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Exceptional Sets in a Product of Harmonic Spaces

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0. Introduction

The sets that are polar have been very important in characterizing the exceptions of several properties, most notably the irregular boundary points in the study of the Dirichlet problem. In this connection the Great Convergence Theorem of Cartan-Brelot, both in the classical case and in the more recent axiomatic study of superharmonic functions, has been extremely useful. In this paper we shall prove a generalization of this well-known result for a family of multiply superharmonic functions. More precisely we consider the set of points where $v \neq \hat{v}$, v being the pointwise infimum of an arbitrary family of uniformly locally lower bounded multiply superharmonic functions and $\hat{v}$ its lower semicontinuous regularization. Contrary to the natural expectation this exceptional set does not seem to be multipolar viz., a set where a positive multiply superharmonic function takes the value ∞. In [9] Gowrisankaran introduced a notion of exceptional sets in the product space $\mathbb{C}^n$. We extend this idea to consider n-negligible sets in a product of n spaces of Brelot. Our principal result (Theorem 4.1) is that the exceptional sets are n-negligible. We give as application a simple proof of a known result concerning the convergence of a decreasing sequence of plurisuperharmonic functions. We also develop some properties of multipolar sets and further properties of n-negligible sets.

1. Preliminaries

We begin by recalling some results associated with a finite product of Brelot spaces. For Details see [7, 8, 14].

Let $\Omega_1, \ldots, \Omega_n$ be Brelot spaces each with a positive potential, a countable base of open sets, and satisfying axiom D. We denote the n-superharmonic (respectively n-harmonic) functions on an open set U of $\Omega_1 \times \ldots \times \Omega_n$ by $n-S(U)$ [respectively $n-H(U)$]. Let $\mathscr{B}_i$ be a base of regular domains for Ω_i, $i=1, \ldots, n$. The following proposition [8] will be of fundamental importance in proving the convergence theorem.

Proposition 1.1. *If v is in $n-S(\mathscr{B}_1, \ldots, \mathscr{B}_n)$ and not identically ∞, then $\hat{v}$ is n-superharmonic and for each $x=(x_1, \ldots, x_n)$*

$$\hat{v}(x)=\sup\{\bar{\int} v d(\varrho_{x_1}^{\delta_1} \times \ldots \times \varrho_{x_n}^{\delta_n}): x_i \in \delta_i \in \mathscr{B}_i\}.$$

We remark that the above family over which we take the supremum can be replaced by any countable subfamily $(\delta_{i,j})_j$ for which $\delta_{i,j} \supset \bar{\delta}_{i,j+1}$, $\bigcap_j \delta_{i,j}=\{x_i\}$, and sup is actually a limit of an increasing sequence.

We now consider the "distinguished" Dirichlet problem on a product of relatively compact domains $\omega_1 \times \omega_2$. For simplicity of notation we are taking $n=2$ although the analogous results clearly hold for any n. Let P_i be the irregular boundary points of ω_i and

$$P=(P_1 \times P_2) \cup (P_1 \times \omega_2) \cup (\omega_1 \times P_2).$$

For f real valued continuous on the distinguished boundary $\partial\omega_1 \times \partial\omega_2$, define ϕ on $\bar{\omega}_1 \times \bar{\omega}_2$ by

$$\begin{aligned}
\phi(x_1, x_2) = & f(x_1, x_2) && \partial\omega_1 \times \partial\omega_2, \\
& \int f(\cdot, x_2) d\mu_{x_1}^{\omega_1} && \omega_1 \times \partial\omega_2, \\
& \int f(x_1, \cdot) d\mu_{x_2}^{\omega_2} && \partial\omega_1 \times \omega_2, \\
& \iint f d(\mu_{x_1}^{\omega_1} \times \mu_{x_2}^{\omega_2}) && \omega_1 \times \omega_2,
\end{aligned}$$

where we integrate with respect to harmonic measure. Following the procedure of Gowrisankaran in [7] where he proved the analogous result for regular domains one can show

Proposition 1.2. *ϕ is continuous on $\bar{\omega}_1 \times \bar{\omega}_2 - P$, it is in $2-H(\omega_1 \times \omega_2)$, and the mappings $\phi(\cdot, x_2)$, $\phi(x_1, \cdot)$ are harmonic for each $x_1 \in \bar{\omega}_1$, $x_2 \in \bar{\omega}_2$.*

Consider now the following minimum principle [14].

Proposition 1.3. *Let $v \in 2-S(\omega_1 \times \omega_2)$ be lower bounded. If for all x in $\partial\omega_1 \times \partial\omega_2$*

$$\liminf_{z \to x} v(z) \geqq 0,$$

then $v \geqq 0$ on $\omega_1 \times \omega_2$.

Using this one can formulate and solve the "distinguished" Dirichlet problem on $\omega_1 \times \omega_2$ by means of the Perron-Wiener-Brelot method. Indeed for f an extended real valued function on $\partial\omega_1 \times \partial\omega_2$ put

$$\begin{aligned}
\mathscr{U}=\{v: v \text{ lower bounded 2-hyperharmonic on } \omega_1 \times \omega_2 \text{ with limit infimum} \\
\geqq f \text{ at each point of } \partial\omega_1 \times \partial\omega_2\}.
\end{aligned}$$

Define the upper solution $\bar{\mathfrak{H}}_f^{\omega_1 \times \omega_2}$ to be the pointwise lower envelope of $\mathscr{U}$ and the lower solution $\underline{\mathfrak{H}}_f^{\omega_1 \times \omega_2} = -\bar{\mathfrak{H}}_{-f}^{\omega_1 \times \omega_2}$. By Proposition 1.3 $\underline{\mathfrak{H}}_f^{\omega_1 \times \omega_2} \leqq \bar{\mathfrak{H}}_f^{\omega_1 \times \omega_2}$. f is said to be resolutive if the upper and lower solutions are identical and 2-harmonic. Using standard methods one can prove

Proposition 1.4. *For any extended real valued function f on $\partial\omega_1 \times \partial\omega_2$ and (x_1, x_2) in $\omega_1 \times \omega_2$*

$$\bar{\mathfrak{H}}_f^{\omega_1 \times \omega_2}(x_1, x_2) = \bar{\int} f d(\mu_{x_1}^{\omega_1} \times \mu_{x_2}^{\omega_2})$$

and if f is $\mu_{x_1}^{\omega_1} \times \mu_{x_2}^{\omega_2}$*-integrable for one point* (x_1, x_2) *then it is integrable with respect to all such measures and resolutive.*

2. *n*-Polar Sets

Let U be an open subset of $\Omega_1 \times \ldots \times \Omega_n$.

