A GLOBAL RIGHT INVERSE OF THE LAPLACE OPERATOR ON TREES WITH A GREEN FUNCTION

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ABSTRACT. Let T be a tree with nearest-neighbor transition probabilities giving rise to a transient potential theory. Let G denote the associated Green function and Δ the Laplace operator. For $f : T \to \mathbb{R}$ satisfying a suitable growth condition, $Gf(u) = \sum_{v \in T} G(u, v)f(v)$ is defined and $\Delta(Gf) = -f$. In this work, we construct a substitute function $Q : T \times T \to \mathbb{R}$ for G such that the mapping $Qf : u \mapsto \sum_{v \in T} Q(u, v)f(v)$ is defined for all functions f on T and which satisfies $\Delta(Qf) = -f$. We then give an explicit formula for Q in the case when T is homogeneous and the transition probability on it is isotropic.

1. INTRODUCTION

In recent years, much research in classical functional analysis and potential theory has been conducted on discrete structures. One such environment popularized by the seminal work of Pierre Cartier in [2] is the tree setting. Some references of recent research of classical problems on trees include [1], [3], [4], [5], [6], [7], [8], and [9].

In this paper, where we focus our attention on the tree setting, motivated by the interest in inverting the Laplace operator Δ, defined as the averaging operator minus the identity I, we investigate how to obtain a right inverse acting on functions on a tree endowed with a Green function G that are unrestricted. Specifically, it is well known that if f is any function such that Gf is well defined, then $\Delta(Gf) = -f$. However, there are plenty of examples of functions f on the tree for which Gf is undefined. The objective of this work is to determine a substitute operator Q, acting on the set of *all* functions on the tree, such that $\Delta Q = -I$.

After giving in Section 2 the needed definitions, notation and background on trees, in Section 3 we provide the construction of such an operator Q.

In Section 4, we give an explicit formula for the operator Q in the case when the tree is homogeneous and the nearest-neighbor transition probability on it is isotropic.

2. PRELIMINARIES

A **tree** is a connected graph with no loops, which, as a set, we identify with the collection of its vertices. Two vertices v and w of a tree are called **neighbors** if there is an edge connecting them, in which case we use the notation $v \sim w$. A vertex v is called **terminal** if it has only one neighbor. A **path** is a finite or infinite sequence of vertices $[v_0, v_1, \dots]$ such that $v_k \sim v_{k+1}$ for all k. A **geodesic path** $[v_0, v_1, \dots]$ is a path such that $v_{k-1} \neq v_{k+1}$ for all k. If u and v are any vertices, we denote by $[u, v]$ the unique geodesic path joining them.

Fixing a vertex e as the root of the tree, the **predecessor** u^- of a vertex u, with $u \neq e$, is the next to the last vertex of the path from e to u. An **ancestor** of u is any vertex in the path from e to u^-. By convention, we set $e^- = e$. We call **children** of a vertex v the vertices u such that $u^- = v$.

A tree T may be endowed with a metric d as follows. If u, v are vertices, $d(u, v)$ is the number of edges in the unique path from u to v. Fixing a root e, the **length** of a vertex v is defined as $|v| = d(e, v)$. We also adopt the notation $|u - v|$ for the distance between vertices u and v. There is a partial ordering $\leq$ on T defined as follows: given two vertices u and v, define $u \leq v$, if $u \in [e, v]$. Given a vertex v, we define the sector determined by v to be the set $S(v)$ of vertices u such that $v \leq u$.

The **boundary** Ω of T is the set of equivalence classes of infinite geodesic paths under the relation $\simeq$ defined by the shift, $[v_0, v_1, \dots] \simeq [v_1, v_2, \dots]$, together with the set of terminal vertices. For any vertex u,

we denote by $[u, \omega)$ the (unique) path starting at u in the class ω. Then Ω can be identified with the set of paths starting at u. If $\omega \in \Omega$ and $v \in T$, define $v \leq \omega$, if $v \in [e, \omega)$, that is, there exists a geodesic path $\gamma = [\omega_0, \omega_1, \dots]$ such that ω is the equivalence class of γ and $v = \omega_k$ for some nonnegative integer k.

Observe that, Ω is a compact space under the topology generated by the sets

$$I_v^u = \{\omega \in \Omega : v \in [u, \omega)\},$$

which we call **intervals**. Clearly, $\Omega = I_u^u$ for any $u \in T$.

