J. London Math. Soc. (2) 75 (2007) 1–17
doi:10.1112/jlms/jdl001

A GLOBAL RIESZ DECOMPOSITION THEOREM ON TREES WITHOUT POSITIVE POTENTIALS

JOEL M. COHEN, FLAVIA COLONNA AND DAVID SINGMAN

Abstract

We study the potential theory of trees with nearest-neighbor transition probability that yields a recurrent random walk and show that, although such trees have no positive potentials, many of the standard results of potential theory can be transferred to this setting. We accomplish this by defining a non-negative function H, harmonic outside the root e and vanishing only at e, and a substitute notion of potential which we call *H-potential*. We define the flux of a superharmonic function outside a finite set of vertices, give some simple formulas for calculating the flux and derive a global Riesz decomposition theorem for superharmonic functions with a harmonic minorant outside a finite set. We discuss the connection of the H-potentials with other notions of potentials for recurrent Markov chains in the literature.

1. *Introduction*

The study of potential theory on trees was introduced by Cartier [**9**], although its consideration on Markov chains goes back to the beginning of Markov chain theory. Subsequent developments include [**10**], [**18**], and [**20**]. Trees, under the harmonic structure determined by nearest-neighbor transition probabilities, are harmonic spaces which may or may not have positive potentials. In [**6**], we used the tools of Brelot theory to derive on trees several properties that were not previously known, and we related these results to the potential-theoretic aspects of trees that had been studied in [**9**] in the special case of trees with positive potentials.

In this paper, we continue the study begun in [**6**] of the potential theory of trees for which the random walk is recurrent, without making explicit use of the Brelot theory.

In the transient case, a Riesz decomposition theorem holds for the positive superharmonic functions, that is, each positive superharmonic function can be written uniquely as the sum of a positive potential and a harmonic function. In the recurrent case, there are no non-constant positive superharmonic functions, and hence there are no positive potentials. Instead, we consider *admissible* superharmonic functions, namely superharmonic functions which are bounded below by a function harmonic outside a compact set (hence, a finite set in this setting). Motivation for this definition is provided by the work of Anandam in axiomatic potential theory (cf. [**1**, **3**]).

Our aim here is to develop a theory of flux and to introduce a class of superharmonic functions (which we call H-potentials) behaving as positive potentials in the transient case, in order to obtain a Riesz decomposition theorem, that is, a complete characterization of admissible superharmonic functions as sums of H-potentials and global harmonic functions. In classical potential theory in the complex plane, there is a notion of potential, more general than logarithmic potential, that yields a Riesz decomposition of all admissible superharmonic functions. Not all such potentials, however, are admissible (cf. [**5**]).

The H-potentials (introduced in [**6**]) are defined in terms of a non-negative unbounded function H that is harmonic and positive outside the root.

Anandam [**1**] gives various definitions of flux on different classes of functions. We condense these into a single definition of the flux of any function on a recurrent tree. We define the *flux*

Received 12 May 2005; revised 13 February 2006; published online 8 January 2007.

1991 *Mathematics Subject Classification* 31C05, 05C05 (primary), 31C20, 60J45 (secondary).

of a function s which is superharmonic outside a finite set as the supremum of all real numbers α such that $s - \alpha H$ is bounded below by a harmonic function on the tree. This allows us to define a kernel G that we call the *H-Green function*, which plays a role similar to that of the Green function on transient trees, except that it is not non-negative. We show that, in analogy to the transient case, the H-potentials are precisely the functions of the form Gf, where f is a non-negative function satisfying a certain growth condition.

To put this paper in the proper perspective from the point of view of Markov chains on discrete structures, we relate the concepts developed in our study to the corresponding ones given in the literature and explain the connection and the differences between our results and those established in earlier works. Specifically, we discuss the connection between our work and the potential theory developed in [**11**–**16**, **19**, **23**].

The topic of recurrent potential theory on denumerable Markov chains has lain dormant for many years. One reason is that in the early studies of trees, the greatest concentration was on homogeneous isotropic trees which can be represented by groups. Kesten [**16**] had completely settled the theory on groups. A second reason is that the notion of admissible superharmonic function came later and was heretofore used almost exclusively in an analytic setting.

1.1. *Trees*

Let (P, T) be a stochastic Markov chain, where T is a countable set of states and $P = \{p(v,u)\}_{v,u\in T}$ is the matrix of transition probabilities of finite type, that is, rows and columns have only finitely many non-zero entries.

A *path* is a finite or infinite sequence of states $[v_0, v_1, \ldots]$ such that $p(v_k, v_{k+1}) > 0$ for all integers $k \geqslant 0$. The path is said to be *geodesic* if furthermore $v_{k+1} \neq v_{k-1}$ for all $k \geqslant 1$. In particular, the paths $[v_0]$ and $[v_0, v_1]$ are always geodesic. An infinite geodesic path is also called a *ray*. A *tree* is the set of states T of such a Markov chain which has the property that for all $v, u \in T$ there is a unique geodesic path from v to u. The states are called the *vertices* of the tree. Throughout this paper, we fix a vertex e called the *root*.

Observe that, for all $v \in T$, $p(v,v) = 0$, otherwise both $[v]$ and $[v,v]$ would be geodesics. Moreover, all finite geodesic paths can be reversed, that is, if $p(u,v) > 0$ then $p(v,u) > 0$, since the unique geodesic path $[v = v_0, \ldots, v_n = u]$ from v to u must be $[v,u]$, otherwise $[u, v_0, \ldots, v_n]$ would be a geodesic from u to u.

Two vertices v and u are *neighbors* and $[v,u]$ is called *an edge* if $p(v,u) > 0$, in which case we use the notation $v \sim u$.

Let $[u,v]$ denote the unique geodesic path between the vertices u and v. A vertex with a single neighbor is called *terminal*.

The distance $d(u,v)$ between the vertices u and v is the number of edges in the unique geodesic path from u to v. The *length* of a vertex v is $|v| = d(e,v)$. The *predecessor* u^- of a vertex $u \neq e$ is the next to the last vertex of the geodesic path from e to u. We call *children* of a vertex v the vertices u such that $u^- = v$. Children of the same vertex are called *siblings*. A vertex v is an *ancestor* of u and u is a *descendant* of v if v is in the geodesic path from e to u^-. The *sector* determined by the vertex v is the set $S(v)$ consisting of v and all its descendants.

By a function on a tree T, we mean a function on its set of vertices. The *Laplacian* of a function f on T at $v \in T$ is defined as $\Delta f(v) = \sum_{u\sim v} p(v,u)f(u) - f(v)$.

A function f on T is said to be *harmonic*, *superharmonic* or *subharmonic* at a vertex v if $\Delta f(v)$ is equal to, at most or at least zero, respectively. The function is said to be harmonic, superharmonic or subharmonic on a set of vertices K if it is harmonic, superharmonic or subharmonic, respectively, at each point of K. If $K = T$, then we will simply call the function harmonic, superharmonic or subharmonic. The *harmonic support* of a function s is the set of vertices where the Laplacian of s is non-zero. A *potential* is a positive superharmonic function whose only non-negative harmonic minorant is the constant zero. A superharmonic function s

on T is said to be *admissible* if there is a finite set K and a function h on T, harmonic at each vertex of $T \setminus K$, such that $h(x) \leqslant s(x)$ for all $x \in T$.

A function harmonic off a finite set of vertices does not necessarily extend to a function harmonic on the whole tree. Indeed, since a single vertex v_0 disconnects the tree, a function harmonic off v_0 cannot necessarily be extended to a function harmonic everywhere.

On the other hand, there is an extension result for functions harmonic off a finite set of vertices and this motivates much of what we do in this paper (see Definition 2.1).

Definition 1.1. Given a finite subset K of T, the *interior* of K is the set $\operatorname{int} K$ consisting of all non-terminal vertices $v \in K$ such that every vertex of T which is a neighbor of v belongs to K. The *boundary* of K in T is defined as the set ∂K of all vertices $v \in K$ such that exactly one neighbor $\tilde{v}$ of v is in $\operatorname{int} K$. We say that K is a *complete* subtree of T if $K = \operatorname{int} K \cup \partial K$.

Remark 1.1 (Dirichlet problem). Any complete finite set K satisfies the following property: if f is any function defined on ∂K, then there exists a unique extension of f to K, non-negative if f is non-negative, which is harmonic on the interior of K.

A useful property of superharmonic functions is the *minimum principle*: let K be a connected complete set and assume that a function s on K attains its minimum at $v_0 \in \operatorname{int} K$ and is superharmonic on $\operatorname{int} K \setminus \{v_0\}$. If s is not constant, then it is not superharmonic at v_0. In particular, if s is superharmonic on $\operatorname{int} K$ and non-constant, then its minimum cannot be attained on $\operatorname{int} K$. Since the negative of a subharmonic function is superharmonic, there is an analogous *maximum principle* for subharmonic functions.

Definition 1.2. Let s be a function on T and K a set of vertices. By a *harmonic (subharmonic) minorant of s on K*, we mean a function h such that $h \leqslant s$ on T and is harmonic (subharmonic) on $\operatorname{int} K$. Note that although these functions are defined globally, the values of h outside of K have no bearing on its harmonicity on $\operatorname{int} K$.

