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A GENERALIZED LITTLEWOOD THEOREM FOR WEINSTEIN POTENTIALS ON A HALFSPACE

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1. Introduction and statement of results

Let $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n > 0\}$ denote the upper halfspace in $\mathbb{R}^n$, $n \geq 2$. We view the boundary of $\mathbb{R}^n_+$ as $\mathbb{R}^{n-1}$. Let $k \in \mathbb{R}$. The Weinstein equation with parameter k is $L_k(f) = 0$ where

$$L_k(f) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} + \frac{k}{x_n}\frac{\partial f}{\partial x_n}.$$

The C^2 functions which satisfy the Weinstein equation form a Brelot harmonic space [He]. We shall refer to these solutions as L_k-harmonic functions. The L_0-harmonic functions are just the classical harmonic functions. An integral representation for all positive L_k-harmonic functions in terms of measures on $\mathbb{R}^{n-1} \cup \{\infty\}$ (when we simply use the term measure, we mean a nonnegative, regular, Borel measure) was given in [BCB1]. There, the uniqueness of such an integral was demonstrated using Choquet's theorem. The same authors have also proved that every positive L_k-harmonic function has finite non-tangential limit at (Lebesgue) almost every point in $\mathbb{R}^{n-1}$ [BCB2].

In our paper we consider the boundary behavior of L_k-potentials. We recall that L_k-superharmonic functions, following the axiomatic study in [He], are precisely those lower semicontinuous, $(-\infty, \infty]$ valued functions v that satisfy $L_k(v) \leq 0$ in the sense of distributions. The L_k-potentials (the Weinstein potentials of the title) are those positive L_k-superharmonic functions that majorise no positive L_k-harmonic function. For every $y \in \mathbb{R}^n_+$ we associate the function

$$G_k(x, y) = a_{n,k} x_n^{1-k} y_n \int_0^\pi \frac{\sin^{1-k} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n-k)/2}}\, dt \quad \text{for } k \leq 1, \tag{1}$$

and

$$G_k(x, y) = a_{n,2-k} y_n^k \int_0^\pi \frac{\sin^{k-1} t}{[|x-y|^2 + 2x_n y_n(1-\cos t)]^{(n+k-2)/2}}\, dt \quad \text{for } k \geq 1, \tag{2}$$

where

$$a_{n,k} = \frac{\Gamma\left(\frac{n-k}{2}\right)}{2\pi^{n/2}\Gamma\left(\frac{2-k}{2}\right)} \quad \text{for } k \leq 1.$$

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We note that this function is L_k-harmonic outside y [BCB1], and it tends to ∞ at y. This choice is such that $(x, y) \mapsto G_k(x, y)$ is continuous outside the diagonal and further we shall show in §3 that it has the normalization property that $L_k(G_k(\cdot, y))$ is exactly $-\delta_y$ in the sense of distribution. We call G_k the L_k-Green's function. Now the results of R.-M. Hervé [He] give us a unique integral representation with a measure corresponding to each L_k-potential

$$p(x) = \int G_k(x, y)\, d\mu_p(y)$$

for every $x \in \mathbb{R}^n_+$. We denote this potential by $G_k\mu_p$. We also prove in §3 that for any measure μ on $\mathbb{R}^n_+$, $G_k\mu$ is an L_k-potential if and only if

$$\int_{\mathbb{R}^n_+} \frac{y_n}{(1 + |y|)^{n-k}}\, d\mu(y) < \infty$$

in case $k < 1$ and a similar condition for $k \geq 1$ (see Proposition 2).

We shall now describe the construction of a non-isotropic Hausdorff measure H^τ_α on the boundary where α and τ are two positive parameters with $\tau \geq 1$. We define the pseudo-distance d^τ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ as follows:

$$d^\tau(x, y) = \sqrt{|x_1 - y_1|^{2\tau} + \sum_{j=2}^{n-1}(x_j - y_j)^2}.$$

We denote by $B^\tau(x, r)$ the open d^τ-ball of radius r with center at x. The (α, τ)-Hausdorff measure H^τ_α of a subset $E \subset \mathbb{R}^{n-1}$ is then defined as

$$H^\tau_\alpha(E) = \sup_{\epsilon>0}\left[\inf\left\{\sum_{j=1}^{\infty} r_j^\alpha : \forall j,\ r_j < \epsilon, \text{ and } \exists x_j \text{ such that } E \subset \bigcup_j B^\tau(x_j, r_j)\right\}\right].$$

Of course when $\tau = 1$, H^τ_α is the usual α-dimensional Hausdorff measure. We also note that when $\alpha = n - 2 + \frac{1}{\tau}$, the corresponding H^τ_α is a multiple of the Lebesgue measure.

We prove in §2 the following key result concerning compact sets of positive H^τ_α-measure. This result is in the spirit of a well known theorem of Frostman ([HK], page 223).

THEOREM 1. *Let $\tau \geq 1$ and $0 < \alpha \leq n - 2 + \frac{1}{\tau}$. Let E be a compact subset of $\mathbb{R}^{n-1}$. Then $H^\tau_\alpha(E) > 0$ if and only if there exists a non-trivial measure σ with support in E such that for every $x \in \mathbb{R}^{n-1}$ and every $r > 0$, $\sigma(B^\tau(x, r)) \leq r^\alpha$.*

Now, we note that the curve Γ^τ_0 consisting of the points $(t, 0, \ldots, t^\tau)$ for all $t \geq 0$ lies in the (x_1, x_n)-plane and meets the boundary with tangency τ. Let Γ^τ_x be the translate $(x, 0) + \Gamma_0^\tau$ of this curve for $x \in \mathbb{R}^{n-1}$; i.e.,

$$\Gamma_x^\tau = \{(t + x_1, x_2, \ldots, x_{n-1}, t^\tau): t \geq 0\}$$

where $x = (x_1, \ldots, x_{n-1})$.

Definition 1. The limit of a function $f(y)$, $f\colon \mathbb{R}_+^n \to [-\infty, \infty]$, when $y \to x$ along the curve Γ_x^τ is called a Γ^τ limit and it is denoted by $\lim_{y \in \Gamma_x^\tau} f(y)$. We define Γ^τ lim sup and Γ^τ lim inf of f in an analogous way.

We now state our first result concerning the behavior of L_k-potentials.

THEOREM 2. *Let $0 < \omega \leq 1$. Let τ be such that $1 \leq \tau \leq \frac{1}{\omega}$. Let μ be a measure on $\mathbb{R}_+^n$ for which $G_k\mu$ is an L_k-potential. Let μ satisfy the growth condition that*

$$\int_F y_n^\omega \, d\mu(y) < \infty \tag{3}$$

in case $k < 1$ and

$$\int_F y_n^{k-1+\omega} \, d\mu(y) < \infty \tag{4}$$

in case $k > 1$, for all Borel sets $F \subset \mathbb{R}_+^n$ such that F is bounded in $\mathbb{R}^n$. Then for all $x \in \mathbb{R}^{n-1}$ except for x in a set E such that $H_{n-2+\omega}^\tau(K) = 0$ for every compact subset K of E, we have $\lim_{\Gamma_x^\tau} G_k\mu = 0$ if $k < 1$ and $\lim_{\Gamma_x^\tau} x_n^{k-1} G_k\mu = 0$ if $k > 1$.

