Carleson measures on a homogeneous tree

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Abstract

We introduce the notion of s-Carleson measure ($s \geq 1$) on a homogeneous tree T and give several characterizations of such measures. In particular, we prove the following discrete version of the extension of Carleson's theorem due to Duren.

For $p > 1$ and $s \geq 1$, a finite measure σ on T is s-Carleson if and only if there exists $C > 0$ such that for all $f \in L^p(\partial T)$,

$$\|Pf\|_{L^{sp}(\sigma)} \leq C\|f\|_{L^p(\partial T)},$$

where Pf denotes the Poisson integral of f.

Here, $L^p(\sigma)$ is the space of functions g defined on T such that $|g|^p$ is integrable with respect to σ and $L^p(\partial T)$ is the space of functions f defined on the boundary of T such that $|f|^p$ is integrable with respect to the representing measure of the harmonic function 1.

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1. Introduction

For $s \geq 1$, a positive measure σ on the open unit disk $\mathbb{D}$ in the complex plane is said to be an s-*Carleson measure* if there exists $C > 0$ such that for

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each $\theta_0 \in \mathbb{R}$ and $h \in (0,1)$,

$$\sigma(S_{\theta_0}) \leq Ch^s,$$

where

$$S_{\theta_0} := \{re^{i\theta} : 1-h \leq r < 1, |\theta - \theta_0| \leq h/2\}.$$

Sets of the form S_{θ_0} are commonly referred to as *Carleson squares.*

In [3], Carleson proved that, for $p > 1$, a positive measure σ is 1-Carleson if and only if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |Pf(z)|^p \, d\sigma(z) \leq C\|f\|_p^p \quad \text{for all } f \in L^p(\partial\mathbb{D}),$$

where Pf is the Poisson integral of f and $\|f\|_p^p = \int_{\partial\mathbb{D}} |f(\zeta)|^p d\lambda(\zeta)$, where λ is the normalized Lebesgue measure on ∂D. In [4], Carleson obtained a similar characterization for functions in the Hardy space H^p for $0 < p < \infty$. In [8], Duren extended the latter result as follows.

Theorem 1.1. [8] *Let σ be a finite measure on $\mathbb{D}$ and let $0 < p \leq q < \infty$. Then the following conditions are equivalent:*

(a) *There exists $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\sigma(z) \right)^{1/q} \leq C\|f\|_p, \quad \textit{for each } f \in H^p. \tag{1}$$

(b) *σ is a q/p-Carleson measure.*

Setting $s = q/p$, (1) can be stated as

$$\|f\|_{L^{sp}(\sigma)} \leq C\|f\|_p, \quad \text{for each } f \in H^p,$$

which is thus equivalent to σ being an s-Carleson measure.

There is an extensive literature on Carleson measures in one and several variables. Carleson measures play a very important role in function theory and operator theory. Such measures were introduced by Carleson to obtain a solution to the corona problem [4]. More recently, Carleson measures have been used to characterize the elements of some Banach spaces of analytic functions such as the Bloch space [16], the space of analytic functions of bounded mean oscillation BMOA and its subspace of functions of vanishing mean oscillation VMOA [11].

Carleson measures are typically defined in terms of some geometric property and then used (as in Theorem 1, for the case of the Hardy spaces) to characterize the boundedness of certain operators between related function spaces. Indeed, Carleson measures have been used to obtain analogues of (a) in Theorem 1 for the classical Dirichlet space [14], for the Bloch space [12], for the analytic Besov spaces [1], and for the Bergman spaces [6].

In this work, we develop a discrete notion of Carleson measure as well as Carleson-type theorems in the setting of a homogeneous tree. While this study has its own intrinsic interest, it is part of an ongoing trend of solving classical problems in a discrete setting, which, due to its combinatorial nature, often yields some useful insights on how to tackle classical open problems, and provides new examples for further analysis.

In [15] and [13], the authors studied the Hardy spaces of harmonic functions on a tree using martingales on the boundary. Moreover, on a large class of trees T (not necessarily homogeneous) endowed with a transient nearest-neighbor transition probability, in [7] two notions of Hardy spaces H^p_N and H^p_A $(0 < p < \infty)$ were given in terms of two functions on the boundary corresponding to a given harmonic function F on T, the nontangential maximal function and the area function of F. The space H^p_N (respectively, H^p_A) was defined as the set of harmonic functions F on T whose nontangential maximal function (respectively, whose area function) is in L^p relative to the representing measure of the harmonic function 1. The authors provided a Green formula and showed that these two notions yield the same Hardy space.

The study of Carleson-type measures in a discrete setting is not novel. For example, in [2], motivated by the work of Stegenga in [14], the authors developed the theory of Carleson measures on a Hilbert space of functions on a dyadic tree, which they call the dyadic Dirichlet space.

1.1. Organization of the paper

After giving some preliminary definitions and notation on trees, in Section 2, we prove a covering lemma that will be used in Section 3 to prove the main result of the paper (Theorem 3.2). We define the harmonic Hardy space $\mathcal{H}^p$ on a homogeneous tree T of degree $q+1$ as the space of harmonic functions h on T such that the average of $|h|^p$ on the set of vertices of length n is bounded. Furthermore, we characterize the elements of $\mathcal{H}^p$ in terms of the corresponding radial limit function on the boundary ∂T of T, in terms of its radial maximal function h^* on ∂T, and in terms of the existence of a harmonic majorant of $|h|^p$ (see Theorem 2.3). We also show the equivalence between the norm of a function f in $L^p(\partial T)$ and the norm in $\mathcal{H}^p$ of its Poisson integral (see Theorem 2.5). The approach we adopt is very different from those in [15], [13], and [7].