We state the following theorem on trees, but it is easy to see that it holds for an arbitrary denumerable Markov chain. We believe that the results are part of the folklore, but we include an elementary proof since we have not been able to locate one in print.

Theorem 1.1. *Let K be a set of vertices. Let s be superharmonic on $\operatorname{int} K$ and suppose that s has a subharmonic minorant t_0 on K. Then s has a greatest harmonic minorant h on K that will be denoted by $\operatorname{ghm}_K(s)$ and $t_0 \leqslant h$. Furthermore, h is equal to the supremum of all subharmonic minorants of s on K.*

Proof. Let $\mathcal{S}$ denote the set of functions that are subharmonic minorants of s on K. Define h by $h(v) = \sup\{t(v) : t \in \mathcal{S}\}$. By assumption, $-\infty < h(v) \leqslant s(v)$. Let $v \in \operatorname{int} K$. For each $t \in \mathcal{S}$, $t(v) \leqslant \sum_{w\sim v} p(v,w)t(w) \leqslant \sum_{w\sim v} p(v,w)h(w)$ so, taking the supremum over all $t \in \mathcal{S}$, we see that h is subharmonic on $\operatorname{int} K$ and thus h is the greatest subharmonic minorant of s on K. Let $v \in \operatorname{int} K$ and define h_v by $h_v(u) = h(u)$ for $u \neq v$ and $h_v(v) = \sum_{w\sim v} p(v,w)h(w)$. Then, h_v is harmonic at v and $h \leqslant h_v \leqslant s$, since h is subharmonic on $\operatorname{int} K$ and s is superharmonic on $\operatorname{int} K$. Let $u \sim v$, $u \in \operatorname{int} K$. Then $\sum_{w\sim u} p(u,w)h_v(w) \geqslant \sum_{w\sim u} p(u,w)h(w) \geqslant h(u) = h_v(u)$, so h_v is subharmonic on $\operatorname{int} K$. Thus, $h_v \in \mathcal{S}$ and thus $h_v = h$. Since v is arbitrary, h is harmonic on $\operatorname{int} K$. Since any harmonic minorant of s on K is in $\mathcal{S}$, h is the greatest harmonic minorant of s on K. □

Corollary 1.1. *Let K be a set of vertices and let s_1 and s_2 be superharmonic on* $\operatorname{int} K$, *each having a subharmonic minorant on K. Let $h_i = \operatorname{ghm}_K(s_i)$, $i = 1, 2$. Then, $h_1 + h_2 = \operatorname{ghm}_K(s_1 + s_2)$.*

Proof. Let $h = \operatorname{ghm}_K(s_1 + s_2)$. Since $h_1 + h_2$ is harmonic on $\operatorname{int} K$, $h_1 + h_2 \leqslant h$. In order to prove the opposite inequality, note that $h - s_1$ is a subharmonic minorant of s_2, so by Theorem 1.1, $h - s_1 \leqslant h_2$. Thus, $h - h_2$ is a harmonic minorant of s_1, so $h - h_2 \leqslant h_1$, that is, $h \leqslant h_1 + h_2$. □

Let T be a tree with a nearest-neighbor transition probability p. The *probability of a path* $\gamma = [v_0, \dots, v_n]$ is defined as $p(\gamma) = \prod_{j=1}^{n} p(v_{j-1}, v_j)$. For $v, w \in T$, let $\Gamma_{v,w}$ be the set of all finite paths from v to w. Define the *Green function* G of T as $G(v, w) = \sum_{\gamma \in \Gamma_{v,w}} p(\gamma)$. Probabilistically, $G(v, w)$ is the expected number of times the associated random walk starting at v visits w. In [**9**], it is shown that if $G(v, w)$ is finite for some vertices v and w, then it is finite for all pairs of vertices. This means that the associated random walk is *transient*. In this case, given $w \in T$, the function $G_w : T \to (0, \infty)$ defined by $G_w(v) = G(v, w)$ satisfies the conditions $\Delta G_w(v) = 0$ for $w \neq v$ and $\Delta G_w(w) = -1$ (see [**9**, Proposition 2.3]). Therefore, G_w is superharmonic (actually, a potential in the sense that its greatest harmonic minorant on T is 0) with harmonic support $\{w\}$. The potentials are precisely the functions of the form $Gf(v) = \sum_{w \in T} G(v, w) f(w)$, where f is a non-negative function on T, not identically zero, and $Gf(e) < \infty$.

If G is infinite, the random walk is *recurrent*, that is, for any pair of vertices v, w the probability that a random walk starting at v will reach w in positive time is one. In this case, there are no (positive) potentials and so, by Theorem 1.1, all positive superharmonic functions are necessarily constant, a well-known result on recurrent irreducible Markov chains. We give here an elementary proof. If s is positive superharmonic and h is its greatest harmonic minorant, then $s - h$ has the greatest harmonic minorant 0. Since there are no positive potentials, $s - h = 0$. This shows that any positive superharmonic function is harmonic. The function $t = \min\{s, s(e)\}$ is positive superharmonic, and hence harmonic. Since t attains its maximum value at e, it must be constant by the maximum principle. Thus, s attains its minimum at e and so, by the minimum principle, s must be constant.

Example 1.1. Let T be a homogeneous tree of degree 3, and define $p(v, v^-) = 1/2$ if $v \neq e$, $p(v^-, v) = 1/4$ if $v^- \neq e$ and $p(e, v) = 1/3$ if $v \sim e$. Then, T is a recurrent tree (see [**6**, Example 4.1]), which we refer to as the $(1/4, 1/2)$-tree.

Example 1.2. Let T_3 be the tree of Example 1.1, fix a ray $\rho = [v_0 = e, v_1, v_2, \dots]$, and fix $p \in (0, 2/3)$. Let T be T_3 with the transition probabilities $p(v, u)$ modified only for $v = v_n$ for each $n \geqslant 1$: let $p(v_n, v_{n-1}) = 1/3$, $p(v_n, v_{n+1}) = p$ and $p(v_n, w_n) = q = 2/3 - p$, where w_n is the other neighbor of v_n. In [**6**, **7**], we proved that T is a recurrent tree for $0 < p \leqslant 1/3$ and a transient tree for $1/3 < p < 2/3$.

1.2. *Outline of results*

In Section 2, we define a non-negative unbounded subharmonic function H harmonic except at e which plays a key role in the study of potential theory on recurrent trees. In Theorem 2.2, we show that H satisfies the property that given any function h harmonic outside a finite set, there exists a unique real number α such that $h - \alpha H$ is a sum of a global harmonic function and a bounded function. The function H and the corresponding family of subharmonic functions H_v with harmonic support at v are used to construct the H-potentials. In Theorem 2.4, we give a direct proof on recurrent trees of an extension theorem (originally due to Nakai in a

Riemann surface setting) that any function superharmonic outside a finite set can be written as a sum of a global superharmonic function, a multiple of H and a function of finite support. This is fundamental for our definition of the flux. In Theorem 2.5, we give the properties of the flux of a function on T that is superharmonic outside a finite set.

In Section 3 (Theorem 3.2), we give an explicit construction of the H-potentials with one-point harmonic support, which we then show (see Proposition 3.1) are unique up to a positive multiplicative constant. We introduce the superharmonic functions $G_v = H_v(e) - H_v$ and use them to define the H-Green function G. Using H-potentials, we also construct (Theorem 3.3) a function on a tree with arbitrary prescribed Laplacian.

In Section 4 (Theorem 4.2), we give an explicit formula for the flux of a function s which is superharmonic outside a finite set in terms of its Laplacian and a formula in terms of its normal derivative.

In Section 5 (Theorem 5.1), we show that the function Gf given by $Gf(w) = \sum_{v\in T} G_v(w)f(v)$ is defined if and only if $\sum_{v\in T} |f(v)|\alpha_v < \infty$. If, in addition, f is non-negative, then Gf is admissible. In Theorem 5.2, we show that the H-potentials are precisely the admissible superharmonic functions s of the form Gf, where $f = -\Delta s$. Furthermore, we prove that every admissible superharmonic function can be written uniquely as the sum of an H-potential and a harmonic function (global Riesz decomposition). After defining the flux of an arbitrary function whose Laplacian satisfies a certain summability condition, we show (Theorem 5.3) that a function of finite flux is the sum of a harmonic function and the difference of two H-potentials.

In Section 6, we give a description of the connections between the H-potentials and the potentials developed for Markov chains by other authors.

2. *Standard and the flux on a recurrent tree*

In this section, we introduce the concept of flux of a superharmonic function outside a finite set of vertices of a recurrent tree.

Definition 2.1. Let T be a recurrent tree, $K \subset T$ a non-empty finite set. A function H harmonic off K is called a *standard* associated with K if the following is true: given any function h which is harmonic off an arbitrary finite set, there exists a function h_T harmonic on the whole space, a unique real number α and a bounded function b such that $h = \alpha H + h_T + b$.