We remark that the growth condition on μ is significant only for bounded Borel sets F of $\mathbb{R}^n$ such that the closure of F meets the boundary $\mathbb{R}^{n-1}$.

The formulae given in [BCB1] for the integral representation for positive L_k-harmonic functions in terms of measures on $\mathbb{R}^{n-1} \cup \{\infty\}$ show that the Lebesgue measure on $\mathbb{R}^{n-1}$ corresponds to a constant function for $k < 1$, and a multiple of $1/x_n^{k-1}$ for $k > 1$. In case $k = 1$, the Lebesgue measure does not give rise to an L_k-harmonic function. In the theorem, the presence of the factor x_n^{k-1} as well as the absence of the case $k = 1$ is explained if we think of the theorem as describing the boundary behavior of the quotient of an L_k-potential and an L_k-harmonic function generated by Lebesgue measure on $\mathbb{R}^{n-1}$.

We observe that Theorem 2 is a generalisation of Littlewood's theorem. For instance, if $\omega = 1$ and $\tau = 1$ we get the Γ^1 limits which are really limits along lines. Further, the same method of proof in the case of $\tau = 1$, $\omega = 1$ gives us the direct generalisation of Littlewood's theorem, namely the potentials have perpendicular limit zero at Lebesgue almost every boundary point. Of course if $k = 0$, this is precisely Littlewood's theorem.

In the last section, we show that the exceptional sets of non-isotropic Hausdorff measure zero are the best possible as far as Theorem 2 is concerned. We shall prove the following. For brevity, we just state it for $k < 1$.

THEOREM 3. *Let $0 < \omega \leq 1$ and $1 \leq \tau \leq 1/\omega$. Suppose E is any subset of $\mathbb{R}^{n-1}$ such that $H^{\tau}_{n-2+\omega}(E) = 0$. Let $k < 1$. Then there exists an L_k-potential $p = G_k\mu$ where μ satisfies the growth condition* (3) *corresponding to ω such that for all $x \in E$,* $\limsup_{\Gamma^{\tau}_x} G_k\mu = \infty$.

In the following sections, we shall use the letter c to represent a quantity that may vary from line to line but does not depend in an important way on the parameters of interest.

2. Non-isotropic Hausdorff measure and a Frostman theorem

We now refer to the H^{τ}_{α} measure defined earlier. We start with a definition.

Definition 2. Let $r > 0$, $\tau \geq 1$ and $x \in \mathbb{R}^{n-1}$. Let

$$R^{\tau}(x,r) = \Big\{(y_1,\dots,y_{n-1}) \in \mathbb{R}^{n-1} : \\ |x_j - y_j| < \frac{r}{2},\ j = 2,\dots,n-1,\ \ |x_1 - y_1| < \frac{r^{1/\tau}}{2}\Big\}.$$

We call the set $R^{\tau}(x,r)$ the non-isotropic rectangle (or in short the τ-rectangle) centered at x and of 'length r'. We denote this length by $l(R^{\tau}(x,r))$.

We observe that such a non-isotropic rectangle is the product of $(n-2)$ intervals of length r in the $x_2,\dots,x_{n-1}$ variables and an interval of length $r^{1/\tau}$ in the x_1-variable. We now state a lemma, the proof of which is easy.

LEMMA 1. *For all $x \in \mathbb{R}^{n-1}$ and for all $r > 0$,*

$$B^{\tau}(x,r) \subset R^{\tau}(x,2^{\tau}r) \subset B^{\tau}\left(x, r\sqrt{1+(n-2)4^{\tau-1}}\right).$$

The following result is proved easily using the last lemma and an elementary argument.

LEMMA 2. *Let $E \subset \mathbb{R}^{n-1}$. Then $H^{\tau}_{\alpha}(E) > 0$ if and only if there exists $\delta > 0$ such that for all $x_j \in \mathbb{R}^{n-1}$ and all $r_j > 0$ with $E \subset \bigcup_{j=1}^{\infty} R^{\tau}(x_j,r_j)$, we have $\sum_{j=1}^{\infty} r_j^{\alpha} > \delta$.*

Proof of Theorem 1. Suppose for a compact set E, $H^{\tau}_{\alpha}(E) = 0$. If there is a non-trivial measure σ on E with the property that $\sigma(B^{\tau}(x,r)) \leq r^{\alpha}$ for all x and r, we use the countable subadditivity of σ to arrive at a contradiction as follows. Choose $\{B^{\tau}(x_i,r_i)\}$ to cover E such that $\sum r_j^{\alpha} < \sigma(E)/2$. Then

$$\sigma(E) \leq \sum_{j=1}^{\infty} \sigma(B^{\tau}(x_j,r_j)) \leq \sum r_j^{\alpha} < \sigma(E)/2.$$

For the converse, suppose E is a compact set with $H_\alpha^\tau(E) > 0$. Since H_α^τ is countably subadditive, we may assume without loss of generality that E is contained in a Euclidean rectangle of diameter less than $1/2$. First, suppose that $\tau = p/q$ is a rational number, where p and q are relatively prime. We take a left closed, right open τ-rectangle Q_0 of length l_0, that is

$$Q_0 = \left\{ x \in \mathbb{R}^{n-1}: -\frac{l_0^{1/\tau}}{2} \le x_1 - y_1 < \frac{l_0^{1/\tau}}{2}, \forall j = 2, \dots, n-1, -\frac{l_0}{2} \le x_j - y_j < \frac{l_0}{2} \right\},$$

for some $y \in \mathbb{R}^{n-1}$, such that E is enclosed in the interior of Q_0. Corresponding to each positive integer m, we shall construct a measure σ_m which lives on Q_0. We shall obtain the required measure σ as a constant multiple of a weakly convergent subsequence of σ_m. To construct these measures, let us fix a positive integer m.

For each $j = 0, \dots, m$, we define a collection $\mathcal{Q}_j^{(m)}$ of left closed, right open τ-rectangles contained in Q_0. $\mathcal{Q}_0^{(m)}$ consists of one τ-rectangle Q_0. Suppose $\mathcal{Q}_0^{(m)}, \dots, \mathcal{Q}_{j-1}^{(m)}$ are already defined. We obtain the collection $\mathcal{Q}_j^{(m)}$ by partitioning each $Q \in \mathcal{Q}_{j-1}^{(m)}$ as follows. We take the τ-rectangle Q (which is a product of intervals) and partition it into $2^{p(n-2)+q}$ sub-τ-rectangles. This is done by dividing each of the $x_2, \cdots, x_{n-1}$ intervals corresponding to Q into 2^p subintervals of equal length and the x_1-interval of Q into 2^q subintervals of equal length and taking their products. It is easy to check that partitioning in this way produces τ-rectangles.