Definition 2.1. $P \subset U$ is said to be n-polar in U if there exists $v \in n - S^+(U)$ such that P is contained in the set where $v = \infty$. We shall say that any such v is associated to E in U. For $U = \Omega_1 \times \ldots \times \Omega_n$ we simply say P is n-polar.

Since an n-superharmonic function is finite on a dense subset of its domain it follows any subset of P is n-polar in any open subset of U containing it. Observe any set n-polar in U is contained in a G_δ set n-polar in U since

$$\{x \in U : v(x) = \infty\} = \bigcap_{m \geqq 1} \{x \in U : v(x) > m\} .$$

As a consequence products of harmonic measures do not charge n-polar sets.

Clearly in Proposition 1.3, if the condition holds for all x except a set 2-polar in $\Omega_1 \times \Omega_2$, then the conclusion still holds. It follows the function ϕ in Proposition 1.2 is the only one in $2 - H(\omega_1 \times \omega_2)$ that is bounded on $\omega_1 \times \omega_2$ and tending to f at all points of $\partial\omega_1 \times \partial\omega_2$ except a subset 2-polar in $\Omega_1 \times \Omega_2$.

One can generate n-polar sets easily since if P is k-polar in $\Omega_1 \times \ldots \times \Omega_k$ then $P \times \Omega_{k+1} \times \ldots \times \Omega_n$ is n-polar in $\Omega_1 \times \ldots \times \Omega_n$. We also have that a countable union of sets n-polar in U is n-polar in U. The proof of this for $n = 1$ extends easily.

For $n = 1$ there is a local property for polar sets which implies the existence of a global associated function [3]. We do not know if such a result holds for $n > 1$ and it is an open question whether or not a set n-polar in an open set is necessarily n-polar.

Definition 2.2. Let E be a subset of $\Omega_1 \times \Omega_2$, $x_1 \in \Omega_1$, and $x_2 \in \Omega_2$. The section $E(1, x)$ of E through x is defined as $\{z \in \Omega_2 : (x_1, z) \in E\}$. The section $E(2, x)$ is $\{z \in \Omega_1 : (z, x_2) \in E\}$.

Definition 2.3. Let E be a subset of and (x_1, x_2, x_3) a point of $\Omega_1 \times \Omega_2 \times \Omega_3$. The 1-section $E(1, x)$ [also denoted by $E(1, x_1)$] of E through x is defined to be $\{z \in \Omega_2 \times \Omega_3 : (x_1, z) \in E\}$. The 2-section $E(1, 2, x)$ $(= E(1, 2, (x_1, x_2))$ is defined as $\{z \in \Omega_3 : (x_1, x_2, z) \in E\}$. The sections $E(2, x)$, $E(3, x)$, $E(1, 3, x)$, and $E(2, 3, x)$ are defined in the obvious way.

For a polar set P in Ω_1 it is true that for any point x of $\Omega_1 - P$ there is a function associated to P that is finite at x [3]. In the case of 2-polar sets this is false since, as is easily seen, it is necessary for the sections $P(1, x)$ and $P(2, x)$ to be polar. We prove this condition is sufficient.

Theorem 2.4. *Let E be* 2*-polar and* $x = (x_1, x_2) \notin E$. *Then there exists v associated to E finite at x if and only if the sections* $E(1, x)$, $E(2, x)$ *are polar.*

Proof. Choose for $j = 1, 2$ a sequence $\{\delta(j, k)\}_k$ of regular domains in Ω_j such that for every k $\bar{\delta}(j, k+1) \subset \delta(j, k)$ and $\bigcap_k \delta(j, k) = \{x_j\}$. Let u be associated to E in $\Omega_1 \times \Omega_2$.

By means of two balayage procedures one easily constructs, for each k, a function $u_k \in 2-S^+(\Omega_1 \times \Omega_2)$ such that $u_k \leqq u$, $u_k = u$ on

$$(\Omega_1 - \delta(1,k)) \times (\Omega_2 - \delta(2,k)), \quad \text{and} \quad u_k \in 2-H^+(\delta(1,k) \times \delta(2,k)).$$

Put $\lambda_k = (2^k u_k(x))^{-1}$ and $w = \sum \lambda_k u_k$. Then w is in $2-S^+(\Omega_1 \times \Omega_2)$, since $w(x) < \infty$, and $w = \infty$ except on

$$(\{x_1\} \times E(1,x)) \cup (E(2,x) \times \{x_2\}).$$

Since $E(1,x)$ and $E(2,x)$ are polar there exist v_i in $S^+(\Omega_i)$, $i=1,2$, with $v_1 = \infty$ on $E(2,x)$, $v_1(x_1) < \infty$, and $v_2 = \infty$ on $E(1,x)$, $v_2(x_2) < \infty$. Let $h_i \in H^+(\Omega_i)$, $i=1,2$, be arbitrary. Then $v = w + h_1 \otimes v_2 + v_1 \otimes h_2$ is clearly associated to E and finite at x. The proof is complete.

Using a similar proof one can show

Corollary 2.5. *Let E be 3-polar and $x = (x_1, x_2, x_3) \notin E$. Then there is an associated function finite at x if and only if all the 1 sections of E through x are 2-polar and all the 2 sections of E through x are polar.*

3. Negligible Sets

Definition 3.1. Let U be an open subset of $\Omega_1 \times \Omega_2$. $E \subset U$ is said to be 2-negligible in U if there exist polar sets P_i in Ω_i, $i=1,2$, such that for all x_1 in $\Omega_1 - P_1$ the section $E(1,x_1)$ is polar in Ω_2 and for all x_2 in $\Omega_2 - P_2$ the section $E(2,x_2)$ is polar in Ω_1 (see Definition 2.2).

We remark that even though these sets are small in a potential theoretic sence there is no Fubini type theorem for 2-negligible sets. For example if

$$H = \{(z_1, z_2) \in U^2 : \operatorname{Im} z_1 = \operatorname{Re}(z_1 + z_2) = 0\},$$

where U is the unit disc in the complex plane and the harmonic functions are the twice continuously differentiable solutions of Laplace's equation, then for any z_2 the section of H through z_2 is polar (a single point) whereas for every real z_1 in U the section of H through z_1 is a line segment. Thus H is not 2-negligible and it is therefore necessary to include both sets of sections in Definition 3.1.

Definition 3.2. Let U be an open subset of $\Omega_1 \times \Omega_2 \times \Omega_3$. $E \subset U$ is said to be 3-negligible in U if there exist polar sets P_i in Ω_i, $i=1,2,3$, such that for all x_i in $\Omega_i - P_i$ the section $E(i, x_i)$ is 2-negligible in $\prod_{j \neq i} \Omega_j$.

It is obvious that E in Definition 3.1 is 2-negligible in U if and only if it is 2-negligible in $\Omega_1 \times \Omega_2$. We may thus refer to sets as being 2-negligible without reference to U. The same remark applies to Definition 3.2.