Let T be any tree. If v is a vertex of length $n \geqslant 1$ and $[v_0, \ldots, v_n]$ is the geodesic path from e to v, let

$$\epsilon_k(v) = \frac{p(v_k, v_{k-1})}{1 - p(v_k, v_{k-1})} \text{ for } 1 \leqslant k \leqslant n-1, \quad \epsilon_0(v) = 1,$$

and define

$$H(v) = \sum_{k=0}^{n-1} \epsilon_0(v)\epsilon_1(v)\cdots\epsilon_k(v) \tag{2.1}$$

for $|v| = n \geqslant 1$, with $H(e) = 0$. Then H is non-negative and constant on siblings (that is, $H(u) = H(w)$ if $u^- = w^-$). If $|u| = 1$, then $H(u) = 1$, so that $\Delta H(e) = 1$. In [**6**, Theorems 5.1 and 5.2], we showed that H is harmonic except at e (so that H is subharmonic), and gave the following result.

Theorem 2.1. *If T is recurrent, then H is unbounded on each sector $S(v)$, and if H is unbounded on each ray, then T is recurrent.*

The special case of $T = \mathbb{Z}_+ \cup \{0\}$ is well known (see [**11**, Theorem 1]).

On the other hand, in Example 2.2 we show examples of both a transient and a recurrent tree whose associated function H is bounded only on a single ray.

If T is a homogeneous tree of degree greater than 2 with isotropic transition probability, then T is transient and H is the function Q of [**8**, p. 454].

Let us now assume that T is recurrent. The following theorem was proved in [**6**, Section 4] using a result from axiomatic potential theory in [**2**,**4**]. We shall give here a simple self-contained proof.

THEOREM 2.2. *H is a standard on T associated with $\{e\}$.*

For the proof, we introduce a family $\{H_v\}_{v\in T}$ of non-negative subharmonic functions with one-point harmonic support. On a recurrent tree, the Green function is identically infinity, and hence it has no practical use. The functions H_v will also be used to take the first step toward defining a substitute notion for the Green function, and hence for potentials.

For $v \in T$, define $\alpha_v = p([e,v])/p([v,e])$ for $v \neq e$, and $\alpha_e = 1$. Set $H_e = H$ and for each $v \in T$ define the function H_v harmonic off v with $\Delta H_v(v) = 1$ and such that $H_v - \alpha_v H$ is bounded. Let $[e,v] = [e = v_0, v_1, \ldots, v_n = v]$ and, for $k = 0, \ldots, n-1$, let $p_k = p(v_k, v_{k+1})$ and $r_{k+1} = p(v_{k+1}, v_k)$. Define

$$S_k = \begin{cases} S(v_k) - S(v_{k+1}) & \text{for } k = 0, \ldots, n-1, \\ S(v_n) & \text{for } k = n, \end{cases} \tag{2.2}$$

so that $T = S(e)$ is the disjoint union of the sets $S_0, \ldots, S_n$. Let $b_n = \alpha_v H(v)$, and

$$b_k = \begin{cases} b_n - \dfrac{1}{r_n}\left(1 + \displaystyle\sum_{m=k+1}^{n-1} \prod_{j=m}^{n-1} p_j/r_j\right) & \text{for } k = 0, \ldots, n-2, \\ b_n - \dfrac{1}{r_n} & \text{for } k = n-1. \end{cases} \tag{2.3}$$

Define H_v to be $\alpha_v H - b_k$ on S_k, $k = 0, \ldots, n$. It is straightforward to show that H_v has the following properties: $H_v(v) = 0$, H_v is harmonic everywhere except at v and $\Delta H_v(v) = 1$. Furthermore, outside the ball of radius n, H_v is strictly increasing as a function of the modulus, since H has this property. Thus, since H_v takes only finitely many values on the ball, it has a minimum which cannot be attained at a point where the function is harmonic. Hence, by the minimum principle, that minimum is attained at v where its value is 0. Thus, H_v is positive except at v.

THEOREM 2.3 [**6**, Theorem 4.3]. *For each $v \in T$, H_v is the unique function harmonic except at v, non-negative, such that $H_v(v) = 0$, $\Delta H_v(v) = 1$ and the function $b^v = \alpha_v H - H_v$ takes on finitely many values. These values are $b_0, \ldots, b_{|v|}$. In particular, $H_e = H$.*

Proof of Theorem 2.2. Let h be a function on T harmonic outside a finite set K. Let $h_T = h - \sum_{v\in K} \Delta h(v) H_v$, $\alpha = \sum_{v\in K} \Delta h(v)\alpha_v$ and $b = -\sum_{v\in K} \Delta h(v) b^v$. Then, h_T is harmonic on T and $h = h_T + \sum_{v\in K} \Delta h(v) H_v = h_T + \alpha H + b$. To prove that α is unique, assume that h can be written as $h'_T + \alpha' H + b'$, where h'_T is harmonic, b' is bounded and $\alpha' \geqslant \alpha$. Then $k = (\alpha - \alpha')H + h_T - h'_T = b' - b$ is a bounded superharmonic function, and hence a constant. Thus, $0 = \Delta k(e) = \alpha - \alpha'$, completing the proof. □

Throughout the rest of the paper, we shall fix the standard to be the function H defined by (2.1), with $K = \{e\}$, the set associated with this standard.

Example 2.1. If $p(v_k, v_{k-1})$ is a constant $r \in [1/2, 1)$ and $\epsilon = r/(1-r)$, then $H(v) = (\epsilon^{|v|} - 1)/(\epsilon - 1)$ for $r \neq 1/2$ and $H(v) = |v|$ for $r = 1/2$. (If $r < 1/2$, then T is transient.) In particular, if T is the $(1/4, 1/2)$-tree, then $H(v) = |v|$.

Example 2.2. Let T be the tree in Example 1.2. Observe that the values x_n of H at a vertex of length n must satisfy the recurrence relation $x_{n+1} + x_{n-1} = 2x_n$ if $x_n \notin \rho$ and the relation $2x_{n+1} + x_{n-1} = 3x_n$ if $x_n \in \rho$. Thus, if $v \in S(v_n) - S(v_{n+1})$, then $H(v) = 2 + (|v| - n - 2)2^{-n}$. Notice that the values of H are completely independent of the probability p on the ray ρ, even though the value of p determines whether T is transient.

A key step toward defining flux is Theorem 2.4. We begin with two lemmas.

Lemma 2.1 (Harnack's property). *Let $\{s_n\}$ be a non-decreasing sequence of functions on T which are superharmonic (respectively, harmonic) on a connected set K. Then, on K, $\lim s_n$ is either identically infinity or superharmonic (respectively, harmonic).*

Proof. Let $s(u) = \lim s_n(u) \leqslant \infty$ for $u \in K$ and assume that $s(v) < \infty$. If $w \sim v$, then $s(v) \geqslant s_n(v) \geqslant \sum_{u \sim v} p(v,u) s_n(u) \geqslant p(v,w) s_n(w) + \sum_{u \sim v, u \neq w} p(v,u) s_1(u)$ so $s_n(w)$ is bounded above by a quantity independent of n. Hence, $s(w) < \infty$. Since K is connected, s is finite on K. The superharmonicity (harmonicity) of s is immediate. $\square$

Lemma 2.2. *For each $n \in \mathbb{N}$, let $B_n = \{v \in T : |v| \leqslant n\}$ and let h_n be the solution to the Dirichlet problem on the interior of B_n with boundary values H on ∂B_n. Then $\lim_{n\to\infty} h_n(v) = \infty$ for each $v \in T$.*

Proof. Fix any positive integer N. Since H is subharmonic, by the maximum principle, it follows that $h_n \geqslant H$ on B_{n-1}, thus $h_n \geqslant h_{n-1}$ on B_{n-1} so $\{h_n\}_{n>N}$ is an increasing sequence of harmonic functions on B_N. By Harnack's property, the limit h is either identically infinity or harmonic on B_N for all N, and hence on T, and $h \geqslant H$. Assume that h is not identically infinity. Then, h is harmonic and so $h - H$ is a non-negative superharmonic function, and hence a constant c. Thus, $h = c + H$, contradicting the fact that H is not harmonic at e. $\square$

Theorem 2.4. *Let s be a function superharmonic outside a finite set in T. Then, there exist $\beta \in \mathbb{R}$ and s_T superharmonic on T such that $s = s_T + \beta H$ outside a finite set.*

Proof. Let $C = \{v \in T : \Delta s(v) > 0\}$, and define $s_1 = s - \sum_{v \in C} \Delta s(v) H_v$. Then, Δs_1 is 0 on C and equal to Δs off C, so s_1 is superharmonic on T. Pick $\alpha > \max\{\alpha_v : v \in C\}$. For $v \in C$, let k_n be the solution to the Dirichlet problem in the interior of B_n with boundary values $H_v - \alpha H$. Note that $H_v - \alpha H = (H_v - \alpha_v H) - (\alpha - \alpha_v) H$. Since $H_v - \alpha_v H$ is bounded, it follows from Lemma 2.2 that n can be chosen so that $k_n \leqslant H_v - \alpha H$ on $\partial B_{|v|+1}$. Since $k_n = H_v - \alpha H$ on ∂B_n, by the minimum principle, it follows that $k_n(u) \leqslant H_v(u) - \alpha H(u)$ for $|v| < |u| \leqslant n$. Define

$$s_v(u) = \begin{cases} k_n(u) & \text{if } |u| \leqslant n, \\ H_v(u) - \alpha H(u) & \text{if } |u| \geqslant n. \end{cases}$$

Then, s_v is superharmonic on T and $H_v = \alpha H + s_v$ outside a finite set. Let $\beta = \alpha \sum_{v \in C} \Delta s(v)$ and $s_T = s_1 + \sum_{v \in C} \Delta s(v) s_v$. Then, outside a finite set $s = s_1 + \sum_{v \in C} \Delta s(v)(\alpha H + s_v) = \beta H + s_T$. $\square$

Remark 2.1. Let s be a function superharmonic outside a finite set. When it is clear from the context that the values of s on any particular finite set are unimportant, we shall use Theorem 2.4 to change s on a finite set so that it is defined globally and superharmonic except at e.