Set $I_v = I_v^e$, for each $v \in T$. If $[v_0, v_1, \dots]$ is an infinite path starting at e, then

$$I_{v_j} - I_{v_{j+1}} = \coprod_{u \in A_j} I_u,$$

where A_j is the set consisting of the children of v_j unequal to v_{j+1}. Thus, for each $v \in T$ with $|v| = n$, we have

$$\Omega = \bigcup_{j=0}^{n-1} \left(I_{v_j} - I_{v_{j+1}} \right) \cup I_v,$$

a finite disjoint union of intervals.

A **distribution** is a finitely additive complex measure on finite unions of the sets I_v^u. Let us denote by $\mathcal{D}$ the space of finite-valued distributions on Ω.

Each $\omega \in \Omega$ induces an orientation on the edges of T: $[u, v]$ is **positively oriented** if $v \in [u, \omega)$. For $\omega \in \Omega$, and $u, v \in T$, define the **horocycle index** $k_\omega(u, v)$ as the number of positively oriented edges minus the number of negatively oriented edges in the path from u to v. If T is rooted at e, we use the notation $k_\omega(v)$ for $k_\omega(e, v)$.

Let T be a tree and let p be a nearest-neighbor transition probability on its vertices. As is customary, a function on a tree T will mean a function on its set of vertices. The **Laplacian** of a function $f : T \to \mathbb{C}$ is defined as

$$\Delta f(v) = \sum_{u \sim v} p(v, u) f(u) - f(v) \quad \text{for all } v \in T.$$

A function f on T is said to be **harmonic** if its Laplacian is identically zero, that is, the value at any vertex is the average of the values at its neighbors.

Let T be a tree endowed with a nearest-neighbor transition probability p. If $\gamma = [v_0, \dots, v_n]$ is a finite path, define

$$p(\gamma) = \begin{cases} \prod_{k=1}^{n-1} p(v_k, v_{k+1}) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Given vertices u and v, let $\Gamma_{u,v}$ be the set of all finite paths from u to v and let $\Gamma'_{u,v}$ be the set of all paths $[v_0, \dots, v_n]$ such that $n > 0$ and $v_j \neq v_n$ for each $j = 1, \dots, n-1$. The functions G and F are defined on $T \times T$ as follows:

$$G(u, v) = \sum_{\gamma \in \Gamma_{u,v}} p(\gamma) \text{ and } F(u, v) = \sum_{\gamma \in \Gamma'_{u,v}} p(\gamma).$$

G is called the **Green function** of T. Probabilistically, $G(u, v)$ represents the expected number of times a random walk starting at u visits v, while $F(u, v)$ is the probability that a random walk starting at u ever visits v in a finite positive time. If $G(u, v) < \infty$ for some $u, v \in T$, then $G(w, w') < \infty$ for any pair of vertices w, w'. We shall assume that the random walk on T associated to the nearest-neighbor transition probability is **transient**. This means that the Green function is finite-valued.

In [2] it is shown that G and F have the following properties:

Lemma 2.1. *For $u, v \in T$,*

$$G(u, v) = \begin{cases} F(u, v) G(v, v) & \text{if } u \neq v, \\ \frac{1}{1 - F(u, v)} & \text{if } u = v. \end{cases}$$

In addition, if w is strictly between u and v (that is, if w is in the interior of the geodesic path from u to v), then

$$F(u, v) = F(u, w)F(w, v).$$

Let T be a tree with root e. The **Poisson kernel** with respect to a boundary point ω is defined by

$$P_\omega(v) = \frac{G(v, v \wedge \omega)}{G(e, v \wedge \omega)},$$

where $v \wedge \omega$ is the unique vertex of largest length which is less than or equal to both v and ω.

The Poisson kernel satisfies the following properties analogous to those that hold in the classical case [9]:

(i) For any fixed $\omega \in \Omega$, P_ω is a harmonic function on T.

(ii) If $\mu \in \mathcal{D}$, then the the Poisson integral

$$P\mu(v) := \int_\Omega P_\omega(v) \, d\mu(\omega)$$

is well-defined and harmonic on T. Conversely, for every harmonic function f on T there is a unique $\mu \in \mathcal{D}$ such that $f = P\mu$. We call μ the representing distribution for f. It can be shown that if f is nonnegative, then its representing distribution is a nonnegative distribution and so extends to a Borel measure on Ω, which we call the **representing measure of** f.

