification of Ω.

The following theorem shows that the H-potentials are the class of potentials best suited to describe all admissible superharmonic functions.

Theorem 3.1 (Global Riesz Decomposition Theorem). *In a BS space Ω every admissible superharmonic function can be written uniquely as the sum of an H-potential and a harmonic function.*

Proof. Let s be an admissible superharmonic function. Since Ds is the greatest harmonic minorant of s outside K_0, by Theorem 2.2(a) and (d), there exist h_Ω harmonic and b bounded such that $Ds=\alpha H+h_\Omega+b$ outside a compact set, where α is the flux of s. By adding the condition that

$$\liminf_{x\to\infty}\ b(x)=0, \tag{1}$$

we get uniqueness in the above decomposition. Then the function $p=s-h_\Omega$ is admissible superharmonic with flux α and $Dp=Ds-h_\Omega=\alpha H+b$. By (1), $Dp-\alpha H$ has inferior limit 0 at infinity, thus p is an H-potential and $s=p+h_\Omega$, proving the existence of the decomposition.

To prove the uniqueness, assume $p_1+h_1=p_2+h_2$ where p_1 and p_2 are H-potentials and h_1 and h_2 are harmonic on Ω. Then $p_1-p_2=h_2-h_1$, which is globally harmonic. In particular, p_1 and p_2 have the same flux α. By definition of H-potential, $Dp_j=\alpha H+b_j$ where b_j $(j=1,2)$ is bounded and harmonic outside a compact set with liminf 0 at infinity. Since $h_2-h_1=p_1-p_2=D(p_1-p_2)=Dp_1-Dp_2=b_1-b_2$, the function b_1-b_2 can be extended to a global bounded harmonic function. Thus, b_1-b_2 is constant. Since $\liminf b_1=\liminf b_2=0$, $b_1=b_2$ and hence $h_1=h_2$. □

Corollary 3.1. *If s is an H-potential with flux α, then $Ds-\alpha H$ is bounded.*

Proof. By the proof of Theorem 3.1, for any admissible superharmonic function s, the unique global harmonic function h_Ω in the decomposition of Ds as $\alpha H+h_\Omega+b$ (with b bounded) is the **same** global harmonic function in the decomposition of s as an H-potential plus a global harmonic function. Thus if s is itself an H-potential, then $h_\Omega = 0$, and so $Ds = \alpha H + b$. □

In [15], where we restricted our attention to recurrent trees, we defined an H-potential to be an admissible superharmonic function s for which $Ds = \alpha H$, where $\alpha = \text{flux}(s)$ and K_0 is taken to be a single point. It can be easily shown that in this setting the two definitions of H-potential are equivalent.

4. Proportionality in BS spaces

DEFINITION 4.1. A Brelot space Ω satisfies the **local axiom of proportionality** if for each $x \in \Omega$ and each relatively compact neighborhood U of x, all potentials on U with harmonic support at x are proportional.

THEOREM 4.1 (Local Riesz Decomposition Theorem). *If Ω is a BS space satisfying the local axiom of proportionality, then the following properties hold.*

(a) *For all $x \in \Omega$ there exists a unique BS potential p_x with harmonic support $\{x\}$ such that* $\text{flux}(p_x) = -1$.

(b) [18] *For all s superharmonic on Ω there exists a unique Radon measure $\mu \geq 0$ such that for each relatively compact domain $U \subset \Omega$ and for all $x \in U$*

$$s(x) = \int_U p_y(x) d\mu(y) + h(x),$$

where h is harmonic on U and p_y is the unique BS potential with harmonic support y and flux equal to -1.

PROOF. (a) By Theorem 1.3, given any $x \in \Omega$, there exists a superharmonic function s_Ω with harmonic support $\{x\}$. By Proposition 3.1, there exists a BS potential p with harmonic support $\{x\}$. Since the flux of an admissible superharmonic function which is not harmonic is negative, the function $p_x = -\frac{p}{\text{flux}(p)}$ is superharmonic, and hence, a BS potential with harmonic support $\{x\}$ and flux -1, proving the existence.