DEFINITION 2.2. If s is defined on T and superharmonic outside a finite set, define the *flux of* s to be $\operatorname{flux}(s) = \sup A_s$, where A_s is the set of $\alpha \in \mathbb{R}$ such that $s - \alpha H$ is bounded below by a function harmonic on T. As is customary, if $A_s = \emptyset$, then the flux of s is $-\infty$.

REMARK 2.2. Using the notation of Theorem 2.4, $\operatorname{flux}(s) \leqslant \beta$ because if $\alpha > \beta$, then $s - \alpha H = s_T - (\alpha - \beta)H$ which is a non-harmonic superharmonic function; hence, it has no harmonic lower bound. Thus, $\alpha \notin A_s$, so $A_s \subset (-\infty, \beta]$ and $-\infty \leqslant \operatorname{flux}(s) < \infty$.

LEMMA 2.3. *For $i = 1, 2$, let h_i be a globally defined function which is the greatest harmonic minorant of some superharmonic function s outside a finite set K_i. Then, $h_1 - h_2$ is bounded.*

Proof. Using Theorem 1.1, without loss of generality, we may assume that $K_1 \subset K_2$. Using the definition of standard, there exists a function h_T harmonic on T and a unique real number α such that $h_2 - \alpha H - h_T$ is bounded. Fix any $v_0 \in K_1$. By Theorem 2.3, $\alpha_{v_0} H - H_{v_0}$ is also bounded, so $h_2 - (\alpha/\alpha_{v_0})H_{v_0} - h_T$ is bounded. Thus, there exists a constant c such that $-c < h_2 - (\alpha/\alpha_{v_0})H_{v_0} - h_T < c$. Therefore, we have $-c < h_2 - (\alpha/\alpha_{v_0})H_{v_0} - h_T \leqslant s - (\alpha/\alpha_{v_0})H_{v_0} - h_T$ outside K_2. By increasing c if necessary, we obtain $s \geqslant (\alpha/\alpha_{v_0})H_{v_0} + h_T - c$ on T. In particular, since $(\alpha/\alpha_{v_0})H_{v_0} + h_T - c$ is harmonic outside K_1, it follows that $(\alpha/\alpha_{v_0})H_{v_0} + h_T - c \leqslant h_1$ outside K_1. Thus, $h_2 - c < (\alpha/\alpha_{v_0})H_{v_0} + h_T \leqslant h_1 + c$ on the complement of K_1. On the other hand, h_1 is harmonic outside K_2, so $h_1 \leqslant h_2$ outside K_2. Thus, $h_2 - h_1 \geqslant 0$ outside K_2 and $h_2 - h_1 \leqslant 2c$ outside K_1. Since K_1 and K_2 are finite, $h_1 - h_2$ is bounded. □

The following is an extensive list of the properties of the flux.

THEOREM 2.5. (a) *If h is harmonic outside a finite set, then the flux of h is the unique constant α of Definition 2.1 such that $h = \alpha H + h_T + b$ with h_T harmonic on T and b bounded. In particular, $A_h = (-\infty, \alpha]$.*

(b) *If h is bounded harmonic outside a finite set K or harmonic everywhere, then the flux of h is zero. If s is superharmonic everywhere, then $\operatorname{flux}(s) \leqslant 0$.*

(c) *Let s be an admissible superharmonic function on T. If h_1 and h_2 are the greatest harmonic minorants of s outside finite sets K_1 and K_2, respectively, then the fluxes of h_1 and h_2 are equal.*

(d) *If s is a function superharmonic outside a finite set K that has a subharmonic minorant on $T \setminus K$ (in particular, if s is admissible), then its flux is equal to the flux of its greatest harmonic minorant on $T \setminus K$. Consequently, admissible superharmonic functions have finite flux.*

(e) *The flux of a non-admissible superharmonic function s is equal to $-\infty$.*

(f) *Let s be superharmonic outside a finite set, and write $s = s_T + \beta H$ outside a finite set, where s_T is superharmonic on T and β is a constant, as in Theorem 2.4. Then $\operatorname{flux}(s) = \operatorname{flux}(s_T) + \beta$.*

(g) *The set of functions that are superharmonic outside a finite set is closed under sums and multiplication by a positive number and flux is linear on that set.*

(h) *If s is superharmonic outside a finite set and $A_s \neq \emptyset$, then $\operatorname{flux}(s) \in A_s$, so that $A_s = (-\infty, \operatorname{flux}(s)]$.*

(j) *If s is superharmonic on T and $\operatorname{flux}(s) = 0$, then s is harmonic on T.*

Proof. We use Corollary 1.1, Theorem 2.4, Remark 2.2, Lemma 2.3, and the following immediate facts.

(i) There are no non-constant positive superharmonic functions, and hence no non-harmonic superharmonic function can be bounded below on T by a harmonic function.

(ii) If s is superharmonic outside a finite set, then for all $\alpha \in \mathbb{R}$ and $c > 0$, $A_{s+\alpha H} = A_s + \alpha$ and $A_{cs} = cA_s$.

(iii) (Monotonicity of the flux) If $s_1 \leqslant s_2$, with s_1, s_2 superharmonic outside a finite set, then $A_{s_1} \subset A_{s_2}$, so that $\text{flux}(s_1) \leqslant \text{flux}(s_2)$.

(iv) If h_T is harmonic on T and $h_T + \gamma H$ is bounded below, then $\gamma \geqslant 0$.

Part (a) follows from (iv). Part (b) follows from (a) and Remark 2.2. Part (c) holds, since Lemma 2.3 implies that $A_{h_1} = A_{h_2}$.

To prove (d), let h be the greatest harmonic minorant of s on $T \setminus K$. By (c), without loss of generality, we may assume that K contains e. By (a), $A_h \neq \emptyset$ and by (iii), $A_h \subset A_s$, so $A_s \neq \emptyset$. Let $\alpha \in A_s$ so that $s - \alpha H \geqslant h_T$, a harmonic function on T. Thus, $s \geqslant \alpha H + h_T$, which is harmonic off $\{e\}$. Then, on $T \setminus K$, $s \geqslant h \geqslant \alpha H + h_T$, so by (iii) and (a), $\text{flux}(s) \geqslant \text{flux}(h) \geqslant \alpha$. Since this is true for all $\alpha \in A_s$, it follows that $\text{flux}(s) = \text{flux}(h)$.

To prove (e), assume that there exists $\alpha \in A_s$. Then, $s \geqslant \alpha H + h_T$, where h_T is a function harmonic on T. Since $\alpha H + h_T$ is harmonic outside e, s is admissible.

Part (f) follows from (ii). To prove (g), let s_1 and s_2 be superharmonic outside the same finite set K, and let h_1 and h_2 be their respective greatest harmonic minorants outside K. Then, by Corollary 1.1, $h_1 + h_2$ is the greatest harmonic minorant of $s_1 + s_2$ outside K. Thus, by (d) and (a), $\text{flux}(s_1 + s_2) = \text{flux}(h_1 + h_2) = \text{flux}(h_1) + \text{flux}(h_2) = \text{flux}(s_1) + \text{flux}(s_2)$. Linearity with respect to multiplication by a positive constant follows from (ii).

Part (h) follows from (d) and (a). Part (j) follows from (h) and (i). □

In [**6**, Corollary 3.1], we proved that a superharmonic function on a transient tree is admissible if and only if it has a global harmonic minorant. This is false on recurrent trees, but we can prove the following result.

PROPOSITION 2.1. *A superharmonic function s on a recurrent tree T is admissible if and only if it has a minorant which is harmonic except at the root.*

Proof. The sufficiency is obvious. To prove the necessity, let s be an admissible superharmonic function and let α be its flux. By part (h) of Theorem 2.5, there exists a harmonic function h_T on T such that $s - \alpha H \geqslant h_T$. However, $\alpha H + h_T$ is harmonic except at e, so s has a harmonic minorant outside e. □

3. *H-potentials and proportionality on recurrent trees*

In this section, unless otherwise specified, T is a recurrent tree.