Observe the fundamental difference between 2 and 3-negligible sets. The difference comes from the fact that in the product of two Brelot spaces a section lies in a single Brelot space while in a product of three Brelot spaces a 1 section still lies in a product space. Thus the case $n=3$ is the model for the general case of n-negligible sets in $\Omega_1 \times \ldots \times \Omega_n$ where there are defined inductively. All results in this paper can be proved in this setting using the same proofs and induction. We consider the particular cases in this paper only for notational simplicity.

It is true, however, that even though there is a fundamental difference between 2 and 3-negligible sets the proofs of theorems in most cases for $n=3$ follow from the $n=2$ case precisely as the proofs for $n=2$ follow from the $n=1$ case. Thus in some cases we will give the proof for $n=2$ and merely indicate any minor changes for $n=3$. Furthermore, even if at times we do not explicitly mention it, all results dealing with 2-negligible sets which have analogues in 3-negligible sets are valid and vice versa.

The following equivalent formulation of 3-negligible is easily demonstrated and we omit the proof.

Proposition 3.3. *E is 3-negligible if and only if the sets $\{x\in\Omega_2\times\Omega_3 : E(2,3,x)$ not polar in $\Omega_1\}$, $\{x\in\Omega_1\times\Omega_2 : E(1,2,x)$ not polar in $\Omega_3\}$, $\{x\in\Omega_1\times\Omega_3 : E(1,3,x)$ not polar in $\Omega_2\}$ are 2-negligible.*

It is easily shown that a subset of a 3-negligible set and a countable union of 3-negligible sets is 3-negligible. Thus, by the Lindelöf property of $\Omega_1\times\Omega_2\times\Omega_3$, a set locally 3-negligible is 3-negligible. Using this fact we show any set $E\subset U$ 3-polar in U is 3-negligible. By the local property we may assume $U=\Omega_1\times\Omega_2\times\Omega_3$. Let $v\in 3-S^+(U)$ be ∞ on E and (x_1,x_2,x_3) any point where v is finite. Then $P=\{z\in\Omega_1 : v(z,x_2,x_3)=\infty\}$ is polar and $v(z,\cdot)\in 2-S^+(\Omega_2\times\Omega_3)$ for $z\in\Omega_1-P$. It follows that for such a z $E(1,z)$ is 2-polar, hence, by repeating the argument again, 2-negligible. By symmetry E is 3-negligible. We remark that it is an open question whether or not 3-negligible implies 3-polar.

One can easily generate 3-negligible sets. For if E is 2-negligible in $\Omega_1\times\Omega_2$ then $F=E\times\Omega_3$ is 3-negligible. Indeed there exists P_1 polar in Ω_1 such that for $x_1\in\Omega_1-P_1$ $E(1,x_1)$ is polar in Ω_2. Then for such an x_1 $F(1,x_1)=E(1,x_1)\times\Omega_3$ is 2-polar hence 2-negligible in $\Omega_2\times\Omega_3$. Similarly there exists P_2 polar in Ω_2 such that for $x_2\in\Omega_2-P_2$ $F(2,x_2)$ is 2-negligible in $\Omega_1\times\Omega_3$. Finally for all $x_3\in\Omega_3$ the section $F(3,x_3)=E$ is 2-negligible in $\Omega_1\times\Omega_2$. Thus F is 3-negligible.

One can also show that the complement of 2 and 3-negligible sets is dense. For polar sets this is well known [3]. Suppose E is 2-negligible and $E\supset U\times V$ where if possible $U\subset\Omega_1$ and $V\subset\Omega_2$ are nonvoid open. There exists a polar set $P\subset\Omega_1$ such that for $z\in\Omega_1-P$ $E(1,z)$ is polar. Since P cannot contain U we may choose x in $U-P$. For this point $E(1,x)$ is polar and it contains V. It follows V is void proving the result.

Proposition 3.4. *Let E be 3-negligible, $U\subset\Omega_1\times\Omega_2\times\Omega_3$ open, $x=(x_1,x_2,x_3)\in U$, and $v\in 3-S(U)$. Then*

$$v(x)=\liminf_{\substack{z\to x\\ z\in U-E}} v(z)\,. \tag{1}$$

Proof. There exists a G_δ polar set $P\subset\Omega_1$ such that for z_1 in Ω_1-P $E(1,z_1)$ is 2-negligible in $\Omega_2\times\Omega_3$. It follows there is a G_δ polar set $Q(z_1)$ in Ω_2 depending on z_1 such that for any z_2 in $\Omega_2-Q(z_1)$ $E(1,2,(z_1,z_2))$ is polar in Ω_3. For such a z_1 and z_2 let $R(z_1,z_2)$ by a G_δ polar set in Ω_3 containing $E(1,2,(z_1,z_2))$. Now suppose if possible (1) fails. Then there exists a neighbourhood W of x and $\varepsilon>0$ such that $v(z)\geqq v(x)+\varepsilon$ on $W\cap(U-E)$. For each $i=1,2,3$ choose a sequence $\{\delta(i,k)\}_k$ of regular domains in Ω_i such that for all k $\bar{\delta}(1,k)\times\bar{\delta}(2,k)\times\bar{\delta}(3,k)\subset W$, $\delta(i,k)\supset\bar{\delta}(i,k+1)$, and $\bigcap_k \delta(i,k)=\{x_i\}$. By Fubini's theorem we obtain

$$\begin{aligned}
&\int v\,d(\varrho_{x_1}^{\delta(1,k)}\times\varrho_{x_2}^{\delta(2,k)}\times\varrho_{x_3}^{\delta(3,k)})\\
&\quad=\int_{z_1\notin P} d\varrho_{x_1}^{\delta(1,k)}(z_1)\int_{z_2\notin Q(z_1)} d\varrho_{x_2}^{\delta(2,k)}(z_2)\int_{z_3\notin R(z_1,z_2)} v(z_1,z_2,z_3)\,d\varrho_{x_3}^{\delta(3,k)}(z_3)\\
&\quad\geqq(v(x)+\varepsilon)\int d(\varrho_{x_1}^{\delta(1,k)}\times\varrho_{x_2}^{\delta(2,k)}\times\varrho_{x_3}^{\delta(3,k)})\,.
\end{aligned}$$

Letting $k\to\infty$ gives $v(x)\geqq v(x)+\varepsilon$ which is impossible. This completes the proof.

4. Principal Results

We will demonstrate the following convergence theorem. Recall that it is necessary to assume axiom D in order that the convergence theorem is valid on each Ω_i.

Theorem 4.1. *Let $U\subset\Omega_1\times\Omega_2\times\Omega_3$ be open and $(v_k)\subset 3-S(U)$ a uniformly locally lower bounded and decreasing sequence with limit function v. Then $\hat{v}$ is in $3-S(U)$ and equals v everywhere except on a 3-negligible set.*

By the local property for 3-negligible sets we may assume without loss of generality that U is $\Omega_1\times\Omega_2\times\Omega_3$ and v is nonnegative. This we do for the remainder of the proof.