We denote by ν the representing measure of the constant harmonic function 1.

The Green operator G is defined on functions $f : T \to \mathbb{C}$ by

$$Gf(u) := \sum_{v \in T} G(u, v)f(v),$$

provided this sum is well defined. This is the case if and only if $G|f|$ is defined, and this holds if and only if $|f|$ satisfies the growth condition $\sum_{v \in T} G(e, v)|f(v)| < \infty$. For such functions f, one has that $\Delta(Gf) = -f$.

3. Construction of the inverse of the Laplace operator

Let T be a tree with root e endowed with a transient nearest-neighbor transition probability. We are going to construct a family of right inverses, $\{Q_\omega : \omega \in \Omega\}$, to the Laplace operator Δ defined for arbitrary functions on T, parametrized by the boundary points of T. We will then use this one-parameter family to obtain a right inverse of Δ that does not depend on the choice of boundary point.

Fix $\omega \in \Omega$. For $v \in T$, let μ_v^ω be the measure on Ω defined by

$$(1) \qquad \mu_v^\omega = \begin{cases} \delta_\omega & \text{if } \omega \in I_v, \\ M_v & \text{if } \omega \notin I_v, \end{cases}$$

having denoted by δ_ω the Dirac delta distribution at ω and M_v the hitting distribution for random walks starting at v and moving forward. Note that the support of μ_v^ω is I_v in case $\omega \notin I_v$.

For $u, v \in T$, let

$$H_\omega(u, v) = \begin{cases} G(e, v)P\mu_v^\omega(u) & \text{if } v \neq e, \\ 0 & \text{if } v = e. \end{cases}$$

We define the kernel function Q_ω by

$$Q_\omega(u, v) = G(u, v) - H_\omega(u, v), \ (u, v \in T).$$

Now, for a function f on T and $u \in T$, define

$$Q_\omega f(u) = \sum_{v \in T} Q_\omega(u, v)f(v) = \sum_{v \in [e, u]} Q_\omega(u, v)f(v).$$

In this definition, the second equality is an immediate consequence of Lemma 3.1 below. Thus the sum defining $Q_\omega f$ can be viewed as extending over only finitely many nonzero terms, and so is defined on T for any function f.

Lemma 3.1. *Given $v \neq e$, let μ be a probability measure with support in I_v. Then for any vertex u such that $v \notin [e, u]$, $G(u, v) = G(e, v)P\mu(u)$.*

Proof. If vertices u and v satisfy $v \notin [e, u]$, then for any $\lambda \in I_v$ we have $u \wedge \lambda = u \wedge v$, so

$$P\mu(u) = \int_{I_v} P_\lambda(u) \, d\mu(\lambda) = \int_{I_v} \frac{G(u, u \wedge v)}{G(e, u \wedge v)} \, d\mu(\lambda) = \frac{G(u, u \wedge v)}{G(e, u \wedge v)} = \frac{G(u, v)}{G(e, v)},$$

where the last equality follows from Lemma 2.1. □

Theorem 3.1. *For each $\omega \in \Omega$ and each function f on T, $\Delta Q_\omega f = -f$.*

Proof. Let f be a function on T, fix $u \in T$ and let N be a positive integer larger than $|u|$. Let f_N be the function on T defined by

$$f_N(v) = \begin{cases} f(v) & \text{if } |v| \le N, \\ 0 & \text{if } |v| > N. \end{cases}$$

It follows that

$$Q_\omega f(u) = \sum_{0 \le |v| \le N} [G(u, v)f(v) - H_\omega(u, v)f(v)] = G f_N(u) - \sum_{0 \le |v| \le N} H_\omega(u, v)f(v).$$

But each function $u \mapsto H_\omega(u, v)$ is harmonic and $\Delta G f_N = -f_N$. So

$$\Delta Q_\omega f(u) = -f_N(u) = -f(u).$$

Hence $\Delta Q_\omega f = -f$. □

Definition 3.1. We define the kernel Q on $T \times T$, by

$$Q(u, v) = \int_\Omega Q_\omega(u, v) \, d\nu(\omega),$$

where we recall that ν denotes the representing measure of the harmonic function which is identically 1. For any function f on T, define

$$(2) \qquad Qf(u) = \sum_{v \in T} Q(u, v)f(v) = \sum_{v \in [e, u]} Q(u, v)f(v) = \int_\Omega Q_\omega f(u) \, d\nu(\omega).$$

The second equality follows since $Q_\omega(u, v) = 0$ independently of ω if $v \notin [e, u]$.