For the uniqueness, assume that p and q are BS potentials on Ω having harmonic support $\{x\}$ and flux -1. Let U be a relatively compact regular neighborhood of x. Since p and q are superharmonic on U and U is a BP space, there exist potentials p_1 and p_2 on U with harmonic support $\{x\}$, and h_1, h_2 harmonic on U, such that $p = p_1 + h_1$, and $q = p_2 + h_2$. By the local axiom of proportionality, there exists $\lambda > 0$ such that $p_1 = \lambda p_2$. Thus $h = p - \lambda q$ is harmonic on U, but it is also harmonic off x, and therefore is harmonic on Ω. Thus

$$-1 = \text{flux}(p) = \text{flux}(h + \lambda q) = -\lambda,$$

so $\lambda = 1$. This shows that $p - q$ is harmonic on Ω. By Observation 3.1 we deduce that $p - q$ is constant. Since p and q are BS-potentials, that constant must be 0. Thus $p = q$ on Ω. □

The following result is a superharmonic extension theorem for BH spaces. It was proved by Hervé [20] in the BP case and later by Anandam (Theorem 3.4 of [2]) in the BS case.

Theorem 4.2. *Let Ω be a BH space and let U be an open subset of Ω. If s is a superharmonic function on U with compact harmonic support K, then there exists a superharmonic function s_Ω on Ω with harmonic support on K such that $s_\Omega - s$ is harmonic in a neighborhood of K.*

The following is a new characterization of the local axiom of proportionality.

Theorem 4.3. *In a BH space, the following statements are equivalent.*

(a) *The local axiom of proportionality holds.*

(b) *For any two superharmonic functions with the same one-point harmonic support, some nonzero linear combination of them is harmonic.*

Proof. Assume that the local axiom of proportionality holds on Ω, and let s_1, s_2 be superharmonic on Ω with harmonic support at $x \in \Omega$. Let U be a relatively compact neighborhood of x. Then $s_1|U$ and $s_2|U$ are superharmonic on U with harmonic support at x, and being lower-semicontinuous, they are bounded below on a relatively compact set, so they have harmonic minorants. Let h_1 and h_2 be the greatest harmonic minorants of $s_1|U$ and $s_2|U$, respectively. Thus $s_1|U - h_1$ and $s_2|U - h_2$ are potentials on U with harmonic support at x, and so for some $\alpha > 0$, $s_1|U - h_1 = \alpha(s_2|U - h_2)$. Hence $(s_1 - \alpha s_2) \,|\, U = h_1 - \alpha h_2$ which is harmonic on U. But $s_1 - \alpha s_2$ is harmonic outside x, thus $s_1 - \alpha s_2$ is harmonic on Ω, proving that (b) holds.

Conversely, suppose (b) holds. Let $x \in \Omega$, U a relatively compact neighborhood of x, and let p_1 and p_2 be potentials on U with harmonic support at x. By Theorem 4.2, there exist superharmonic functions s_1 and s_2 on Ω with harmonic support at x such that $s_1 - p_1$ and $s_2 - p_2$ are harmonic on a neighborhood of x. Thus, there exist nonzero $\alpha, \beta \in \mathbf{R}$ such that $\alpha s_1 + \beta s_2$ is harmonic on Ω. Then $\alpha(s_1 - p_1) + \beta(s_2 - p_2)$ is harmonic on U, so $\alpha p_1 + \beta p_2$ is harmonic on U. Letting $\lambda = -\frac{\beta}{\alpha}$, we have that $p_1 - \lambda p_2$ is a harmonic function h on U. Notice that λ cannot be negative since otherwise $-\lambda p_2$ would be a potential on U and the sum of two potentials cannot be harmonic. Thus, $\lambda > 0$ and $p_1 = \lambda p_2 + h$. But the greatest harmonic minorant of p_1 and of p_2 is zero. Hence $h = 0$ and so $p_1 = \lambda p_2$ on U, proving the local axiom of proportionality. □

EXAMPLE 4.1. *Let Ω be an open subset of $\mathbf{R}^n$ whose harmonic structure is defined by the Laplace operator Δ. Then Ω satisfies part* (b) *of Theorem* 4.3, *hence satisfies the local axiom of proportionality, since if s_1 and s_2 are superharmonic functions on Ω with support at $x \in \Omega$, then $\Delta s_2(x)s_1 - \Delta s_1(x)s_2$ is harmonic on Ω.*

References

[1] K. Abodayeh, V. Anandam, Dirichlet problem and Green's formula on trees, Hiroshima Math. J., **35** (2005), 413–424.