We begin by proving the analogous result on $\Omega_1\times\Omega_2$.

Theorem 4.2. *Let $(v_k)\subset 2-S^+(\Omega_1\times\Omega_2)$ be a decreasing sequence with limit function v. Then $\hat{v}$ is in $2-S^+(\Omega_1\times\Omega_2)$ and equals v except on a 2-negligible set.*

We introduce the following notation. If f is an extended real valued function on $\Omega_1\times\Omega_2$, $\hat{f}^1$ and $\hat{f}^2$ are defined on $\Omega_1\times\Omega_2$ by

$$\hat{f}^1(x_1,x_2)=\liminf_{z\to x_1} f(z,x_2)\,,\qquad \hat{f}^2(x_1,x_2)=\liminf_{z\to x_2} f(x_1,z)\,.$$

Lemma 4.3. *Let $(w_k)\subset 2-S^+(\Omega_1\times\Omega_2)$ be a decreasing sequence with w_1 locally bounded. Denote the limit function by w. Suppose for every k and x_1 in Ω_1 $x_2\to w_k(x_1,x_2)$ is harmonic. Then $w=\hat{w}$ everywhere except a 2-polar set of the form $P\times\Omega_2$ where P is polar in Ω_1. In addition $\hat{w}=\hat{w}^1$.*

Proof. Let $x=(x_1,x_2)$ be any point of $\Omega_1\times\Omega_2$ and for $i=1,2$ let $\{\omega(i,l)\}_l$ be a sequence of regular domains in Ω_i such that $\bar{\omega}(i,l+1)\subset\omega(i,l)$ and $\bigcap_l \omega(i,l)=\{x_i\}$. Since w is nearly 2-superharmonic [8], Proposition 1.1 says

$$\hat{w}(x)=\sup_l \iint w\,d\varrho_{x_1}^{\omega(1,l)}d\varrho_{x_2}^{\omega(2,l)}\,. \tag{1}$$

Now by assumption we have for each k and l

$$\iint w_k\,d\varrho_{x_1}^{\omega(1,l)}d\varrho_{x_2}^{\omega(2,l)}=\int w_k(\cdot,x_2)\,d\varrho_{x_1}^{\omega(1,l)}$$

thus the Monotone Convergence Theorem implies it is also true for w. Hence by (1) and Proposition 1.1 we get the last assertion $\hat{w}(x)=\hat{w}^1(x)$.

The sequence $\{\int w(\cdot, x_2) d\varrho_{x_1}^{\omega(1,l)}\}_l$ is nondecreasing. Since for every z the mapping $x_2 \to w(z, x_2)$ is harmonic (axiom 3) it follows that for every x_1 in Ω_1 and positive integer l, $x_2 \to \int w(\cdot, x_2) d\varrho_{x_1}^{\omega(1,l)}$ is harmonic and by axiom 3 we deduce that $x_2 \to \hat{w}^1(x_1, x_2)$ is a harmonic function on Ω_2. We show now that the set

$$E = \{x \in \Omega_1 \times \Omega_2 : \hat{w}^1(x) < w(x)\}$$

is contained in a set of the form $P \times \Omega_2$ where P is polar in Ω_1. For x_2 in Ω_2 define

$$E(x_2) = \{y \in \Omega_1 : \hat{w}^1(y, x_2) < w(y, x_2)\}.$$

This set is polar by the convergence theorem on Ω_1. Now fix any x_2' in Ω_2. We claim $E \subset E(x_2') \times \Omega_2$. For if (x_1, x_2) is in E, since $z \to w(x_1, z)$ and $z \to \hat{w}^1(x_1, z)$ are both harmonic on Ω_2 and $\hat{w}^1(x_1, x_2) < w(x_1, x_2)$ it follows that $\hat{w}^1(x_1, z) < w(x_1, z)$ for all z in Ω_2, in particular for $z = x_2'$. Thus x_1 is in $E(x_2')$ and the claim is proved. This completes the proof.

Lemma 4.4. *Let (v_k) and v be as in Theorem 4.2 with v_1 locally bounded. Then $\hat{v}^1$ is Borel measurable.*

Proof. Let a be any real number. We must show $E = \{x : \hat{v}^1(x) \geqq a\}$ is a Borel set. Let $\{\omega_l\}$ be a countable base of relatively compact open subsets of Ω_1. Then it is easy to see that

$$E = \bigcap_m \bigcup_l \left\{(x_1, x_2) \in \Omega_1 \times \Omega_2 : x_1 \in \omega_l, v(z, x_2) \geqq a - \frac{1}{m} \text{ for all } z \text{ in } \bar{\omega}_l\right\}.$$

It thus suffices to show that for any relatively compact open set ω in Ω_1 and real number b

$$\{(x_1, x_2) \in \Omega_1 \times \Omega_2 : x_1 \in \omega, v(z, x_2) \geqq b \text{ for all } z \text{ in } \bar{\omega}\}$$

is Borel. But this set is just

$$\omega \times \{x_2 \in \Omega_2 : v(z, x_2) \geqq b \text{ for all } z \text{ in } \bar{\omega}\}.$$

Thus we need consider only the latter set in this product and that is

$$\bigcap_{k,l} \left\{x_2 \in \Omega_2 : v_k(z, x_2) > b - \frac{1}{l} \text{ for all } z \text{ in } \bar{\omega}\right\}.$$

By a simple compactness argument each set in this intersection is open. This completes the proof.

Lemma 4.5. *Let ω_1, ω_2 be relatively compact domains in Ω_1, Ω_2 respectively and v a nonnegative locally bounded 2-superharmonic function defined on a neighbourhood of $\bar{\omega}_1 \times \bar{\omega}_2$. Then the mapping*

$$w : (x_1, x_2) \to \iint v \, d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2}$$

is the greatest 2-harmonic minorant of v on $\omega_1 \times \omega_2$.

Proof. Since v is 2-superharmonic on a neighborhood of $\omega_1 \times \omega_2$ it follows from Proposition 1.4 that w is in $2 - H^+(\omega_1 \times \omega_2)$ and it is a minorant of v on $\omega_1 \times \omega_2$.