Now let us consider the collection $\mathcal{Q}_m^{(m)}$. Let Q be a τ-rectangle in $\mathcal{Q}_m^{(m)}$ such that $E \cap Q \neq \emptyset$. We take a multiple of Lebesgue measure on Q so that the total measure of Q is $(l(Q))^\alpha$. The measure ν_1 is defined as the sum of these measures restricted to all those τ-rectangles Q which intersect E. Assume that the measures $\nu_1, \dots, \nu_{j-1}$ are constructed for $j \le m+1$. We now construct the measure ν_j. The measure ν_j is defined by prescribing it for each $Q \in \mathcal{Q}_{m-j+1}^{(m)}$. Let $Q \in \mathcal{Q}_{m-j+1}^{(m)}$. If $\nu_{j-1}(Q) \le (l(Q))^\alpha$, we set $\nu_j = \nu_{j-1}$ on Q. If, however, $\nu_{j-1}(Q) > (l(Q))^\alpha$, we set $\nu_j = \frac{(l(Q))^\alpha}{\nu_{j-1}(Q)} \nu_{j-1}$ on Q, so that in this case, $\nu_j(Q) = (l(Q))^\alpha$. Let $\sigma_m = \nu_{m+1}$.

We observe that for each $Q \in \mathcal{Q}_j^{(m)}$, $\sigma_m(Q) \le (l(Q))^\alpha$ for all $j = 0, 1, \dots, m$, and in particular, $\|\sigma_m\| \le (l(Q_0))^\alpha$. Further, for each $x \in E$, there is at least one τ-rectangle in a $\mathcal{Q}_j^{(m)}$, for some j, such that $\sigma_m(Q) = (l(Q))^\alpha$. It is clear that two τ-rectangles Q, Q' in $\bigcup_{j=1}^m \mathcal{Q}_j^{(m)}$ are either disjoint or one of them is contained in the other. Hence, containing any point $x \in E$, there is a "biggest" τ-rectangle Q_x (belonging to $\mathcal{Q}_j^{(m)}$ evidently with the smallest j value) such that $\sigma_m(Q_x) = (l(Q_x))^\alpha$. Further, a similar argument shows that the τ-rectangles in the collection $\{Q_x\}$ are pairwise disjoint. It now follows from Lemma 2 that there is a constant $\delta > 0$ depending on τ such that $\|\sigma_m\| \ge \sum \sigma_m(Q_x) = \sum (l(Q_x))^\alpha > \delta$, where we sum over a set of disjoint Q_x which cover E.

To complete the proof for the case of rational τ we note that one can choose a subsequence of σ_m which converges weakly to a measure σ on Q_0. The rest of the proof, namely to show that σ has support in E and that it satisfies the inequality $\sigma(B^\tau(x,r)) \le r^\alpha$ for every x and every r, follows on the same lines as in the (classical) case when $\tau = 1$ and as given in [HK], p. 223. We note that σ also satisfies

$$\delta \le \sigma(E) \le (l(Q_0))^\alpha. \tag{5}$$

Before we proceed to prove the theorem for irrational τ, let us make the following observations. Suppose $1 \le \tau_1 < \tau_2$. Then $d^{\tau_1}(x,y) \ge d^{\tau_2}(x,y)$ if $|x_1 - y_1| < 1$, and in particular, if $d^{\tau_1}(x,y)$ or $d^{\tau_2}(x,y) < 1$. Hence, if $r < 1$, then $B^{\tau_1}(x,r) \subset B^{\tau_2}(x,r)$ and $R^{\tau_1}(x,r) \subset R^{\tau_2}(x,r)$. It follows that if we choose a $\delta > 0$ for τ_2 as in Lemma 2, the same constant will serve the purpose of Lemma 2 for τ_1.

Now let $\tau > 1$ be any irrational number. Let $H_\alpha^\tau(E) > 0$. Let $\tau_1 < \tau_2 \ldots < \tau$ be an increasing sequence of rational numbers with limit τ. We choose a $\delta > 0$ as per the Lemma 2 for τ. Let λ_j, for each j, be the choice of the measure on E corresponding to τ_j given by the earlier part of the proof (where it was referred to as σ). This is possible since $H_\alpha^\tau(E) > 0$ implies that $H_\alpha^{\tau_j}(E) > 0$ for all j.

By (5), we conclude that the measures λ_j on E verify

$$\delta \le \lambda_j(E) \le (l(Q_0))^\alpha, \tag{6}$$

hence some subsequence of λ_j will converge weakly to a measure σ on E. By relabelling, if necessary, we assume that $\lambda_j \to \sigma$ weakly. It is clear from (6) that σ is non-trivial.

Let $r < 1$ and fix a positive integer, l. Then we have $\lambda_j(B^{\tau_l}(x,r)) \le \lambda_j(B^{\tau_j}(x,r)) \le r^\alpha$ for all $j \ge l$. It follows that

$$\sigma(B^{\tau_l}(x,r)) \le \liminf_{j\to\infty} \lambda_j(B^{\tau_l}(x,r)) \le r^\alpha$$

since $B^{\tau_l}(x,r)$ is an open set in the Euclidean topology. In view of the continuity of the functions involved, we get $B^\tau(x,r) = \bigcup_{k=1}^\infty B^{\tau_k}(x,r)$, which is an increasing union. Hence

$$\sigma(B^\tau(x,r)) = \lim_{k\to\infty} \sigma(B^{\tau_k}(x,r)) \le r^\alpha.$$

Finally, if $r \ge 1$ and $B^\tau(x,r)$ is a τ-ball, then

$$\sigma(B^\tau(x,r)) \le \|\sigma\| \le (l(Q_0))^\alpha < 1 \le r^\alpha. \qquad \square$$

3. Green's function and potentials

In this section, we prove that the Green's function for L_k (as defined below) is given by (1) and (2). We also give the estimates we shall use for G_k, and prove the precise growth condition on a measure μ which guarantees that $G_k\mu$ defines an L_k-potential.

The Green's function for L_k is defined to be the function $G\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to (0, \infty]$ such that (i) G is continuous outside the diagonal, (ii) for each $y \in \mathbb{R}^n_+$, $x \mapsto G(x, y)$ is an L_k-potential with support at $\{y\}$ (meaning it is L_k-harmonic outside of $\{y\}$), and (iii) for each $y \in \mathbb{R}^n_+$, $L_k(G(\cdot, y)) = -\delta_y$ in the sense of distribution. It is shown in [He] that L_k-potentials of point support are proportional, so the Green's function is unique.