From Theorem 3.1, we deduce the following result.

Corollary 3.1. *For each function f on T, $\Delta Qf = -f$.*

Proof. Let f be a function on T and fix $u \in T$. From (2) and the linearity of integration, we get

$$\Delta Qf(u) = \Delta \left(\int_\Omega Q_\omega f(u) \, d\nu(\omega) \right) = \int_\Omega \Delta Q_\omega f(u) \, d\nu(\omega) = -f(u)\|\nu\| = -f(u),$$

as desired. □

4. Explicit formulas in the homogeneous case

By a **homogeneous tree** of degree $q + 1$ (with $q \ge 1$) we mean a tree T all of whose vertices have exactly $q+1$ neighbors. Let T be homogeneous of degree $q+1$ whose associated nearest-neighbor transition probability is isotropic, i.e. $p(v, u) = \frac{1}{q+1}$ if v and u are neighbors. Choose a vertex e as the root. Then [2] the Poisson kernel with respect to $\omega \in \Omega$ is given by

$$(3) \qquad P_\omega(v) = q^{k_\omega(v)} = q^{2|v \wedge \omega| - |v|}.$$

For $q \ge 2$, the Green function is given by

$$(4) \qquad G(u, v) = \frac{q}{q-1} q^{-|u-v|}, \quad \text{for } u, v \in T.$$

When $q = 1$, the tree does not have a Green function. Thus, from now on we shall assume $q \ge 2$.

The representing measure ν of the constant harmonic function 1 satisfies

$$\nu(S(v)) = \frac{1}{(q+1)q^{|v|-1}},$$

for each $v \ne e$.

We shall make use of the following result from [5] in the calculation below of $P\mu_v^\omega$.

Lemma 4.1. [Lemma 4.1 of [5]] *Let $v \in T$ with $v \neq o$. Let $g : \{v^-\} \cup S(v) \to \mathbb{R}$ be a radial function (meaning that its value at each $w \in S(v)$ depends only on $d(w, v)$) which in addition is harmonic on $S(v)$. Let $A = g(v^-)$, $B = g(v)$. Then for any $u \in \{v^-\} \cup S(v)$ and $j = d(u, v)$,*

$$(5) \qquad g(u) = \frac{Bq - A}{q - 1} + \frac{A - B}{q - 1} q^{-j}.$$

Conversely, for any real numbers A and B, the formula in (5) defines a function on $\{v^-\} \cup S(v)$ which is radial and harmonic at each vertex of $S(v)$.

We now state and prove our main result.

Theorem 4.1. *For each $\omega \in \Omega$, and $u, v \in T$,*

$$(6) \qquad Q_\omega(u, v) = \begin{cases} 0 & \text{if } v \neq e, u \notin S(v), \\ \frac{q}{q-1} q^{-|u-v|} & \text{if } v = e, \\ -\frac{q}{q-1} q^{-|u-v|} \left(q^{2(|u \wedge \omega| - |v|)} - 1 \right) & \text{if } v \neq e, u \in S(v), \omega \in I_v, \\ -\left(\frac{q+1}{q-1} \right) \left(1 - q^{-|u-v|} \right) & \text{if } v \neq e, u \in S(v), \omega \notin I_v, \end{cases}$$

$$(7) \qquad Q(u, v) = \begin{cases} \frac{q}{q-1} q^{-|u|} & \text{if } v = e, \\ -\left(\frac{q+1}{q-1} \right) \left(1 - q^{-|u-v|} \right) & \text{if } v \neq e \text{ and } u \in S(v), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix $\omega \in \Omega$ and let $u, v \in T$. Recall that

$$Q_\omega(u, v) = \begin{cases} G(u, v) & \text{if } v = e, \\ G(u, v) - G(e, v) P \mu_v^\omega(u) & \text{if } v \neq e, \end{cases}$$

where μ_v^ω is defined in (1). We first prove that for $v \neq e$ and $u \in S(v)$,

$$(8) \qquad P \mu_v^\omega(u) = \begin{cases} q^{2|u \wedge \omega| - |u|} & \text{if } \omega \in I_v, \\ q^{|v|-1}(q + 1 - q^{-|u-v|}) & \text{if } \omega \notin I_v. \end{cases}$$