Now let u be a 2-harmonic minorant of v on $\omega_1 \times \omega_2$. By axiom D, for any x_2 in ω_2 the greatest harmonic minorant of $x_1 \to v(x_1, x_2)$ on ω_1 is $x_1 \to \int v(\cdot, x_2) d\mu_{x_1}^{\omega_1}$. Thus

$$u(x_1, x_2) \leqq \int v(\cdot, x_2) d\mu_{x_1}^{\omega_1} \quad \text{on} \quad \omega_1 \times \omega_2 .$$

If $\{\delta_k\}$ is a relatively compact exhaustion of ω_2 then for all (x_1, x_2) in $\omega_1 \times \omega_2$ and all large k

$$\begin{aligned} u(x_1, x_2) &= \int u(x_1, \cdot) d\mu_{x_2}^{\delta_k} \\ &\leqq \iint v d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\delta_k} . \end{aligned} \tag{1}$$

Let $k \to \infty$. Since for each x_1 in ω_1 the mapping $g: z \to \int v(\cdot, z) d\mu_{x_1}^{\omega_1}$ is locally bounded and in $S^+(\omega_2)$ the last expression in (1) converges to the greatest harmonic minorant of g on ω_2 evaluated at x_2. Axiom D implies this is just

$$x_2 \to \int g d\mu_{x_2}^{\omega_2} = \iint v d(\mu_{x_1}^{\omega_1} \times \mu_{x_2}^{\omega_2}) .$$

Thus (1) gives

$$u(x_1, x_2) \leqq \iint v d(\mu_{x_1}^{\omega_1} \times \mu_{x_2}^{\omega_2}) = w(x_1, x_2)$$

and we are done.

Corollary 4.6. *Let ω_1, ω_2 be relatively compact domains in Ω_1, Ω_2 respectively and (v_k), v as in Theorem 4.2 with v_1 locally bounded. Then for all (x_1, x_2) in $\omega_1 \times \omega_2$*

$$\iint v d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} = \iint \hat{v} d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} \tag{1}$$

and hence for $i = 1, 2$

$$\iint v d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} = \iint \hat{v}^i d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} .$$

Proof. For each k and $x = (x_1, x_2)$ in $\omega_1 \times \omega_2$ we have $\iint v_k d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} \leqq v_k(x)$. Letting $k \to \infty$ gives the same result for v. Furthermore, since

$$g: (x_1, x_2) \to \iint v d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2}$$

is in $2 - H^+(\omega_1 \times \omega_2)$, it is in particular continuous, hence it minorizes $\hat{v}$. Now $\hat{v}$ is locally bounded and 2-superharmonic. Thus g minorizes the greatest 2-harmonic minorant of $\hat{v}$ on $\omega_1 \times \omega_2$. The lemma gives us then

$$\iint v d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} \leqq \iint \hat{v} d\mu_{x_1}^{\omega_1} d\mu_{x_2}^{\omega_2} .$$

The reverse inequality being obvious, we have (1). Finally (2) holds since $\hat{v} \leqq \hat{v}^i \leqq v$.

Proof of Theorem 4.2. We suppose first v_1 is locally bounded. Let δ be a regular domain in Ω_2. Define

$$F = \{(x_1, x_2) \in \Omega_1 \times \delta : \int \hat{v}^1(x_1, \cdot) d\varrho_{x_2}^{\delta} < \int v(x_1, \cdot) d\varrho_{x_2}^{\delta}\} .$$

We show F is 2-polar. Define, for each positive integer k and (x_1, x_2) in $\Omega_1 \times \delta$,

$$w_k(x_1, x_2) = \int v_k(x_1, \cdot) d\varrho_{x_2}^{\delta} \quad \text{and} \quad w(x_1, x_2) = \int v(x_1, \cdot) d\varrho_{x_2}^{\delta} .$$

Then we may apply Lemma 4.3 to $\{w_k\}$ and w and deduce $w = \hat{w}$ except on a 2-polar subset of $\Omega_1 \times \delta$. In order to show F is 2-polar it therefore suffices to prove that

$$\hat{w}(x_1, x_2) = \int \hat{v}^1(x_1, \cdot) d\varrho_{x_2}^{\delta} . \tag{1}$$

Let (x_1, x_2) be any point in $\Omega_1 \times \delta$ and let $\{\omega_l\}$ be a sequence of regular domains with $\bar{\omega}_{l+1} \subset \omega_l$ and $\bigcap_l \omega_l = \{x_1\}$. Again by Lemma 4.3

$$\begin{aligned}\hat{w}(x_1, x_2) &= \hat{w}^1(x_1, x_2) \\ &= \sup_l \int w(\cdot, x_2) d\varrho_{x_1}^{\omega_l} \\ &= \sup_l \iint v d\varrho_{x_2}^{\delta} d\varrho_{x_1}^{\omega_l} \\ &= \sup_l \iint \hat{v}^1 d\varrho_{x_2}^{\delta} d\varrho_{x_1}^{\omega_l} \quad \text{(Corollary 4.6)} \\ &= \int \hat{v}^1(x_1, \cdot) d\varrho_{x_2}^{\delta}.\end{aligned}$$

[This last equality holds by applying Proposition 1.1 to $y \to \int \hat{v}^1(y, \cdot) d\varrho_{x_2}^{\delta}$.] We have therefore (1) and shown F is 2-negligible.

Now for $i = 1, 2$ let $\mathscr{B}_1$ be a countable base of open sets of Ω_i consisting of regular domains. Define the sets G_1, G_2 by

$$G_1 = \bigcup_{\omega \in \mathscr{B}_1} \{(x_1, x_2) \in \omega \times \Omega_2 : \int \hat{v}^2(\cdot, x_2) d\varrho_{x_1}^{\omega} < \int v(\cdot, x_2) d\varrho_{x_1}^{\omega}\},$$

$$G_2 = \bigcup_{\omega \in \mathscr{B}_2} \{(x_1, x_2) \in \Omega_1 \times \omega : \int \hat{v}^1(x_1, \cdot) d\varrho_{x_2}^{\omega} < \int v(x_1, \cdot) d\varrho_{x_2}^{\omega}\}.$$

We have shown each G_i is 2-polar. Define $K = \{x \in \Omega_1 \times \Omega_2 : \hat{v}^1(x) < v(x)$ and $\hat{v}^2(x) < v(x)\}$. K is 2-negligible by the convergence theorem on Ω_1 and Ω_2. Thus $E = K \cup G_1 \cup G_2$ is 2-negligible and we will be done if we can show that for $x = (x_1, x_2)$ in $\Omega_1 \times \Omega_2 - E$, $\hat{v}(x) = v(x)$. Without loss of generality we may assume

$$\hat{v}^1(x) = v(x). \tag{2}$$

There are sequences $\{\omega(i, l)\}_l$ in $\mathscr{B}_i$, $i = 1, 2$, such that $\bar{\omega}(i, l+1) \subset \omega(i, l)$ and $\bigcap_l \omega(i, l) = \{x_i\}$. Now for every k the doubly indexed sequence $\int v_k d(\varrho_{x_1}^{\omega(1,l)} \times \varrho_{x_2}^{\omega(2,m)})$ increases in one index if the other index is fixed. The Monotone Convergence Theorem then gives the same result for v. Therefore

$$\begin{aligned}\hat{v}(x) &= \sup_{l,m} \int v d(\varrho_{x_1}^{\omega(1,l)} \times \varrho_{x_2}^{\omega(2,m)}) \\ &= \sup_l \int d\varrho_{x_1}^{\omega(1,l)} \sup_m \int v d\varrho_{x_2}^{\omega(2,m)} \\ &= \sup_l \int \hat{v}^2(\cdot, x_2) d\varrho_{x_1}^{\omega(1,l)} \\ &\geqq \sup_l \int v(\cdot, x_2) d\varrho_{x_1}^{\omega(1,l)}\end{aligned}$$