The Correspondence Principle [W] states that u is L_k-harmonic if and only if $x_n^{k-1}u$ is L_{2-k}-harmonic. It allows us to deduce results for the L_k potential theory for $k \geq 1$ from analogous results for $k \leq 1$ and vice versa. The Correspondence Principle follows from the fact, easily established using the chain rule, that for any C^2 function, f,

$$L_{2-k}(x_n^{k-1} f) = x_n^{k-1} L_k(f). \tag{7}$$

As a consequence, the adjoint operator, $\widetilde{L}_k$ satisfies

$$\widetilde{L}_k(f) = x_n^{k-1} \widetilde{L}_{2-k}(x_n^{1-k} f). \tag{8}$$

From (7), (8), and (iii) of the definition of the Green's function, we deduce that

$$\left(\frac{x_n}{y_n}\right)^{k-1} G_k(x, y) = G_{2-k}(x, y). \tag{9}$$

PROPOSITION 1. *The Green's function $G_k(x, y)$ is given by equations* (1), *and* (2).

Remark. Notice that the integral defining G_0 is an elementary one, and when integrated, gives $\frac{1}{(n-2)\omega_n}(|x-y|^{n-2} - |x-\overline{y}|^{n-2})$, where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface measure of the unit sphere in $\mathbb{R}^n$ and $\overline{y} = (y_1, \dots, y_{n-1}, -y_n)$ is the reflection of $y = (y_1, \dots, y_n)$ in the $\mathbb{R}^{n-1}$-plane. This is the familiar Green's function for the Laplace equation on $\mathbb{R}^n_+$, and so our results generalize classical results for potentials with respect to the Laplacian on a halfspace.

Proof of Proposition 1. Fix $y = (y', y_n) \in \mathbb{R}^n_+$ once and for all. By (9), it suffices to prove the proposition for $k \geq 1$. It was shown in [BCB2] that for each fixed $y \in \mathbb{R}^n_+$, $x \to G_k(x, y)$ majorizes no positive L_k-harmonic function. Clearly G_k is jointly continuous outside the diagonal and tends to ∞ at the diagonal. It thus remains to show that for any C^∞ function ϕ with compact support in $\mathbb{R}^n_+$,

$$-\phi(y) = \int G_k(x, y)\widetilde{L}_k\phi(x)dx.$$

For $\epsilon > 0$, let

$$v_\epsilon(x,t) = a_{n,2-k} y_n^k \frac{\sin^{k-1} t}{\left[|x-y|^2 + \epsilon^2 + 2x_n y_n (1-\cos t)\right]^{(n+k-2)/2}}.$$

By (17) and (18) below, $G_k(\cdot, y)$ is locally Lebesgue integrable on $\mathbb{R}^n_+$ for each fixed y. Together with the fact that v_ϵ is smooth, this implies

$$\begin{aligned}\int G_k(x,y)\widetilde{L}_k\phi(x)dx &= \lim_{\epsilon\to 0}\int\int_0^\pi v_\epsilon(x,t)\widetilde{L}_k\phi(x)\,dt\,dx \\ &= \lim_{\epsilon\to 0}\int\int_0^\pi L_k(v_\epsilon(x,t))\phi(x)\,dt\,dx.\end{aligned}$$

A tedious computation shows

$$\begin{aligned}L_k(v_\epsilon) &= (-n-k+2)a_{n,2-k}y_n^k \sin^{k-1} t \qquad (10)\\ &\times\left\{\frac{(-ky_n x_n^{-1}\cos t)}{\left[|x'-y'|^2 + x_n^2 + y_n^2 + \epsilon^2 - 2x_n y_n \cos t\right]^{(n+k)/2}}\right.\\ &\left. + (n+k)\frac{(\epsilon^2 + y_n^2 \sin^2 t)}{\left[|x'-y'|^2 + x_n^2 + y_n^2 + \epsilon^2 - 2x_n y_n \cos t\right]^{(n+k+2)/2}}\right\}.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}&\int_0^\pi \frac{-ky_n x_n^{-1}\sin^{k-1} t\cos t}{\left[|x'-y'|^2 + x_n^2 + y_n^2 + \epsilon^2 - 2x_n y_n \cos t\right]^{(n+k)/2}}\,dt\\ &\quad = (n+k)\int_0^\pi \frac{-y_n^2 \sin^{k+1} t}{\left[|x'-y'|^2 + x_n^2 + y_n^2 + \epsilon^2 - 2x_n y_n \cos t\right]^{(n+k+2)/2}}\,dt, \qquad (11)\end{aligned}$$

and so if we integrate (10) from 0 to π with respect to t, we are left with

$$\begin{aligned}\int_0^\pi L_k(v_\epsilon)(x,t)\,dt &= (-n-k+2)(n+k)a_{n,2-k}y_n^k\\ &\quad\times\int_0^\pi \frac{\epsilon^2\sin^{k-1} t}{\left[|x'-y'|^2 + x_n^2 + y_n^2 + \epsilon^2 - 2x_n y_n \cos t\right]^{(n+k+2)/2}}\,dt\\ &= (-n-k+2)(n+k)a_{n,2-k}y_n^k\\ &\quad\times\int_0^\pi \frac{\epsilon^2\sin^{k-1} t}{\left[|x-y|^2 + \epsilon^2 + 2x_n y_n(1-\cos t)\right]^{(n+k+2)/2}}\,dt.\end{aligned}$$

Thus, we must show

$$\begin{aligned}-\phi(y) &= (-n-k+2)\,(n+k)\,a_{n,2-k}\,y_n^k\\ &\quad\times\lim_{\epsilon\to 0}\int\int_0^\pi \frac{\epsilon^2\,\sin^{k-1} t\,\phi(x)\,dt\,dx}{\left[|x-y|^2 + \epsilon^2 + 2x_n y_n(1-\cos t)\right]^{(n+k+2)/2}}. \qquad (12)\end{aligned}$$

We remark that in what follows, our use of the notation "$\lim_{\epsilon\to 0}$" is justified by the fact that the limit in (14) below exists.