[2] V. Anandam, Espaces harmoniques sans potentiel positif, Ann. Inst. Fourier (Grenoble), **22** (1972), no. 4, 97–160.

[3] V. Anandam, Harmonic spaces with positive potentials and nonconstant harmonic functions, Rend. Circ. Mat. Palermo (2), **21** (1972), 149–167.

[4] V. Anandam, Potentials in a BS harmonic space, Bull. Math. de la Soc. Sci. Math. de la R. S. de Roumanie, **18** (1974), 233–248.

[5] V. Anandam, Pseudo–potentiels dans un espace harmonique sans potentiel positif, Bull. Sc. Math. 2^e série, **100** (1976), 369–376.

[6] V. Anandam, Admissible subharmonic functions and associated measures, J. London Math. Soc. (2), **19** (1979), no. 1, 65–78.

[7] V. Anandam, Subharmonic extensions outside a compact set in R^n, Proc. Amer. Math. Soc., **84** (1982), no. 1, 52–54.

[8] V. Anandam, Superharmonic functions and the flux, Bull. Sc. Math., **106** (1982), 85–92.

[9] V. Anandam, S. I. Othman, Flux in an infinite network, Mem. Fac. Sci. Eng. Shimane Univ. Series B. Mathematical Sciences, **38** (2005), 7–16.

[10] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, 2nd ed., Graduate Texts in Mathematics **137**, Springer-Verlag, New York, (2001).

[11] I. Bajunaid, J. M. Cohen, F. Colonna, D. Singman, Trees as Brelot spaces, Adv. in Appl. Math. **30** (2003), 706–745.

[12] M. Brelot, Lectures on Potential Theory, Tata Institute, Bombay (1960).

[13] M. Brelot, Historical introduction, Potential Theory (C.I.M.E., I Ciclo, Stresa, 1969) Edizioni Cremonese, Rome (1970), pp. 1–21.

[14] M. Brelot, Axiomatique des Fonctions Harmoniques, Séminaire de Mathématiques Supérieures, No. **14** (Été, 1965), Les Presses de l'Université de Montréal, Montréal, (1969).

[15] J. M. Cohen, F. Colonna, D. Singman, A global Riesz decomposition theorem on trees without positive potentials, J. London Math. Soc., (2), **75** (2007), 1–17.

[16] J. M. Cohen, F. Colonna, D. Singman, Riesz decomposition and flux in potential theory, preprint.

[17] N. Du Plessis, An Introduction to Potential Theory, Oliver & Boyd, Edinburgh (1970).

[18] H. Guessous, Sur le procédé alterné et le prolongement surharmonique, Thèse, Université Paris VI, 1977.

[19] H. Guessous, Prolongement surharmonique dans un espace harmonique. Notion de flux, Potential Theory, Copenhagen 1979, Berlin, Springer-Verlag, 1980 (Lecture Notes in Mathematics, 787), 121–143.

[20] R. M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, Ann. Inst. Fourier, Grenoble, **12** (1962), 415–571.

[21] P. A. Loeb, An axiomatic treatment of pairs of elliptic differential equations, Ann. Inst. Fourier, **16/2** (1966), 167–208.

[22] B. Rodin, L. Sario, Principal functions, In collaboration with M. Nakai, Van Nostrand, Princeton, NJ, 1968.

Ibtesam Bajunaid
P. O. box 60618, *Riyadh* 11555
Kingdom of Saudi Arabia
E-mail: ibajunaid@ksu.edu.sa

Joel M. Cohen
Department of Mathematics
University of Maryland
College Park, Maryland, 20742-4015
USA
E-mail: jcohen@umd.edu

Flavia Colonna & David Singman
Dept. of Mathematical Sciences
George Mason University
4400 *University Dr.*
Fairfax, VA 22030
USA
E-mail: fcolonna@gmu.edu
E-mail: dsingman@gmu.edu