[since $\omega(1, l) \in \mathscr{B}_1$ and $x \notin G_1$]

$$\begin{aligned}&= \hat{v}^1(x) \\ &= v(x) \quad \text{[Eq. (2)]}.\end{aligned}$$

This completes the proof of Theorem 4.2 in case v_1 is locally bounded. The general case is deduced precisely as the general case for the analogous theorem on Ω_1 is deduced [3, Theorem 27]. The proof is complete.

Proof of Theorem 4.1. We merely sketch the proof, it being similar to the above. Using Theorem 4.2 we can prove versions of Lemma 4.3 and Corollary 4.6 on $\Omega_1 \times \Omega_2 \times \Omega_3$ where the analogue of the 2-polar set in Lemma 4.3 is a 3-negligible set. Consider now the following notation. For f an extended real valued function on $\Omega_1 \times \Omega_2 \times \Omega_3$ define $\hat{f}^1$ and $\underline{\hat{f}}^1$ on $\Omega_1 \times \Omega_2 \times \Omega_3$ by

$$\hat{f}^1(x_1, x_2, x_3) = \liminf_{z \to x_1} f(z, x_2, x_3), \qquad \underline{\hat{f}}^1(x_1, x_2, x_3) = \liminf_{z \to (x_2, x_3)} f(x_1, z).$$

$\hat{f}^2, \hat{f}^3, \underline{\hat{f}}^2, \underline{\hat{f}}^3$ are also defined in the obvious ways. The functions $\underline{\hat{f}}^1, \underline{\hat{f}}^2, \underline{\hat{f}}^3$ are shown to be Borel measurable precisely as in Lemma 4.4. Now for $i = 1, 2, 3$ let $\mathscr{B}_i$ be a countable base of open sets of Ω_i consisting of regular domains. The set G_1 is defined by

$$G_1 = \bigcup_{\omega \in \mathscr{B}_1} \{(x_1, x_2, x_3) \in \omega \times \Omega_2 \times \Omega_3 : \int \underline{\hat{v}}^1(\cdot, x_2, x_3) d\varrho_{x_1}^{\omega} < \int v(\cdot, x_2, x_3) d\varrho_{x_1}^{\omega}\}$$

and the sets G_2, G_3 defined in the obvious ways as a union over $\mathscr{B}_2, \mathscr{B}_3$ respectively. Put

$$K = \bigcap_{i=1}^{3} \{x \in \Omega_1 \times \Omega_2 \times \Omega_3 : \hat{v}^i(x) < v(x)\}.$$

Using Theorem 4.2 and the techniques involved in its proof each of the sets G_i and K are 3-negligible and one can show that outside of $K \cup \bigcup_{i=1}^{3} G_i$, $\hat{v} = v$. The proof is complete.

Now by using the topological lemma of Choquet [3] we deduce easily the general form of the convergence theorem.

Theorem 4.7. *Let $(v_i) \subset 3 - S(U)$ be a uniformly locally lower bounded family with pointwise infimum v. Then $\hat{v}$ is in $3 - S(U)$ and equals v except on a 3-negligible set.*

We now prove the following extension theorem.

Theorem 4.8. *Let N be a closed 3-negligible subset of an open U and $v \in 3 - S(U - N)$ locally lower bounded near N. Then there exists a unique element of $3 - S(U)$ which equals v on $U - N$.*

A theorem of this type was investigated in a product of Euclidean spaces $\mathfrak{R}^p \times \mathfrak{R}^p$, $p, q \geqq 3$, by Avanissian [0] where he extended 2-harmonic functions across sets of the form $\{(x, y) \in \mathfrak{R}^p \times \mathfrak{R}^q : x = 0\}$. Our methods are quite different.

As in the case of the convergence theorem we begin by proving a similar result on $\Omega_1 \times \Omega_2$.

Theorem 4.9. *Let N be a closed 2-negligible subset of an open set U and $v \in 2 - S(U - N)$ locally lower bounded near N. Then there is a unique element of $2 - S(U)$ which equals v on $U - N$.*

Remark. We first remark that the analogous result on a single Brelot space is easily proved [3].

Observe that by the local property for 2-superharmonic functions we may assume $U = \Omega_1 \times \Omega_2$ and this we do for the remainder of the proof.

Define h on U to be v on $U-N$ and $\liminf\limits_{\substack{z\to x\\ z\in U-N}} v(z)$ for x in N. Note h is Borel measurable. Proposition 3.4 implies that if the existence part of the theorem holds then the extension must be h. This gives the uniqueness assertion. However, we cannot show directly that h is in $2-S(U)$. The idea of the proof is to extend $v(x_1,\cdot)$ hyperharmonically for "most" x_1 using the fact that the theorem holds on Ω_2. This extended function is of course just limit infimum in the last variable. Notice this gives a function which majorizes $h(x_1,\cdot)$.

Proof of Theorem 4.9. Since N is 2-negligible there is a G_δ polar set P such that if z_1 is in $\Omega_1 - P$ $N(1,z_1)$ is polar in Ω_2. Thus, for each such z_1, the mapping $z_2 \to v(z_1,z_2)$ can be extended to one that is hyperharmonic on Ω_2. Thus, by a slight abuse of notation, we will assume v is defined everywhere on U except

$$M = N \cap \{(z_1,z_2)\in U : z_1 \in P\}$$

and, for each z_1 in $\Omega_1 - P$, the mapping $z_2 \to v(z_1,z_2)$ is hyperharmonic on Ω_2. As we observed in the remark this mapping majorizes $h(z_1,\cdot)$ for such z_1.