THEOREM 3.1. *Let s be superharmonic on $T \setminus \{e\}$. Let h_n be the solution to the Dirichlet problem on $B_n \setminus \{e\} = \{v \in T : 0 < |v| \leqslant n\}$ with boundary values s on $\partial B_n \cup \{e\}$. Then, for all v, $D(s)(v) = \lim_{n\to\infty} h_n(v)$ exists and on each connected component of $T \setminus \{e\}$ is either harmonic or identically $-\infty$. If s has a subharmonic minorant off e, then its greatest harmonic minorant on $T \setminus \{e\}$ is $D(s)$. Furthermore, the flux of s is 0 if and only if the greatest harmonic minorant of s on T exists and is equal to $D(s)$.*

Proof. Since s is superharmonic on $T \setminus \{e\}$, by the minimum principle, $s \geqslant h_{n+1}$ on ∂B_n, so (again by the minimum principle) $h_n \geqslant h_{n+1}$ on B_n. Thus, by Lemma 2.1, on each connected component of $T \setminus \{e\}$, $D(s)$ is either harmonic or identically $-\infty$.

Now assume that s has a subharmonic minorant t off e. By the minimum principle, $t \leqslant h_n$ on B_n. Thus, $t \leqslant D(s) \leqslant s$, which implies that $D(s)$ is harmonic. Hence, $D(s)$ is the greatest harmonic minorant of s off e.

Next, suppose that the flux of s is zero. Then, by part (d) of Theorem 2.5, the flux of $D(s)$ is zero. Since $D(s)$ is harmonic on $T \setminus \{e\}$, it is either harmonic, superharmonic or subharmonic on T, depending on the value of $\Delta D(s)$ at e. Since its flux is zero, by part (j) of Theorem 2.5, $D(s)$ must be harmonic on T. Since $D(s)$ is the greatest harmonic minorant of s on $T \setminus \{e\}$, and $D(s)(e) = s(e)$, it follows immediately that if h is any harmonic minorant of s on T, then $h \leqslant D(s)$ on T. Thus, $D(s)$ is the greatest harmonic minorant of s on T.

Conversely, if the greatest harmonic minorant of s on T exists, then the flux of s equals the flux of its greatest harmonic minorant, hence it is 0. □

DEFINITION 3.1. Let s be admissible superharmonic on T. If α is the flux of s, then $s - \alpha H$ is superharmonic off $\{e\}$ with flux zero. By Theorem 3.1 applied to $s - \alpha H$, the greatest harmonic minorant of $s - \alpha H$ on T exists and is equal to $D(s - \alpha H)$. We say that s is an *H-potential* on T if the greatest harmonic minorant of $s - \alpha H$ on T is zero. By the linearity of D, this means that $D(s) = \alpha H$.

THEOREM 3.2. *The function G_v defined by $G_v = H_v(e) - H_v$ is an H-potential.*

Proof. Let $s = G_v - \mathrm{flux}(G_v)H$, which is superharmonic off e. By definition, G_v is harmonic everywhere except at v, $G_v(e) = 0$ and $\Delta G_v(v) = -1$. So G_v is superharmonic on T, and by Theorem 2.3 and the linearity of the flux, $s = G_v + \alpha_v H = b^v + H_v(e) = b^v - b_0$. From equation (2.3), we see that $b_0 < b_1 < \cdots < b_n$. Thus, the minimum value of s is 0, which is attained on the entire set S_0 defined in (2.2). By Theorem 1.1, s has a greatest harmonic minorant h on $T \setminus S_0$, with $0 \leqslant h \leqslant s$. Extending h to be 0 on S_0, we obtain a harmonic function except possibly at e. Since $h(e) = 0$ and h is a bounded subharmonic function on T, h must be identically 0. Now observe that the harmonic minorants of s on the connected components of $T \setminus \{e\}$ are completely independent of one another. Thus, the greatest harmonic minorant outside the root of s is 0, proving that G_v is an H-potential. □

For vertices v and w, denote by $v \wedge w$ the vertex of least modulus along the geodesic path joining v and w.

DEFINITION 3.2. We define the H-Green function G on $T \times T$ by

$$G(w, v) = G_v(w) = H_v(e) - H_v(w) = -\alpha_v H(w) + b^v_{|v \wedge w|} - b^v_0,$$

where for $k = 0, \ldots, |v|$, the numbers b^v_k are the constants b_k defined in (2.3). Here, the superscript is used to emphasize the dependence on v.

If f is a function on T, then we define the function Gf by

$$Gf(w) = \sum_{v \in T} G(w, v) f(v),$$

provided this series converges absolutely for all $w \in T$.

We shall study G and such functions in detail in Section 5. In particular, we shall show that all H-potentials are of the form Gf.

PROPOSITION 3.1. *Any two H-potentials with the same one-point harmonic support on a recurrent tree T are proportional.*

Proof. Let s_1 and s_2 be H-potentials with the same one-point harmonic support at v. It suffices to show that if s_1 and s_2 have the same flux α (a negative number), then $s_1 = s_2$. Let $\lambda = \Delta s_1(v)/\Delta s_2(v)$. Since s_1 and s_2 are harmonic except at v, the function $h = s_1 - \lambda s_2$ is harmonic everywhere, so its flux is zero. Thus, $0 = \operatorname{flux}(s_1) - \lambda \operatorname{flux}(s_2) = (1-\lambda)\alpha$, whence $\lambda = 1$ and $s_1 = s_2 + h$. Consequently, $\alpha H = D(s_1) = D(s_2 + h) = \alpha H + h$, so $h = 0$ and $s_1 = s_2$. □

We now use H-potentials on recurrent trees (respectively, potentials with one-point harmonic support on transient trees) to show that it is always possible to find a function on a tree with arbitrarily prescribed Laplacian.

THEOREM 3.3. *Let T be an infinite tree, possibly transient. Given any function μ on T, the equation $\Delta\varphi = -\mu$ has a solution. In particular, if $\mu \geqslant 0$, then any solution φ is superharmonic on T.*

Proof. If T is a recurrent tree, let G_v be the H-potential of Theorem 3.2. If T is a transient tree, let G_v be the potential $G_v(w) = G(w, v)$, where G is the Green function on T. In both cases, we have $\Delta G_v = -\delta_v$, where δ_v is the Dirac δ function at v. Then the function $s_n = \sum_{|v|=n} \mu(v) G_v$ is harmonic in the interior of the ball B_n. Let h_n be a harmonic extension of $s_n|B_n$ to all of T. Then $(s_n - h_n)|B_n = 0$, so the function $\varphi = \sum_{n=0}^{\infty}(s_n - h_n)$ is a finite sum at any vertex, and hence is defined on T. Then

$$\Delta\varphi = \sum_{n=0}^{\infty} \Delta s_n = \sum_{n=0}^{\infty} \sum_{|v|=n} \mu(v)(-\delta_v) = -\sum_{v\in T} \mu(v)\delta_v = -\mu.$$

4. *Superharmonic functions and calculation of the flux on a recurrent tree*

The main result of this section is Theorem 4.2 in which we relate the flux given by Definition 2.2 to the limiting value of a more familiar and classical form.

Let $[v, w]$ be an edge with $v = w^-$. Then

$$\alpha_w = \frac{p([e,v])p(v,w)}{p(w,v)p([v,e])} = \alpha_v \frac{p(v,w)}{p(w,v)},$$

that is,

$$\alpha_v p(v,w) = \alpha_w p(w,v). \tag{4.1}$$

This is the condition that allows us to view T as a *reversible Markov chain* (electric network). The quantity $c([v,w]) = \alpha_w p(w,v)$, called the *conductance* of the edge $[v,w]$, is a measure of the amount of current flow.

In the transient case, $\alpha_v G(v,w) = \alpha_w G(w,v)$, where G is the ordinary Green function. In the recurrent case, however, this reversibility does not hold for the H-Green function as can be seen by considering Example 2.1 for $r = 1/2$. In this case, $H_v(w) = d(v,w)$, α_v is constant and $G(v,w) = |w| - d(v,w)$.

For each $v \in T$, $\alpha_v = \sum_{w\sim v} \alpha_v p(v,w) = \sum_{w\sim v} \alpha_w p(w,v)$, and so $v \mapsto \alpha_v$ is a positive regular measure on T in the sense of [**15**]. It is the unique positive regular measure whose value at e is 1.

Following [**24**, p. 14], we let E be the set of edges of the tree. Since every edge γ has the form $\gamma = [v^-, v]$ for a unique vertex $v \neq e$, we may set $\gamma_+ = v, \gamma_- = v^-$. Put inner products on the sets of functions on T and on E as follows: if f, g are functions on T and F, G are functions on

E, then let $\langle f, g\rangle = \sum_v \alpha_v f(v)g(v)$ and $\langle F, G\rangle = \sum_\gamma (1/c(\gamma))F(\gamma)G(\gamma)$, if these series converge absolutely. Let $\ell^2_\alpha(T)$ and $\ell^2_{1/c}(E)$ be the corresponding Hilbert spaces. For any function f on T, let ∇f be the function on E given by $\nabla f(\gamma) = (f(\gamma_+) - f(\gamma_-))c(\gamma)$. A direct calculation shows that with respect to the above inner products the adjoint of ∇ is given by

$$\nabla^*(F)(v) = \frac{1}{\alpha_v}\left(\sum_{\gamma_+=v} F(\gamma) - \sum_{\gamma_-=v} F(\gamma)\right).$$

DEFINITION 4.1. Fix a finite complete set of vertices K. For each $w \in \partial K$, there is a unique neighbor $\widetilde{w}$ of w with $\widetilde{w} \in \operatorname{int} K$. For a function f on K set $(\partial f/\partial n)(w) = f(w) - f(\widetilde{w})$, the normal derivative of f at w.