Fix a small, but unspecified positive number, δ. Because of the ϵ^2 in the numerator, the limit is the same if we integrate over $\{|x-y|<\delta\}\times\{0<t<\delta\}$, and so by the continuity of ϕ, the limit in (12) is

$$\phi(y)\lim_{\epsilon\to 0}\int_{|x-y|<\delta}\int_0^\delta \frac{\epsilon^2 \sin^{k-1} t\, dt\, dx}{\left[|x-y|^2+\epsilon^2+2x_n y_n(1-\cos t)\right]^{(n+k+2)/2}}. \tag{13}$$

Make the substitution $u^2 = 2x_n y_n(1-\cos t)$. We get (since the limit is independent of which δ we use)

$$\lim_{\epsilon\to 0}\int_{|x-y|<\delta}\int_0^\delta \frac{\epsilon^2 \sin^{k-1} t}{\left[|x-y|^2+\epsilon^2+2x_n y_n(1-\cos t)\right]^{(n+k+2)/2}}\, dt\, dx$$

$$= \lim_{\epsilon\to 0}\int_{|x-y|<\delta}\int_0^\delta \frac{\epsilon^2\left(\frac{u^2}{x_n y_n}\left(1-\frac{u^2}{4x_n y_n}\right)\right)^{(k-2)/2}\frac{u}{x_n y_n}}{\left(|x-y|^2+\epsilon^2+u^2\right)^{(n+k+2)/2}}\, du\, dx.$$

Since δ may be chosen arbitrarily small, the limit is the same if we replace the term $1-\frac{u^2}{4x_n y_n}$ by 1 and the x_n terms by y_n. After replacing x by polar coordinates we thus have (since the integrals are independent of δ)

$$\lim_{\epsilon\to 0}\frac{\omega_n\epsilon^2}{y_n^k}\int\int_{r^2+u^2<\delta^2}\frac{r^{n-1}u^{k-1}}{\left(r^2+u^2+\epsilon^2\right)^{(n+k+2)/2}}\, du\, dr,$$

where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface measure of the unit ball in $\mathbb{R}^n$. If we treat r, u as rectangular coordinates and switch to polar coordinates, that is put $r = w\cos\theta$ and $u = w\sin\theta$, we get

$$\lim_{\epsilon\to 0}\frac{\omega_n\epsilon^2}{y_n^k}\int_0^{\pi/2}\int_0^\delta \frac{w^{n+k-1}\cos^{n-1}\theta\,\sin^{k-1}\theta}{\left(w^2+\epsilon^2\right)^{(n+k+2)/2}}\, dw\, d\theta.$$

A substitution of $z = w/\epsilon$ gives

$$\lim_{\epsilon\to 0}\frac{\omega_n}{y_n^k}\int_0^{\pi/2}\int_0^{\delta/\epsilon}\frac{z^{n+k-1}}{(1+z^2)^{(n+k+2)/2}}\cos^{n-1}\theta\ \sin^{k-1}\theta\, dz\, d\theta$$

$$= \frac{\omega_n}{y_n^k}\left(\int_0^{\pi/2}\cos^{n-1}\theta\,\sin^{k-1}\theta\, d\theta\right)\left(\int_0^\infty \frac{z^{n+k-1}}{(1+z^2)^{(n+k+2)/2}}\, dz\right). \tag{14}$$

Make the change of variables $x = \sin^2\theta$ in the first integral and $1 + z^2 = 1/t$ in the second integral. The first becomes the beta integral

$$(1/2)\ B(k/2, n/2) \quad = \quad (1/2)\ \frac{\Gamma(n/2)\Gamma(k/2)}{\Gamma((n+k)/2)}$$

(recall that for $x, y > 0$, $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ [K]). The second integral becomes an elementary integral with value $1/(n+k)$. Now, combining (12), (13), and (14) gives an expression for $a_{n,2-k}$. Replacing k by $2-k$ and using the fact that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ gives the required expression for $a_{n,k}$. □

The estimates on G_k which we need are given by the following result.

LEMMA 3. *Let $k \le 1$. Then*

$$G_k(x, y) \le c\frac{x_n^{1-k} y_n}{|x-y|^{n-k}} \tag{15}$$

and

$$G_k(x, y) \ge c\frac{x_n^{1-k} y_n}{|x-\overline{y}|^{n-k}} \tag{16}$$

where $\overline{y}$ is the reflection of $y \in \mathbb{R}^n_+$ in the hyperplane boundary. The following upper bounds, which are stronger in case $k < 1$, hold for all k:

$$G_k(x, y) \le c\frac{x_n^{-k/2} y_n^{k/2}}{|x-y|^{n-2}} \ \textit{for}\ n \ge 3 \tag{17}$$

and

$$G_k(x, y) \le c x_n^{-k/2} y_n^{k/2}\left(1 + \left|\log\frac{\sqrt{x_n y_n}}{|x-y|}\right|\right) \ \textit{for}\ n = 2. \tag{18}$$

Proof. If we ignore all but the $|x-y|$ term in the denominator of (1) we get (15). Since $|x-\overline{y}|^2 - |x-y|^2 = 4x_n y_n$, we can rewrite G_k as

$$a_{n,k}\ x_n^{1-k}\ y_n \int_0^\pi \frac{\sin^{1-k} t}{[|x-y|^2\cos^2 t/2 + |x-\overline{y}|^2 \sin^2 t/2]^{(n-k)/2}}\, dt. \tag{19}$$

Now (16) follows from the fact that $|x-y| \le |x-\overline{y}|$.

Suppose $k \le 1$. We have

$$\begin{aligned} G_k(x, y) &= a_{n,k} x_n^{1-k} y_n \int_0^\pi \frac{\sin^{1-k} t}{\left[|x-y|^2 + 2x_n y_n(1-\cos t)\right]^{(n-k)/2}}\, dt \\ &\le c x_n^{1-k} y_n \int_0^\pi \frac{t^{1-k}}{\left[|x-y|^2 + x_n y_n t^2\right]^{(n-k)/2}}\, dt. \end{aligned}$$

After a substitution of $s = \sqrt{x_n y_n} t / |x - y|$ and a simplification we get

$$G_k(x, y) \leq c \frac{x_n^{-k/2} y_n^{k/2}}{|x - y|^{n-2}} \int_0^{\pi \sqrt{x_n y_n}/|x-y|} \frac{s^{1-k}}{(1 + s^2)^{(n-k)/2}} \, ds.$$

If $n \geq 3$, the integrand is integrable over $[0, \infty)$, and the result follows. The case $n = 2$ also follows easily. That the inequalities hold for $k > 1$ follows from an application of (9). □

PROPOSITION 2. *Let μ be a regular Borel measure on $\mathbb{R}^n_+$. If $k < 1$ then $G_k \mu$ is a potential if and only if*

$$\int \frac{y_n}{(1 + |y|)^{n-k}} \, d\mu(y) < \infty. \tag{20}$$

If $k \geq 1$, $G_k \mu$ is a potential if and only if

$$\int \frac{y_n^k}{(1 + |y|)^{n+k-2}} \, d\mu(y) < \infty. \tag{21}$$

Proof. By (9), it is enough for us to show the proof for $k \leq 1$. Suppose first that $G_k \mu(z) < \infty$ for some $z \in \mathbb{R}^n_+$. Using (16),

$$\begin{aligned} \infty &> \int G_k(z, y) \, d\mu(y) \\ &\geq c \, z_n^{1-k} \int \frac{y_n}{|z - \bar{y}|^{n-k}} \, d\mu(y) \\ &\geq c \, \frac{z_n^{1-k}}{(\max\{1, |z|\})^{n-k}} \int \frac{y_n}{(1 + |y|)^{n-k}} d\mu(y). \end{aligned}$$