Since M is contained in $P\times\Omega_2$ it is 2-polar in $\Omega_1\times\Omega_2$. Therefore there exists $u\in 2-S^+(U)$ such that $u=\infty$ on M. Define, for each positive integer k, u_k on U by

$$\begin{aligned} u_k(z) &= v(z)+k^{-1}u(z) \quad && U-M \\ &\infty && M. \end{aligned}$$

We claim u_k is nearly 2-superharmonic on U. Clearly it is locally lower bounded. Let δ_1,δ_2 be regular domains and $x=(x_1,x_2)\in\delta_1\times\delta_2$. We must show that

$$\overline{\int} u_k d(\varrho_{x_1}^{\delta_1}\times\varrho_{x_2}^{\delta_2}) \leqq u_k(x). \tag{1}$$

N is a Borel set and so it follows easily by Fubini's theorem that it has product measure zero. Since u_k equals the locally lower bounded Borel function $h+k^{-1}u$ on $U-N$ (here they are both just $v+k^{-1}u$) we deduce from Fubini's theorem that

$$\begin{aligned} \overline{\int} u_k d(\varrho_{x_1}^{\delta_1}\times\varrho_{x_2}^{\delta_2}) &= \int (h+k^{-1}u)d(\varrho_{x_1}^{\delta_1}\times\varrho_{x_2}^{\delta_2}) \\ &= \int_{z_1\notin P} d\varrho_{x_1}^{\delta_1}(z_1)\int (h+k^{-1}u)d\varrho_{x_2}^{\delta_2} \\ &\leqq \overline{\int_{z_1\notin P}} d\varrho_{x_1}^{\delta_1}(z_1)\int u_k d\varrho_{x_2}^{\delta_2} \\ &\leqq \overline{\int_{z_1\notin P}} u_k(z_1,x_2)d\varrho_{x_1}^{\delta_1}(z_1) \end{aligned}$$

since for $z_1\in\Omega_1-P$ the mapping $z\to u_k(z_1,z)$ is hyperharmonic. Thus (1) will be proved if we can show

$$\overline{\int} u_k(\cdot,x_2)d\varrho_{x_1}^{\delta_1} \leqq u_k(x). \tag{2}$$

Suppose first x is in $U-N$. Since N is 2-negligible there exists a polar set $Q\subset\Omega_2$ such that if z_2 is in Ω_2-Q $N(2,z_2)$ is polar in Ω_1. If x_2 is in Ω_2-Q then (2) holds. Indeed there exists a hyperharmonic function on Ω_1 which equals $z\to u_k(z,x_2)$ except on $N(2,x_2)$, a Borel (closed) set of $\varrho_{x_1}^{\delta_1}$ measure zero. Now use the fact that x_1 is not in $N(2,x_2)$. If x_2 is in Q, since u_k is 2-hyperharmonic on $U-N$, N is closed, and Q is polar, it follows that there is a sequence $\{y_l\}$ in Ω_2 converging to x_2 such that for every l $y_l \notin Q$, $(x_1, y_l) \notin N$, and

$$u_k(x) = \liminf_{l \to \infty} u_k(x_1, y_l).$$

Thus from the previous case we have for each l

$$u_k(x_1, y_l) \geqq \overline{\int_{z \notin P}} u_k(z, y_l) d\varrho_{x_1}^{\delta_1}. \tag{4}$$

Letting $l \to \infty$ gives

$$u_k(x) = \liminf_{l \to \infty} u_k(x_1, y_l) \tag{5}$$

$$\geqq \liminf_{l \to \infty} \overline{\int_{z \notin P}} u_k(z, y_l) d\varrho_{x_1}^{\delta_1}(z) \tag{6}$$

$$\geqq \overline{\int_{z \notin P}} u_k(z, x_2) d\varrho_{x_1}^{\delta_1}(z) \tag{7}$$

by Fatou's lemma and the fact that for $z \notin P$ the mapping $u_k(z, \cdot)$ is lower semicontinuous. Thus (2) holds if x is in $U - N$. If x is in M it is trivial that (2) holds. Finally suppose x is in $N - M$. Then x_1 is not in P and therefore $N(1, x_1)$ is polar. Since $u_k(x_1, \cdot)$ is hyperharmonic there is a sequence $\{y_l\}$ in Ω_2 such that $y_l \notin N(1, x_1)$, $\{y_l\}$ converges to x_2, and $u_k(x) = \liminf_{l \to \infty} u_k(x_1, y_l)$. Since $(x_1, y_l) \notin N$ we may apply a previous case and proceed exactly as in inequalities (4)–(7) to deduce (2) holds. Thus u_k is nearly 2-superharmonic.

Define w on U by $w(x) = \liminf_{k \to \infty} u_k(x)$. Then w is easily seen to be nearly 2-superharmonic and hence $\hat{w}$ is 2-superharmonic and $\hat{w} = v$ on $U - N$. Indeed $\hat{w} = v$ on the subset of $U - N$ where u is finite, that is everywhere on $U - N$ except a 2-polar set, hence a 2-negligible set. Thus by Proposition 3.4 $\hat{w}$ is the required extension and we are done.

With this result Theorem 4.8 can now by proved by first extending $v(x_1, \cdot)$ 2-hyperharmonically for x_1 outside a polar set and then proceeding exactly as in the proof of Theorem 4.9.

Corollary 4.10. *Let h be 3-harmonic on $U - N$ and locally bounded near N. Then h has a unique 3-harmonic extension to U.*

5. Some Applications

We consider now the special case of open subsets of $\mathbb{C}^n$. Although there is no positive potential in $\mathbb{C}$ we still have the notion of a polar set as a set on which a superharmonic function (no longer positive) takes the value ∞. With this definition of polar we can define the 2 and 3-negligible sets exactly as before. One shows easily that a countable union of 3-negligible sets is 3-negligible. With this fact we can extend all properties of 3-negligible sets that are of a local character to this setting. In particular Theorems 4.1 and 4.8 are valid.

Let c denote the interior logarithmic capacity on $\mathbb{C}$. For $E \subset \mathbb{C}^n$ the quantity $\gamma_n(E)$ is defined inductively as follows:

$$\gamma_1(E) = c(E), \quad \gamma_n(E) = c\{z \in \mathbb{C} : \gamma_{n-1}\{z' \in \mathbb{C}^{n-1} : (z, z') \in E\} > 0\}.$$

Ronkin's Γ-capacity $\Gamma_n(E)$ is defined to be

$$\sup\{\gamma_n(\alpha E) : \alpha \text{ a complex unitary transformation of } \mathbb{C}^n\}.$$

Cegrell [5] has proved the following result.

Proposition 5.1. *If E is universally capacitable then*

$$\gamma_n(E) = \overline{\mathrm{cap}_2}\{z \in \mathbb{C} : \gamma_{n-1}\{z' \in \mathbb{C}^{n-1} : (z, z') \in E\} > 0\},$$

where $\overline{\mathrm{cap}_2}$ is the outer logarithmic capacity on $\mathbb{C}$.

We consider now the relationship between the 3-negligible sets and sets of vanishing Ronkin Γ-capacity.