DEFINITION 4.2. Let $K \subset T$ be complete. Define the *Dirichlet sum* of two functions f and g on K as $\mathbf{D}_K(f, g) = \sum_{v,v^-\in \operatorname{int} K}(f(v) - f(v^-))(g(v) - g(v^-))c([v^-, v])$ provided that the sum converges absolutely. For f, g defined on T, set $\mathbf{D}(f, g) = \mathbf{D}_T(f, g) = \langle\nabla f, \nabla g\rangle$.

The *Dirichlet space* is the space $\mathcal{D}$ of all functions f on T such that $\nabla f \in \ell^2_{1/c}(E)$ or, equivalently, $\mathbf{D}(f, f) = \sum_{v\neq e}(f(v) - f(v^-))^2 c([v^-, v]) < \infty$. Define the inner product $\langle f, g\rangle_{\mathcal{D}} = \mathbf{D}(f, g) + f(e)g(e)$, whose associated norm $\| \|_{\mathcal{D}}$ (known as the *Dirichlet norm*) makes $\mathcal{D}$ a Hilbert space. (Of course, ∇^* is not the adjoint of ∇ with respect to the inner product in $\mathcal{D}$.)

It is easily shown by direct calculation (see [**24**, Lemma 2.4]) that $-\nabla^*\nabla$ is exactly the Laplace operator Δ. In particular, for any $f \in \mathcal{D}$ and $g \in \ell^2_\alpha(T)$,

$$\langle\Delta f, g\rangle = -\langle\nabla f, \nabla g\rangle = -\mathbf{D}(f, g). \tag{4.2}$$

Notice that equation (4.2) also holds if either f or g has finite support and the other is arbitrary. To see this, assume that K is a complete set and g is a function with support contained in the interior of K. Then, let $\tilde{f}$ be the function which equals $f|K$ on K and is zero outside. Then $\langle\Delta f, g\rangle = \langle\Delta\tilde{f}, g\rangle = -\langle\nabla\tilde{f}, \nabla g\rangle = -\langle\nabla f, \nabla g\rangle$.

Applying equation (4.2) to a function g times the characteristic function of $\operatorname{int} K$ and an arbitrary function f, we obtain the following result.

THEOREM 4.1 (Green's theorem). *Let K be a finite complete set of vertices. Let f, g be functions on T. Then*

$$\sum_{v\in \operatorname{int} K} \Delta f(v)g(v)\alpha_v + \mathbf{D}_K(f, g) = \sum_{w\in\partial K} \frac{\partial f}{\partial n}(w)g(\tilde{w})c([\widetilde{w}, w]). \tag{4.3}$$

In particular, letting $K = B_n$ with $n \to \infty$ and $f = g$ harmonic on T, we get

$$\|f\|^2_{\mathcal{D}} = \lim_{n\to\infty} \sum_{|w|=n} \frac{\partial f}{\partial n}(w)f(\widetilde{w})c([w, w^-]) + f(e)^2.$$

PROPOSITION 4.1. *If $f \in \ell^2_\alpha(T)$, then $\|f\|_{\mathcal{D}} \leqslant 2\|f\|_\alpha + |f(e)|$ so $f \in \mathcal{D}$.*

Proof. $\langle\nabla f, \nabla f\rangle \leqslant \sum_\gamma |f(\gamma_+)|^2 c(\gamma) + \sum_\gamma |f(\gamma_-)|^2 c(\gamma) + 2\sum_\gamma |f(\gamma_+)||f(\gamma_-)|c(\gamma)$, and each of these sums is at most $\|f\|^2_\alpha$. □

THEOREM 4.2. *If s is superharmonic outside a finite set, then*

$$\operatorname{flux}(s) = \sum_{v\in T} \Delta s(v)\alpha_v = \lim_{n\to\infty} \sum_{|v|=n} \frac{\partial s}{\partial n} c([v, v^-]). \tag{4.4}$$

Thus, s has a harmonic minorant outside a finite set if and only if the above sum is finite.

Proof. Let s be a function on T which is superharmonic outside a finite set. For each $n \in \mathbb{N}$, let $s_n = s + \sum_{|v|\leqslant n}(-\Delta s(v))H_v$. Then, $\{s_n\}$ is an increasing sequence of superharmonic functions for n sufficiently large, and each s_n is harmonic on B_n. By part (b) of Theorem 2.5, $\operatorname{flux}(s_n) \leqslant 0$. By the monotonicity and the linearity of the flux, it follows that $\sum_{|v|\leqslant n} \Delta s(v)\alpha_v \geqslant \operatorname{flux}(s)$. Since $\Delta s(v) \leqslant 0$ for all $|v|$ sufficiently large, it follows that if s is admissible, then $\sum_{v\in T} \Delta s(v)\alpha_v$ converges and $\operatorname{flux}(s) \leqslant \sum_{v\in T} \Delta s(v)\alpha_v$. Of course, if s is not admissible, then this holds trivially, since $\operatorname{flux}(s) = -\infty$. Since the second equality in equation (4.4) follows from Theorem 4.1, we will be done if we prove that

$$\sum_{v\in T} \Delta s(v)\alpha_v \leqslant \operatorname{flux}(s). \tag{4.5}$$

For any function of finite support, the first and third terms of equation (4.4) are zero. By Theorem 4.1 with $g = 1$, the second term is zero as well. Since by Theorem 2.4, $s = s_T + \beta H$ outside a finite set, where s_T is superharmonic on T and β is a constant, it follows that we can replace s by $s_T + \beta_H$ in equation (4.4). However, by Theorem 4.1 for the special case of $g = 1$ and $K = B_n$, the result holds for $s = H$, so without loss of generality we may assume that s is superharmonic on T.

For each $n \geqslant 2$, let h_n be as in Theorem 3.1. As shown there, $h_n(v) = s(v)$ for $v = e$ or $|v| = n$, $h_n(v) \leqslant s(v)$ for $0 < |v| < n$ and $D(s)(v)$ is the pointwise limit of the decreasing sequence $\{h_n(v)\}_{n=1}^{\infty}$. If s is admissible, then $D(s)$ is the greatest harmonic minorant of s on $T \setminus \{e\}$, and if s is not admissible, then $D(s) \equiv -\infty$ on at least one connected component of $T \setminus \{e\}$. In either case, by Theorem 4.1, we have

$$\sum_{|v|<n} \Delta s(v)\alpha_v = \sum_{|v|=n} (s(v) - s(v^-))\alpha_v p(v, v^-) \leqslant \sum_{|v|=n} (h_n(v) - h_n(v^-))\alpha_v p(v, v^-) = \Delta h_n(e).$$

Letting $n \to \infty$, the left side converges to $\sum_{v\in T} \Delta s(v)\alpha_v$. If s is not admissible, since $h_n(e) = s(e)$, it follows that $\lim_{n\to\infty} \Delta h_n(e) = -\infty$, so inequality (4.5) holds. If s is admissible, $\lim_{n\to\infty} \Delta h_n(e) = \Delta D(s)(e)$. Since $D(s) - \Delta D(s)(e)H$ is harmonic on T, its flux is zero, so $\operatorname{flux}(D(s)) = \Delta D(s)(e)$. By Theorem 2.5(d), we have $\sum_{v\in T} \Delta s(v)\alpha_v \leqslant \Delta D(s)(e) = \operatorname{flux}(D(s)) = \operatorname{flux}(s)$, proving inequality (4.5) in this case. This completes the proof. □

A classical theorem on Riemann surfaces states that parabolic Riemann surfaces have the property that there are no non-constant harmonic functions in the Dirichlet space (cf. [**21**, p. 162]). On recurrent trees and, more generally, in the setting of infinite networks, a stronger result holds: there are no non-constant superharmonic functions in the Dirichlet space (cf. [**22**, Theorem 3.34]).

5. *H-Green function and the Riesz decomposition theorem*

We now study in more detail the properties of the H-Green function G of Definition 3.2. Notice that it takes on positive and negative values. Indeed, $G(e, v) = 0$, $G(v, v) = H_v(e) > 0$ for $v \neq e$, and since H_v is not upper bounded, for each vertex v there exists a vertex w such that $G(w, v) < 0$.

Let $\ell^1_\alpha(T)$ be the Banach space determined by the norm $\|f\|_{1,\alpha} = \sum_{v\in T} |f(v)|\alpha_v$. Given a function f on T, let f_+ and f_- be the the positive and negative parts of f.

THEOREM 5.1. *Let f be a function on T.*

(i) *Then Gf is defined if and only if $f \in \ell^1_\alpha(T)$.*

(ii) *If $f \in \ell^1_\alpha(T)$, then Gf_+ and Gf_- are admissible superharmonic functions such that $\Delta Gf_+ = -f_+$ and $\Delta Gf_- = -f_-$, and their fluxes are $-\|f_+\|_{1,\alpha}$ and $-\|f_-\|_{1,\alpha}$. In particular, $\Delta Gf = -f$ and $\mathrm{flux}(Gf) = -\sum_{v\in T} \alpha_v f(v)$. Furthermore, if f is non-negative, then Gf is admissible.*

Proof. Observe that Gf is defined if and only if Gf_+ and Gf_- are and $Gf = Gf_+ - Gf_-$. Thus, without loss of generality, we may assume that f is non-negative.