Suppose conversely that (20) is satisfied. Fix $z \in \mathbb{R}^n_+$, and write μ as $\mu_1 + \mu_2$, where μ_1 is the restriction of μ to the complement of the Euclidean ball $B(z, z_n/4)$. Suppose $x \in B(z, z_n/8)$, and y is in the complement of $B(z, z_n/4)$. We leave it to the reader to show that there is a c depending only on $|z|$ and z_n such that $|x - y| \geq c\,(1 + |y|)$. Thus, by (15),

$$G_k \mu_1(x) \leq c \int \frac{x_n^{1-k} y_n}{|x - y|^{n-k}} \, d\mu_1(y) \leq c \, x_n^{1-k} \int \frac{y_n}{(1 + |y|)^{n-k}} \, d\mu(y) < \infty. \tag{22}$$

Now consider $G_k \mu_2$. For $x \in B(z, z_n/8)$ and $y \in B(z, z_n/4)$ we have $x_n \geq (7/8) z_n$, and $(3/4) z_n \leq y_n \leq (5/4) z_n$ and so by (17) and (18), $G_k(x, y) \leq c|x - y|^{2-n}$ in case $n \geq 3$, and $G_k(x, y) \leq c(1 + |\log z_n/|x - y||)$ in case $n = 2$. We also have $B(z, z_n/8) \subset B(y, y_n/2)$. For $n \geq 3$, an integration with respect to polar coordinates shows

$$\begin{aligned}\int_{B(z,z_n/8)} G_k\mu_2(x)\,dx &= \int_{B(z,z_n/4)}\int_{B(z,z_n/8)} G_k(x,y)\,dx\,d\mu_2(y)\\ &\le c\int_{B(z,z_n/4)}\int_{B(y,y_n/2)} |x-y|^{2-n}\,dx\,d\mu_2(y) < \infty,\end{aligned}$$

so $G_k\mu_2(x) < \infty$ for Lebesgue a.e. $x \in B(z, z_n/8)$. A similar argument shows the $n = 2$ case. By (22), $G_k\mu(x) < \infty$ at any such x. □

4. Generalised Littlewood Theorem

We start the section with some auxiliary results. The first is proved using an elementary method.

LEMMA 4. *Let $c > 0$. Then there exists a constant C_c such that for all $a, b > 0$,*

$$C_c^{-1}(a^c + b^c) \le (a+b)^c \le C_c(a^c + b^c)$$

From the above lemma we easily deduce:

LEMMA 5. *Let $\tau \ge 1$. Then d^τ satisfies a pseudo-triangle inequality. That is, there is a constant c_τ such that, for all $x, y, z \in \mathbb{R}^{n-1}$,*

$$d^\tau(x,z) \le c_\tau\left[d^\tau(x,y) + d^\tau(y,z)\right].$$

Remark. Let $a, b > 0$. By using standard calculus techniques, we can show that, if $c > 1$, then $(a+b)^c \le 2^{c-1}(a^c + b^c)$ and if $0 < c \le 1$, then $(a+b)^c \le a^c + b^c$.

Repeated applications of Lemma 5 allows us to deduce

LEMMA 6. *There is a constant c depending only on n and τ which enables the following cover by homothetics: whenever $r \ge s$ and the τ-balls $B^\tau(x,r)$ and $B^\tau(y,s)$ intersect, then $B^\tau(y,s) \subset B^\tau(x,cr)$.*

We easily deduce:

LEMMA 7. *There is a constant d depending only upon n and τ for which we have the following property: given any finite number of τ-balls, there is a pairwise disjoint subcollection of these τ-balls which when expanded by a factor of d (that is, the radii are multiplied by the factor d) cover all the original τ-balls.*

Indeed just arrange the balls in order of decreasing radii, take the first one, then the next which does not meet the first, then the next which does not meet the first two, etc. The resulting collection is easily shown to have the required property.

We now introduce the τ-projection map $P^\tau\colon \mathbb{R}^n_+ \to \mathbb{R}^{n-1}$ which is a key device in our proof of Theorem 2 and whose properties motivated the consideration of the non-isotropic pseudo-distance and the associated Hausdorff measure.

Definition 3. Let $\tau \geq 1$. Let $x \in \mathbb{R}^n_+$, $x = (x_1, \ldots, x_n)$. Then

$$P^\tau(x) = (x_1 - x_n^{1/\tau}, x_2, \ldots, x_{n-1}).$$

Observe that $\Gamma^\tau_{P^\tau(x)}$ passes through the point x. We also note that for all positive values of t, all the points $(x_1 + t, x_2, \ldots, x_{n-1}, t^\tau)$ will have P^τ-projection equal to $(x_1, x_2, \ldots, x_{n-1})$. We now have the following result concerning the projection.

LEMMA 8. *Let $B(x, r)$ be the open Euclidean ball, centered at $x \in \mathbb{R}^n_+$ and radius $0 < r < 1$. Then,*

$$P^\tau\left[B(x,r)\right] \subset B^\tau\left(P^\tau(x), 2^{\tau-\frac{1}{2}}r\right).$$

Proof. Let $y \in B(x, r)$. Then

$$\begin{aligned} d^\tau\left(P^\tau(y), P^\tau(x)\right) &= \sqrt{\left|(y_1 - y_n^{1/\tau}) - (x_1 - x_n^{1/\tau})\right|^{2\tau} + \sum_{j=2}^{n-1}(x_j - y_j)^2} \\ &\leq \sqrt{\left[|y_1 - x_1| + |y_n^{1/\tau} - x_n^{1/\tau}|\right]^{2\tau} + \sum_{j=2}^{n-1}(x_j - y_j)^2} \\ &\leq \sqrt{2^{2\tau-1}\left[|y_1 - x_1|^{2\tau} + |y_n^{1/\tau} - x_n^{1/\tau}|^{2\tau}\right] + \sum_{j=2}^{n-1}(x_j - y_j)^2}. \end{aligned}$$

Noting that $|y_1 - x_1|^{2\tau} \leq |y_1 - x_1|^2$ and $|y_n^{1/\tau} - x_n^{1/\tau}| \leq |y_n - x_n|^{1/\tau}$, we deduce

$$d^\tau\left(P^\tau(y), P^\tau(x)\right) \leq 2^{\tau-\frac{1}{2}}\sqrt{\sum_{j=1}^{n}(x_j - y_j)^2} < 2^{\tau-\frac{1}{2}}r. \qquad \square$$

Before we proceed to the proof of the theorem, we define the following generalisation of the Hardy-Littlewood maximal function [S].

Definition 4. Let $\tau \geq 1$ and $\alpha > 0$. Let λ be a measure on $\mathbb{R}^{n-1}$. We define

$$\left(M_\alpha^\tau \lambda\right)(x) = \sup\left\{\frac{\lambda(B^\tau(x,r))}{r^\alpha}:\ r > 0\right\}$$

for all $x \in \mathbb{R}^{n-1}$.