For $\omega \in I_v$, $\mu_v^\omega = \delta_\omega$, so by (3), we have

$$P \mu_v^\omega(u) = \int_\Omega P_\lambda(u) \, d\delta_\omega(\lambda) = P_\omega(u) = q^{2|u \wedge \omega| - |u|}.$$

In the case where $\omega \notin I_v$, $\mu_v^\omega = M_v$, so for each vertex $u' \in (T \backslash S(v)) \cup \{v\}$,

$$P \mu_v^\omega(u') = \int_\Omega P_\lambda(u') \, dM_v(\lambda) = \int_{I_v} q^{2|u' \wedge \lambda| - |u'|} \, dM_v(\lambda) = q^{|u'|} \|M_v\| = q^{|u'|}.$$

In particular, $P \mu_v^\omega(v) = q^{|v|}$ and $P \mu_v^\omega(v^-) = q^{|v|-1}$. Since $P \mu_v^\omega$ is harmonic and radial on $S(v)$, from Lemma 4.1 with $A = q^{|v|-1}$ and $B = q^{|v|}$, it follows that

$$P \mu_v^\omega(u) = \frac{q^{|v|+1} - q^{|v|-1}}{q - 1} - \frac{q^{|v|} - q^{|v|-1}}{q - 1} q^{-|u-v|} = q^{|v|-1}(q + 1 - q^{-|u-v|}),$$

completing the proof of (8).

We now turn to the proof of the formulas for $Q_\omega(u, v)$. The first formula in display (6) is a consequence of Lemma 3.1, and the second formula follows from (4). For the third formula, note that since $|u - v| = |u| - |v|$,

$$Q_\omega(u, v) = G(u, v) - G(e, v) q^{2|u \wedge \omega| - |u|}$$

$$= \frac{q}{q - 1} \left(q^{-|u-v|} - q^{-|v|} q^{2|u \wedge \omega| - |u|} \right)$$

$$= \frac{q}{q - 1} \left(q^{-|u-v|} - q^{-2|v| - |u-v| + 2|u \wedge \omega|} \right)$$

$$= \frac{q}{q - 1} q^{-|u-v|} \left(1 - q^{-2|v| + 2|u \wedge \omega|} \right).$$

For the fourth formula, by (4), we have

$$Q_\omega(u,v) = G(u,v) - G(e,v)P\mu_v^\omega(u)$$

$$= \frac{q}{q-1}\left(q^{-|u-v|} - q^{-|v|}q^{|v|}\left(\frac{q+1}{q} - \frac{q^{-|u-v|}}{q}\right)\right)$$

$$= \frac{q}{q-1}\left(\left(1 + \frac{1}{q}\right)q^{-|u-v|} - \left(\frac{q+1}{q}\right)\right)$$

$$= -\left(\frac{q+1}{q-1}\right)\left(1 - q^{-|u-v|}\right).$$

Finally, we give the proof of the formulas for $Q(u,v)$. The only nontrivial case to check is the one with $v \neq e$ and $u \in S(v)$. Let $\{u_k : |v| \leq k \leq |u|\}$ denote the vertices of $[v,u]$. Then noting that

$$\nu\left(I_{u_k}\right) = \frac{1}{(q+1)q^{k-1}} \text{ and } \nu\left(I_{u_k} - I_{u_{k+1}}\right) = \left(\frac{q-1}{q+1}\right)q^{-k},$$

and using (6), we obtain

$$Q(u,v) = \int_\Omega Q_\omega(u,v)\,d\nu(\omega)$$

$$= -\left(\frac{q}{q-1}\right)q^{-|u|+|v|}\left[\sum_{k=|v|}^{|u|-1}\left(q^{2k-2|v|} - 1\right)\left(\frac{q-1}{q+1}\right)q^{-k} + q^{2(|u|-|v|)}\frac{1}{(q+1)q^{|u|-1}}\right]$$

$$- \left(\frac{q+1}{q-1}\right)\left(1 - q^{-|u|+|v|}\right)\left(1 - \frac{1}{(q+1)q^{|v|-1}}\right)$$

$$= -\left(\frac{q+1}{q-1}\right)\left(1 - q^{-|u-v|}\right),$$

where the last equality was obtained using Mathematica [10]. One can confirm independently that Q as given in (7) has the property that $\Delta Q f = -f$ for any function f on T. $\qquad\square$

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