(i) Since $G_v(e) = 0$, it follows that $Gf(e) = 0$ as well. For each $v, w \in T$ with $w \neq e$, $H_v(w) = \alpha_v H(w) - b^v_k(w)$, where $k = |v \wedge w|$. Let $v, w \in T$ such that $n = |v| > |w|$. By (2.3), for $1 \leqslant k \leqslant |v \wedge w|$ we have

$$b^v_k - b^v_0 = \frac{1}{r^v_n} \sum_{m=1}^{k} \prod_{j=m}^{n-1} \frac{p^v_j}{r^v_j} = \frac{\alpha_v}{p^v_0} \left[1 + \sum_{m=1}^{k-1} \prod_{j=1}^{m} \frac{r^v_j}{p^v_j}\right] = \frac{\alpha_v}{p^v_0} \left[1 + \sum_{m=1}^{k-1} \prod_{j=1}^{m} \frac{r^w_j}{p^w_j}\right],$$

so

$$\frac{\alpha_v}{p^v_0} \leqslant b^v_k - b^v_0 \leqslant \frac{\alpha_v}{p_0}(1 + c_w), \tag{5.1}$$

where $p_0 = \min\{p(e,u) : |u| = 1\}$ and $c_w = \sum_{m=1}^{|w|-1} \prod_{j=1}^{m} (r^w_j / p^w_j)$.

Suppose first that $f \notin \ell^1_\alpha(T)$. Then $\sum_{v\in S(w)} \alpha_v f(v) = \infty$ for some w with $|w| = 1$. Since $H(w) = 1$, for each $v \in S(w)$, $p^v_0 = p^w_0$, $v \wedge w = w$, and $G_v(w) = -\alpha_v H(w) + (b^v_1 - b^v_0) = -\alpha_v + \alpha_v/p^w_0 = \alpha_v(1/p^w_0 - 1)$, so $\sum_{v\in S(w)} G(w,v)f(v) = (1/p^w_0 - 1)\sum_{v\in S(w)} \alpha_v f(v) = \infty$, hence Gf is not defined.

Suppose now that $f \in \ell^1_\alpha(T)$ and fix $w \in T$, $w \neq e$. Then

$$\sum_{v\in T} G_v(w)f(v) = -H(w)\sum_{v\in T} \alpha_v f(v) + \sum_{|v|\leqslant|w|} \left(b^v_{|v\wedge w|} - b^v_0\right) f(v) + \sum_{|v|>|w|} \left(b^v_{|v\wedge w|} - b^v_0\right) f(v).$$

The first sum on the right-hand side is finite by assumption, the second consists of finitely many terms, and hence is finite; the third is finite since, by (5.1), it is a non-negative sum dominated by $(1/p_0)(1 + c_w)\sum_{v\in T} \alpha_v f(v)$. We have thus shown that Gf is finite on T if and only if $\sum_{v\in T} \alpha_v f(v)$ is finite.

(ii) Since each term in the sum defining Gf is superharmonic, it follows that Gf is superharmonic on T and $\Delta Gf(w) = -f(w)$. The value of $\mathrm{flux}(Gf)$ and the fact that Gf is admissible comes from Theorem 4.2. □

THEOREM 5.2 (Global Riesz decomposition theorem). *Every admissible superharmonic function can be written uniquely as the sum of an H-potential and a harmonic function on T. The H-potentials are precisely the functions of the form Gf, where f is a non-negative function in $\ell^1_\alpha(T)$. The flux of an admissible superharmonic function s is equal to $-\|\Delta s\|_{1,\alpha}$.*

Proof. We begin by showing that if $f \in \ell^1_\alpha(T)$ and $f \geqslant 0$, then Gf is an H-potential. By Theorem 5.1, Gf is admissible. Let $s_0 = Gf - \mathrm{flux}(Gf)H$. Since $b^v_k - b^v_0 \geqslant 0$ for all k, $s_0(w) = \sum_{v\in T}(b^v_{|v\wedge w|} - b^v_0)f(v) \geqslant 0$. Thus, 0 is a harmonic minorant of s_0. Since $\mathrm{flux}(s_0) = 0$, $D(s_0)$ is the greatest harmonic minorant of s_0 on T, by Theorem 3.1. So $D(s_0) \geqslant 0$, and thus must be a constant. Since $0 = s_0(e) = D(s_0)(e)$, $D(s_0)$ must be identically 0. This proves that Gf is an H-potential.

Let us now assume that s is an admissible superharmonic function on T. Then, the function $f = -\Delta s$ is non-negative and by Theorem 4.2, $\sum_{v\in T} -\Delta s(v)\alpha_v$ is finite, so $f \in \ell^1_\alpha(T)$ and Gf is admissible. By the previous part, Gf is an H-potential. Also $h = s - Gf$ is harmonic on T. Thus, $s = Gf + h$, where f is given uniquely as $-\Delta s$. Hence, s can be written uniquely as the sum of a harmonic function and a function of the form Gf with f non-negative in $\ell^1_\alpha(T)$. Moreover, $\mathrm{flux}(s) = \mathrm{flux}(Gf) = -\sum_{v\in T} f(v)\alpha_v = -\|f\|_{1,\alpha}$.

Finally, if s is an H-potential, then it is an admissible superharmonic function, so it can be written as $Gf + h$, where h is harmonic on T. Since $D(s - \mathrm{flux}(s)H) = D(Gf - \mathrm{flux}(Gf)H + h) = h$ and s is an H-potential, h must be 0, and so $s = Gf$. Thus, the H-potentials are precisely the admissible superharmonic functions of the form Gf, with f non-negative in $\ell^1_\alpha(T)$. □

Definition 5.1. Let g be an arbitrary function on T. We say that g has *finite flux* at ∞ (or simply, that g has finite flux) if $\sum_{v\in T} \alpha_v \Delta g(v)$ converges absolutely, in which case we define $\mathrm{flux}(g) = \sum_{v\in T} \alpha_v \Delta g(v) = \|(\Delta g)_+\|_{1,\alpha} - \|(\Delta g)_-\|_{1,\alpha}$.

Theorem 5.3. *A function has finite flux if and only if it can be written uniquely as the sum of a global harmonic function and the difference of two H-potentials with disjoint harmonic supports. Furthermore, the flux of a non-negative function with finite flux is non-negative.*

Proof. Assume that g has finite flux. The functions $f_1(v) = (\Delta g)_+$ and $f_2 = (\Delta g)_- = f_1 - \Delta g$ are in $\ell^1_\alpha(T)$, have disjoint supports and are non-negative. By Theorem 5.2, Gf_1 and Gf_2 are H-potentials with disjoint harmonic supports. By Theorem 5.1, we have $\Delta(Gf_1 - Gf_2) = -f_1 + f_2 = -\Delta g$, so the function $g + Gf_1 - Gf_2$ is harmonic on T, proving the existence of the stated decomposition. The converse follows from Theorem 5.2, and the linearity of finite flux and the uniqueness follows by applying the Laplacian and using Theorem 5.1.

Now let g be a non-negative function with finite flux. By the first part, $g = p_1 - p_2 + h$, where p_1 and p_2 are H-potentials and h is harmonic on T. Since $p_1 + h$ and p_2 are admissible superharmonic functions and $p_1 + h \geqslant p_2$, by the monotonicity of the flux, $\mathrm{flux}(p_1 + h) \geqslant \mathrm{flux}(p_2)$ so by linearity $\mathrm{flux}(g) = \mathrm{flux}(p_1 + h) - \mathrm{flux}(p_2) \geqslant 0$. □

6. *Relation to the Markov chain literature*

Various authors have studied the theory of recurrent Markov chains (see [**11**–**17**, **19**, **23**]). In these works, the notion of potential and potential kernel appear and so it is appropriate to ask why we have constructed yet another such notion. We shall look at some of these other treatments to see their relation to our work.

In [**12**–**15**], the authors use the term *potential* on a recurrent Markov chain (P, T) in various ways. A function f on T is a column vector (with entries parametrized by the states) and a measure μ on T is a row vector. The function identically 1 is denoted by $\mathbf{1}$. Given a positive regular measure $\boldsymbol{\alpha}$ on T (that is, a positive measure such that $\boldsymbol{\alpha}P = P$), a function f is called a (*right*) charge if $g = \sum_{n=0}^{\infty} P^n f$ is finite on T, and a measure μ is called a (*left*) charge if $\nu = \sum_{n=0}^{\infty} \mu P^n$ is finite. The function g and the measure ν are called the potentials of the respective charges and the quantities $\boldsymbol{\alpha}f (= \sum_v f(v)\alpha_v)$ and $\mu\mathbf{1}$ are called the total charges of f and μ, respectively. Observe that the total charge of f is $-\mathrm{flux}(g)$ as in Definition 5.1. Since in the recurrent case $\sum P^n$ is identically ∞, there are no non-negative non-zero charges. In fact, f and μ are charges if and only if $\boldsymbol{\alpha}f = 0$ and $\mu\mathbf{1} = 0$, respectively.