Proof of Theorem 2. By (9), it is enough to prove the theorem for $k < 1$. Since the union of a countable number of sets of vanishing H_α^τ-measure has vanishing H_α^τ-measure, we may assume that x is in the intersection of $\mathbb{R}_+^n$ and an open Euclidean ball B in $\mathbb{R}^n$ of radius r. If we take the restriction of μ to the complement of a closed ball containing the closure of B in its interior, it follows easily from (15) that we can dominate the integrand of the resulting L_k-potential by a multiple of $x_n^{1-k} y_n (1 + |y|)^{-(n-k)}$, and so (20) implies that the unrestricted limit is 0 at every point in $\mathbb{R}^{n-1}$ in the closure of B. Thus we may assume that μ has support in B.

Let $d\nu = y_n^\omega\, d\mu$, so that ν is a finite measure. For $\epsilon > 0$, write $\nu = \nu_\epsilon + \nu_\epsilon'$, where ν_ϵ is the restriction of ν to the strip $\{y \in \mathbb{R}_+^n\colon\ y_n < \epsilon\}$. Let λ_ϵ be the P^τ-image of ν_ϵ; that is, for each $E \subset \mathbb{R}^{n-1}$, $\lambda_\epsilon(E) = \nu_\epsilon((P^\tau)^{-1}(E))$. Since $\|\lambda_\epsilon\| = \|\nu_\epsilon\|$, we have $\|\lambda_\epsilon\| \to 0$ as $\epsilon \to 0$.

Notice that

$$\lim_{\Gamma_z^\tau} \int G_k(x, y) y_n^{-\omega}\, d\nu_\epsilon'(y) = 0, \quad \text{for all } z \in \mathbb{R}^{n-1},$$

since, by (15) and our assumption that x and y lie in a bounded set,

$$G_k(x, y) y_n^{-\omega} \leq c \frac{x_n^{1-k} y_n^{1-\omega}}{|x - y|^{n-k}} < c x_n^{1-k}$$

for x_n smaller than $\epsilon/2$.

Write $G_k\mu(x)$ as

$$G_k\mu(x) = \left(\int_I + \int_{II}\right) G_k(x, y) y_n^{-\omega}\, d\nu(y) = f_1(x) + f_2(x),$$

where

$$I = \left\{y \in \mathbb{R}_+^n\colon\ |x - y| \geq \frac{x_n}{2}\right\}, \text{ and } II = \left\{y \in \mathbb{R}_+^n\colon\ |x - y| < \frac{x_n}{2}\right\}.$$

We need to show that $\limsup_{\Gamma_z^\tau} f_j(x) = 0$, $j = 1, 2$, for H_α-a.e. $z \in \mathbb{R}^{n-1}$. By the above, we get the same lim sup if we replace ν by ν_ϵ for any $\epsilon > 0$. Let ϵ be chosen, unspecified for the moment. Fix $z \in \mathbb{R}^{n-1}$ and let $x \in \mathbb{R}_+^n$ with $P^\tau(x) = z$ and $|x - z| < 1/2$.

We first estimate $f_1(x)$. If $2^m x_n \leq |x - y| \leq 2^{m+1} x_n$ for $m \geq -1$, then

$$G_k(x, y) y_n^{-\omega} \leq c \frac{x_n^{1-k} y_n^{1-\omega}}{|x - y|^{n-k}}$$

and

$$y_n \le |y_n - x_n| + x_n \le |y - x| + x_n \le (1 + 2^{m+1})x_n \le 2^{m+2}x_n,$$

so

$$\begin{aligned} f_1(x) &\le c\sum_m \int_{2^m x_n \le |x-y| \le 2^{m+1}x_n} \frac{x_n^{1-k} y_n^{1-\omega}}{|x-y|^{n-k}}\, d\nu_\epsilon(y) \\ &\le c\sum_m \frac{x_n^{1-k}(2^{m+2}x_n)^{1-\omega}}{(2^m x_n)^{n-k}} \nu_\epsilon(B(x, 2^{m+1}x_n)), \end{aligned}$$

where $B(x, 2^{m+1}x_n)$ denotes a Euclidean ball in $\mathbb{R}^n$ and is contained in $\mathbb{R}^n_+$ by the choice. By our assumption concerning the support of μ, we may sum over those m for which $2^{m+1}x_n < 1$. By Lemma 8,

$$P^\tau(B(x, 2^{m+1}x_n)) \subset B^\tau(z, 2^{\tau+m+1}x_n),$$

and so

$$\nu_\epsilon(B(x, 2^{m+1}x_n)) \le \lambda_\epsilon(B^\tau(z, 2^{\tau+m+1}x_n)).$$

Thus

$$f_1(x) \le c\sum_m 2^{-m(1-k)} \frac{\lambda_\epsilon(B^\tau(z, 2^{\tau+m+1}x_n))}{(2^{\tau+m+1}x_n)^{n-2+\omega}} \le cM^\tau_{n-2+\omega}(\lambda_\epsilon)(z). \tag{23}$$

Now consider $f_2(x)$. If $|x - y| \le x_n/2$, then $x_n/2 \le y_n \le (3/2)x_n$. By (17), if $n \ge 3$,

$$\begin{aligned} f_2(x) &\le c\sum_{m=0}^{\infty} \int_{\frac{x_n}{2^{m+1}} \le |x-y| \le \frac{x_n}{2^m}} G_k(x, y) y_n^{-\omega} d\nu_\epsilon(y) \\ &\le c\sum_m \frac{x_n^{-\omega}}{|x-y|^{n-2}}\, \nu_\epsilon(B(x, x_n/2^m)) \\ &\le c\sum_m \frac{x_n^{-\omega}}{(x_n/2^{m+1})^{n-2}}\, \lambda_\epsilon(B^\tau(z, 2^{\tau-m}x_n)) \\ &\le c\sum_m 2^{-m\omega} \frac{\lambda_\epsilon(B^\tau(z, 2^{\tau-m}x_n))}{(2^{\tau-m}x_n)^{n-2+\omega}} \\ &\le cM^\tau_{n-2+\omega}(\lambda_\epsilon)(z), \end{aligned} \tag{24}$$

since $\omega > 0$. Using (18) we get a similar result for $n = 2$.