Proposition 5.2. *Let E be Borel with $\Gamma_3(E) = 0$. Then for every complex unitary transformation α of $\mathbb{C}^3$ $\alpha(E)$ is 3-negligible.*

Proof. Observe first that if $F \subset \mathbb{C}^2$ is Borel with $\gamma_2(F) = 0$ and

$$\gamma_2\{(z_1, z_2) \in \mathbb{C}^2 : (z_2, z_1) \in F\} = 0$$

then F is 2-negligible. Suppose α is complex unitary. Let σ be one of the two permutations of $\{2, 3\}$. The mapping $\beta(z_1, z_2, z_3) = (z_1, z_{\sigma 2}, z_{\sigma 3})$ is complex unitary hence $\gamma_3(\beta^{-1}\alpha E) = 0$. Therefore there exists P_σ polar such that if z_1 is in $\mathbb{C} - P_\sigma$

$$\gamma_2\{(z_{\sigma^{-1}2}, z_{\sigma^{-1}3}) \in \mathbb{C}^2 : (z_1, z_2, z_3) \in \alpha E\} = 0.$$

Put $Q_1 = \bigcup_\sigma P_\sigma$. By the observation at the beginning of the proof we have that if z_1 is in $\mathbb{C} - Q_1$ $\{(z_2, z_3) \in \mathbb{C}^2 : (z_1, z_2, z_3) \in \alpha E\}$ is 2-negligible. Now consider $\beta_1(z_1, z_2, z_3) = (z_2, z_1, z_3)$. Since $\beta_1\alpha$ is unitary, from what we have just seen there exists a polar set Q_2 such that if z_1 is in $\mathbb{C} - Q_2$ $\{(z_2, z_3) \in \mathbb{C}^2 : (z_1, z_2, z_3) \in \beta_1\alpha E\}$ is 2-negligible. By relabelling we see that for z_2 in $\mathbb{C} - Q_2$ $\{(z_1, z_3) \in \mathbb{C}^2 : (z_1, z_2, z_3) \in \alpha E\}$ is 2-negligible. Finally, by taking $\beta_2(z_1, z_2, z_3) = (z_1, z_3, z_2)$, as above we see there exists Q_3 polar such that if z_3 is in $\mathbb{C} - Q_3$ $\{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2, z_3) \in \alpha E\}$ is 2-negligible. This completes the proof.

The converse is trivial to prove. Note that it does not require E to be Borel.

Proposition 5.3. *Let E be a subset of $\mathbb{C}^3$. If E is 3-negligible then $\gamma_3(E) = 0$. Consequently, if for every complex unitary transformation α $\alpha(E)$ is 3-negligible, $\Gamma_3(E) = 0$.*

Denote the plurisuperharmonic functions on $U \subset \mathbb{C}^3$ by $P\,\mathrm{Sup}(U)$. Clearly they are contained in $3 - S(U)$. We shall prove a convergence theorem for decreasing sequences of such functions. The first such result was proved by Lelong [12] with the additional requirement that the regularized limit function be pluriharmonic. In this case the exceptional set is pluripolar. The result we shall show was proved by Ronkin [13] without additional assumption. Following the work of Favorov [6] Cegrell also proved this result using a general theory of product capacity. For a stronger version of this theorem see Bedford and Taylor [1].

Theorem 5.4. *Let $U \subset \mathbb{C}^3$ be open and $(v_k) \subset P\,\mathrm{Sup}(U)$ a decreasing sequence that is uniformly locally lower bounded. Then the limit function v differs from $\hat{v}$ at most on a set of vanishing Ronkin Γ-capacity.*

Proof. Let $E = \{z \in U : \hat{v}(z) < v(z)\}$ and α any complex unitary transformation of $\mathbb{C}^3$. Then by Theorem 4.1 $\alpha^{-1}E = \{z \in \alpha^{-1}U : \hat{v}\alpha(z) < v\alpha(z)\} = \{z \in \alpha^{-1}U : \widehat{v\alpha}(z) < v\alpha(z)\}$ is 3-negligible. Thus by Proposition 5.3 $\Gamma_3(E) = 0$ and we are done.

Finally we prove an extension theorem in this setting. It was originally proved by Cegrell [4].

Theorem 5.5. *Let $U \subset \mathbb{C}^3$ be open, $E \subset U$ closed, and $v \in P\,\mathrm{Sup}(U-E)$ locally lower bounded near E. Then there exists a unique w in $P\,\mathrm{Sup}(U)$ such that $w = v$ on $U - E$.*

Proof. The uniqueness follows immediately from Proposition 3.4.

We know from Theorem 4.8 that there is at least an extension w in $3-S(U)$. We will show for every complex unitary transformation α of $\mathbb{C}^3$ that $w\alpha$ is in $3-S(\alpha^{-1}U)$. Since $w\alpha$ is lower semicontinuous it is enough to show, for every $x = (x_1, x_2, x_3)$ in $\alpha^{-1}U$ and $\delta_1, \delta_2, \delta_3$ regular neighbourhoods of x_1, x_2, x_3 respectively with $\bar{\delta}_1 \times \bar{\delta}_2 \times \bar{\delta}_3 \subset \alpha^{-1}U$, that

$$\iiint w\alpha\, d\varrho_{x_1}^{\delta_1} d\varrho_{x_2}^{\delta_2} d\varrho_{x_3}^{\delta_3} \leqq w\alpha(x).$$

Suppose first x is in $\alpha^{-1}U - \alpha^{-1}E$. By Theorem 4.8 there exists $v_1 \in 3-S(\alpha^{-1}U)$ which equals $v\alpha$ on $\alpha^{-1}U - \alpha^{-1}E$. It follows

$$\begin{aligned}\iiint w\alpha\, d\varrho_{x_1}^{\delta_1} d\varrho_{x_2}^{\delta_2} d\varrho_{x_3}^{\delta_3} &= \iiint v_1\, d\varrho_{x_1}^{\delta_1} d\varrho_{x_2}^{\delta_2} d\varrho_{x_3}^{\delta_3} \\ &\leqq v_1(x) \\ &= (v\alpha)(x) \\ &= (w\alpha)(x).\end{aligned}$$

In general there is a sequence $\{y^k\} = \{(y^{1,k}, y^{2,k}, y^{3,k})\}$ in U converging to $\alpha(x)$ such that y^k is in $U - E$ and $w(\alpha(x)) = \liminf_{k \to \infty} w(y^k)$. Put $x^k = \alpha^{-1}(y^k)$. By the previous case

$$\iiint w\alpha\, d\varrho_{x^{k,1}}^{\delta_1} d\varrho_{x^{k,3}}^{\delta_2} d\varrho_{x^{k,3}}^{\delta_3} \leqq w\alpha(x^k)$$

and letting $k \to \infty$ gives the same result for x. Thus $w\alpha$ is in $3-S(\alpha^{-1}U)$. It follows w is plurisuperharmonic. The proof is complete.

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