The authors define the potential kernels

$$\mathcal{C}(u,v)=\sum_{n=0}^{\infty}\left[P^n(v,v)-P^n(u,v)\right],\quad \mathcal{G}(u,v)=\sum_{n=0}^{\infty}\left[P^n(u,u)\frac{\alpha_v}{\alpha_u}-P^n(u,v)\right],$$

$$\mathcal{K}(u,v)=\sum_{n=0}^{\infty}\left[P^n(e,e)\alpha_v-P^n(u,v)\right]$$

and ${}^{e}N(u,v)$ which is the expected number of visits to v before hitting e beginning at u. The latter is always finite and is zero if u or v is e. All the above series clearly converge in the transient case since $\sum_{n=0}^{\infty}P^n(u,v)$ is finite for all $u,v\in T$. In the recurrent case, the chains for which $\mathcal{C}$ and $\mathcal{G}$ exist (equivalently, for which $\mathcal{K}$ exists) are the normal chains.

All potentials are described by means of the kernel ${}^{e}N$ as $g={}^{e}Nf+g(e)\mathbf{1}$ and $\Delta g=-f$. Since f is never non-negative, potentials are never superharmonic. In particular, potentials are never H-potentials. The potentials g (or ν) whose charges f (or μ, respectively) have finite support are of the form $\mathcal{G}f$ (or $\mu\mathcal{C}$).

The reverse chain $(\hat{P},T)$ is defined by $\hat{P}(u,v)=(\alpha_v/\alpha_u)P(v,u)$. A chain is called reversible if $P=\hat{P}$. As observed in Section 4, trees are always reversible. Matrices defined with respect to $(\hat{P},T)$ are written with a circumflex (for example, $\hat{\mathcal{G}}$, $\hat{\mathcal{C}}$). The dual of a square matrix E, a function f and a measure μ are defined by $(\operatorname{dual}E)_{u,v}=(\alpha_v/\alpha_u)E_{v,u}$, $(\operatorname{dual}f)_u=\alpha_u f_u$, $(\operatorname{dual}\mu)_u=(1/\alpha_u)\mu_u$. The dual of a right charge is a left charge, $\operatorname{dual}\mathcal{C}=\hat{\mathcal{G}}$ and $\operatorname{dual}\mathcal{G}=\hat{\mathcal{C}}$. Thus, theorems about right charges yield corresponding theorems about left charges using duality. In order to unify the theory, the authors introduce $\mathcal{K}$ which has the property that $\operatorname{dual}\mathcal{K}=\hat{\mathcal{K}}$. The potential of a right charge f (or left charge μ) is $\mathcal{K}f$ (or $\mu\mathcal{K}$, respectively) provided that the charges are finitely supported. In addition to charges, $\mathcal{K}f$ is defined for any function f of finite support (in particular, for f non-negative).

A *pure potential* is a function of the form $-\mathcal{K}f$ for some non-negative function f of finite support. Pure potentials are used to develop various potential principles, for example, balayage and capacity. Pure potentials are superharmonic but their harmonic supports are always finite. By contrast, the harmonic support of H-potentials may be infinite. In fact, by the global Riesz decomposition theorem, given any subset of T, there exists an H-potential having that set as harmonic support.

Of the above integral kernels, only the kernel ${}^{e}N$ is used in a Riesz-type representation theorem (see [**14**; **15**, Theorems 11–17]) to characterize the non-negative functions h (or more generally, those with a global harmonic minorant) such that $\Delta h\in\ell^1_\alpha(T)$. This compares with Theorem 5.3 above in which we characterize all functions h such that $\Delta h\in\ell^1_\alpha(T)$. In particular, since there are no non-constant non-negative superharmonic functions, their result does not cover any admissible superharmonic function (except a constant), whereas Theorem 5.2 characterizes it uniquely as the sum of an H-potential and a global harmonic function.

In the case when T is a group and P is group invariant, that is, $p(gu,gv)=p(u,v)$ for all $g\in T$, the kernel $\mathcal{C}$ is the potential kernel A studied in [**16, 17, 23**].

In a different direction, Orey [**19**] develops a theory of potential kernels, as opposed to a theory of potentials based on a specific kernel as in the Kemeny–Snell work and the present paper. He develops the concepts of balayage and capacity for such kernels. Again, as in [**12**–**15**], potentials must have finite harmonic support.

In the simple case of the integers viewed as a homogeneous tree of degree 2 with $p(v,w)=1/2$ if and only if $w=v\pm1$, the three kernels $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{K}$ are all the same. They correspond to the kernel $(u,v)\mapsto H_v(u)$ (which turns out to be $|u-v|$). The pure potentials and the H-potentials of finite harmonic support differ by a constant. We suspect that these functions agree in this case only because G_e is constant on siblings and its value at each vertex v depends only on the values of p on the ball of radius $|v|$, while the function $v \mapsto \mathcal{C}(e, v)$ depends on the probability distribution on the whole tree.

In conclusion, the H-potentials introduced in this paper are different from previous definitions of potentials and serve completely different purposes.

References

1. V. Anandam, 'Espaces harmoniques sans potentiel positif', *Ann. Inst. Fourier* (*Grenoble*) 22 (1972) no. 4, 97–160.

2. V. Anandam, 'Harmonic spaces with positive potentials and nonconstant harmonic functions', *Rend. Circ. Mat. Palermo* (2) 21 (1972) 149–167.

3. V. Anandam, 'Admissible subharmonic functions and associated measures', *J. London Math. Soc.* (2) 19 (1979) 65–78.

4. V. Anandam, 'Superharmonic functions and the flux', *Bull. Sci. Math.* 106 (1982) 85–92.

5. V. Anandam and M. Brelot, 'On a generalized logarithmic kernel and its potentials', *Ann. Polon. Math.* 39 (1981) 15–29.

6. I. Bajunaid, J. M. Cohen, F. Colonna and D. Singman, 'Trees as Brelot spaces', *Adv. in Appl. Math.* 30 (2003) 706–745.

7. I. Bajunaid, J. M. Cohen, F. Colonna and D. Singman, 'Function series, Catalan numbers, and random walks on trees', *Amer. Math. Monthly* 112 (2005) 765–785.

8. C. A. Berenstein, E. Casadio Tarabusi, J. M. Cohen and M. A. Picardello, 'Integral geometry on trees', *Amer. J. Math.* 113 (1991) 441–470.

9. P. Cartier, 'Fonctions harmoniques sur un arbre', *Sympos. Math.* 9 (1972) 203–270.

10. J. M. Cohen, F. Colonna, K. GowriSankaran and D. Singman, 'Polyharmonic functions on trees', *Amer. J. Math.* 124 (2002) 999–1043.

11. S. Karlin and J. McGregor, 'The classification of birth and death processes', *Trans. Amer. Math. Soc.* 86 (1957) 366–400.

12. J. G. Kemeny and J. L. Snell, 'Potentials for denumerable Markov chains', *J. Math. Anal. Appl.* 3 (1961) 196–260.

13. J. G. Kemeny and J. L. Snell, 'A new potential operator for recurrent Markov chains', *J. London Math. Soc.* 38 (1963) 359–371.

14. J. G. Kemeny and J. L. Snell, 'Boundary theory for recurrent Markov chains', *Trans. Amer. Math. Soc.* 106 (1963) 495–520.

15. J. G. Kemeny, J. L. Snell and A. W. Knapp, 'Denumerable Markov chains', Graduate Texts in Mathematics 40 (Springer, New York, 1976).

16. H. Kesten, 'Martin boundary of recurrent random walks on countable groups', Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability (University of California Press, Berkeley, 2 1967) 51–74.

17. H. Kesten and F. Spitzer, 'Random walk on countably infinite Abelian groups', *Acta Math.* 114 (1965) 237–265.

18. A. Korányi, M. A. Picardello and M. Taibleson, 'Hardy spaces on nonhomogeneous trees', *Symposia Mathematica XXIX*, Cortona, 1984 (Academic Press, New York, 1987) 255–265.

19. S. Orey, 'Potential kernels for recurrent Markov chains', *J. Math. Anal. Appl.* 8 (1964) 104–132.

20. M. A. Picardello and W. Woess, 'Finite truncations of random walks on trees', Appendix to A. Korányi, M. A. Picardello and M. Taibleson, 'Hardy spaces on nonhomogenous trees', *Symposia Mathematica XXIX*, Cortona, 1984 (Academic Press, New York, 1987) 255–265.

21. L. Sario and M. Nakai, *Classification theory of Riemann surfaces* (Springer, New York, 1970).

22. P. M. Soardi, *Potential theory on infinite networks*, Lecture Notes in Mathematics 1590 (Springer, Berlin-Heidelberg, 1994).

23. F. Spitzer, *Principles of random walk*, 2nd edn (Springer, New York, 1976).

24. W. Woess, *Random walks on infinite graphs and groups* (Cambridge University Press, Cambridge, 2000).

Joel M. Cohen
University of Maryland
College Park
MD 20742
USA

jcohen@umd.edu

Flavia Colonna and David Singman
George Mason University
Fairfax
VA 22030
USA

fcolonna@gmu.edu
dsingman@gmu.edu