Let $\delta > 0$. By (23) and (24), we deduce

$$\left\{z \in \mathbb{R}^{n-1}\colon \limsup_{x \in \Gamma^\tau_z} G_k\mu(x) > \delta\right\} \subset \left\{z \in \mathbb{R}^{n-1}\colon M^\tau_{n-2+\omega}(\lambda_\epsilon)(z) > c\delta\right\}, \tag{25}$$

where c depends only on n, k, τ. Notice that the left side of this inclusion is independent of ϵ. Let K be a compact subset of the set on the left side of (25). We must show that $H^{\tau}_{n-2+\omega}(K) = 0$. Suppose not. By Theorem 1, there exists a non-trivial measure σ with support in K such that for every τ-ball $B^{\tau}(x,r)$, $\sigma(B^{\tau}(x,r)) \le r^{n-2+\omega}$. There exists a finite number of τ-balls whose union contains K such that for each one, the λ_{ϵ} measure is greater than $c\delta$ multiplied by the τ-radius raised to the $n-2+\omega$ power. By Lemma 7, we can pass to a disjoint subcollection $\{B^{\tau}(x_j, r_j)\}$ of these balls such that $K \subset \bigcup_j B^{\tau}(x_j, dr_j)$. Then

$$\begin{aligned}\sigma(K) &\le \sum_j \sigma(B^{\tau}(x_j, dr_j)) \le \sum_j d^{n-2+\omega} r_j^{n-2+\omega} \\ &\le \sum_j \frac{d^{n-2+\omega}\lambda_{\epsilon}(B^{\tau}(x_j, r_j))}{c\delta} \le c\frac{\|\lambda_{\epsilon}\|}{\delta}.\end{aligned}$$

By letting $\epsilon \to 0$, we see $\sigma(K) = 0$ and we arrive at a contradiction. □

5. Characterisation of exceptional sets

The following result is the key element in the construction of the potential with the required bad behavior.

LEMMA 9. *Let $R^{\tau}(x,r)$ be a τ-rectangle in $\mathbb{R}^{n-1}$ of radius $r < 1/4$. Let*

$$h = \frac{r}{2^{\tau-1}(3^{1/\tau} - 2^{1/\tau})^{\tau}}, \qquad \tilde{h} = \frac{r}{2^{\tau-1}(2^{1/\tau} - 1)^{\tau}}.$$

Let $p^L = (x_1 + h^{1/\tau}, x_2, \ldots, x_{n-1}, h)$, and $p^R = (x_1 + \tilde{h}^{1/\tau}, x_2, \ldots, x_{n-1}, \tilde{h})$. Define $G = G^L \cup G^R$, where

$$G^L = \{x_1 + h^{1/\tau}\} \times \left(x_2 - \frac{r}{2}, x_2 + \frac{r}{2}\right) \times \ldots \times \left(x_{n-1} - \frac{r}{2}, x_{n-1} + \frac{r}{2}\right) \times \left[h, \frac{3}{2}h\right)$$

and

$$G^R = \left\{x_1 + \tilde{h}^{1/\tau}\right\} \times \left(x_2 - \frac{r}{2}, x_2 + \frac{r}{2}\right) \times \ldots \times \left(x_{n-1} - \frac{r}{2}, x_{n-1} + \frac{r}{2}\right) \times \left(\frac{\tilde{h}}{2}, \tilde{h}\right).$$

Then the τ-projection of G contains $R^{\tau}(x,r)$. We have the lower bound on the L_k-Green's function G_k that for each $y \in G^L$ and $z \in G^R$, $G_k(y, p^L) \ge cr^{2-n}$ and $G_k(z, p^R) \ge cr^{2-n}$, where c depends only on n, k, τ.

Proof. Let $y \in R^{\tau}(x,r)$. If $y_1 \le x_1$ let $y^L = (x_1 + h^{1/\tau}, y_2, \ldots, y_{n-1}, (x_1 - y_1 + h^{1/\tau})^{\tau})$. If $y_1 > x_1$, let $y^R = (x_1 + \tilde{h}^{1/\tau}, y_2, \ldots, y_{n-1}, (\tilde{h}^{1/\tau} - (y_1 - x_1))^{\tau})$.

Now suppose $y_1 \leq x_1$. Notice that $P^\tau(y^L) = y$ and

$$0 \leq (x_1 - y_1 + h^{1/\tau})^\tau - h < \left(\frac{r^{1/\tau}}{2} + h^{1/\tau}\right)^\tau - h$$
$$= h\left[\left(\frac{(r/h)^{1/\tau}}{2} + 1\right)^\tau - 1\right]$$
$$= \frac{h}{2}.$$

Thus $y^L \in G^L$. A similar argument shows that $y^R \in G^R$ in case $y_1 > x_1$. Let $y \in G^L$. Then $y_n \geq h$, $p_n^L = h$, and $|y - p^L|$ is bounded above by a multiple of r. Since h is a multiple of r, the required lower bound on $G_k(y, p^L)$ follows from (16). A similar argument for $G_k(z, p^R)$ completes the proof. □

Proof of Theorem 3. Let $E \subset \mathbb{R}^{n-1}$, be contained in a τ-ball of radius less than $1/4$. We first prove the result in this case. For each positive integer, m, we can cover E by τ-rectangles $\{R^\tau(x_{mj}, r_{mj}) \colon j \in \mathbb{Z}^+\}$ such that $\sum_j r_{mj}^{n-2+\omega} < 2^{-m}$, (so $r_{mj} \to 0$ as $m \to \infty$ uniformly in j). Associate the numbers h_{mj}, $\tilde{h}_{mj}$, the points p_{mj}^L, p_{mj}^R, and the sets G_{mj} as in the lemma. Let μ_{mj} be the measure concentrated on the two point set $\{p_{mj}^L, p_{mj}^R\}$ with mass $m \cdot r_{mj}^{n-2}$ at each of the two points. Let $\mu = \sum_{m,j} \mu_{mj}$. Then μ has bounded support and

$$\int y_n^\omega \, d\mu \leq c \sum_{m,j} r_{mj}^\omega \, r_{mj}^{n-2} \, m < \sum_m 2^{-m} m < \infty.$$

For any $m \in \mathbb{Z}^+$, if we take $x \in R^\tau(x_{mj}, r_{mj})$, by the lemma we can find $y \in G_{mj}$ such that $P^\tau(y) = x$ and $G_k\mu(y) \geq c r_{mj}^{2-n} m r_{mj}^{n-2} = cm \to \infty$ as $m \to \infty$. Thus $\limsup_{\Gamma_x^\tau} G_k\mu = \infty$ for all $x \in E$. This completes the proof in case E is contained in a τ-ball of radius less than $1/4$.

Now suppose we have a set $E \subset \mathbb{R}^{n-1}$ of H_α^τ-measure zero. Since τ-balls are open in the Euclidean topology, we may cover E by a countable number of τ-balls B_l^τ of radius smaller than $1/4$ for each $l = 1, 2, \ldots$. For each set $E_l = B_l \cap E$ by the above argument we can find an L_k-potential v_l on $\mathbb{R}^{n-1}$ which has the property that

$$\limsup_{\Gamma_x^\tau} v_l = \infty$$

for all $x \in E_l$. Now, it is easily seen that a suitable sum of the form $\sum_{l=1}^\infty c_l v_l$ will satisfy the requirements of the theorem. □

Added in proof. In a recently published article (*A note on capacity and Hausdorff measure in homogeneous spaces*, Potential Analysis **6** (1997), pp. 87–97), T. Sjödin has proved a general version of our result (Theorem 1) concerning H_α^τ-measure.

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