In Section 3, we define the notion of s-Carleson measure on a homogeneous tree and, in Theorem 3.2, we provide several characterizations of such a measure, including the discrete analogue of Duren's generalization stated in the introduction.

The methods used in our work are only valid for $1 \leq p < \infty$. In the classical case treated in [8], the proof for the case $0 < p < 1$ is based on the factorization of functions in H^p into a Blaschke product and a nonvanishing function in H^p. In the discrete setting, no such factorization exists and entirely new methods will need to be developed to extend the results to $0 < p < 1$.

1.2. Homogeneous trees

By a *tree* T we mean a locally finite, connected, and simply-connected graph, which, as a set, we identify with the collection of its vertices. Two vertices u and v are called *neighbors* if there is an edge connecting them, and we use the notation $u \sim v$. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \dots]$ such that $v_k \sim v_{k+1}$, for all k. It is a *geodesic path* if in addition $v_{k-1} \neq v_{k+1}$, for all k. For any pair of vertices u, v there is a unique geodesic path from u to v, which we denote by $[u, v]$.

Given a tree T rooted at e and a vertex $u \in T$, a vertex v is called a *descendant* of u if u lies in the unique path from e to v. The vertex u is then called an *ancestor* of v. Given a vertex $v \neq e$, we denote by v^- the unique neighbor which is an ancestor of v. For $v \in T$, The set S_v consisting of v and all its descendants is called the *sector* determined by v. The sectors are the sets of vertices that will play the role of Carleson squares in the tree setting.

Define the *length* of the finite path $[u_0, u_1, \dots, u_n]$ to be n. The *distance*, $d(u, v)$, between vertices u and v is the length of the geodesic path $[u, v]$.

The tree T is a metric space under the distance d. Fixing e as the root of the tree, we define the *length* of a vertex v, by $|v| = d(e, v)$. We shall also adopt the notation $|v - w|$ for the distance between vertices v and w.

A tree is termed *homogeneous* of degree $q + 1$ (with $q \in \mathbb{N}$) if all its vertices have $q + 1$ neighbors. The number of vertices of T of length n is

$$c_n = \begin{cases} (q+1)q^{n-1} & \text{if } n \geq 1, \\ \quad 1 & \text{if } n = 0. \end{cases}$$

The *boundary* ∂T of T is the set of infinite geodesic paths ω of the form $[e = \omega_0, \omega_1, \omega_2, \dots]$. We denote by $[e, \omega)$ the set of vertices in the path ω. Then, ∂T is a compact space under the topology generated by the sets

$$I_v = \{\omega \in \partial T : v \in [e, \omega)\},$$

which yields a compactification of T. Clearly, $\partial T = I_e$. Furthermore, for any $n \in \mathbb{N}$, ∂T is the disjoint union of the sets I_v over the vertices v of length n. Under this topology, $\partial S_v = I_v$ for each $v \in T$.

For $v \in T$ with $0 \leq n \leq |v|$, define v_n to be the vertex of length n in the path $[e, v]$.

Define a partial order $\leq$ on $T \cup \partial T$ as follows: For $v \in T$ and $x \in T \cup \partial T$, $v \leq x$ if $v \in [e, x]$. Since $T \cup \partial T$ has e as the minimum, for any $x, y \in T \cup \partial T$ the greatest lower bound of x and y is well defined. We denote this greatest lower bound by $x \wedge y$.

We assume throughout that T is a homogeneous tree of degree $q+1$ ($q \geq 2$) rooted at e. The case $q = 1$ is not considered here because on a homogeneous tree of degree 2 there is no notion of Poisson integral, which is instrumental in our study.

By a *function on a tree* we mean a real-valued function on the set of its vertices.

The Laplacian operator Δ is defined as the averaging operator minus the identity operator: for a function f on T,

$$\Delta f(v) = \frac{1}{q+1} \sum_{w \sim v} f(w) - f(v), \quad v \in T.$$

A function f on T is *harmonic* if Δf is identically 0.

For $v \in T$, $\omega \in \partial T$ the Poisson kernel is given by $P_\omega(v) = q^{2|\omega \wedge v| - |v|}$. Note that $P_\omega(e) = 1$. Every positive harmonic function on T can be written as

$$P\mu(v) := \int_{\partial T} P_\omega(v)\, d\mu(\omega)$$

for a unique Borel measure μ on ∂T.

We let m denote the measure on ∂T for which

$$m(I_v) = \frac{1}{c_{|v|}}, \text{ for } v \in T.$$

If μ is absolutely continuous with respect to m with density function f, we write Pf instead of $P\mu$. In particular, m is the representing measure of the harmonic function 1, and thus is the discrete analogue of the Lebesgue measure on the unit circle.

Observe that if h is harmonic on T, then

$$\sum_{|w|=n} h(w) = c_n h(e), \; n \geq 1. \tag{2}$$

This is proved by induction on n as follows. For $n = 1$, (2) is true by definition of harmonicity at e. Assume (2) holds for all $k \in \{1, \dots, n\}$, where n is a fixed positive integer. Then

$$\begin{aligned}(q+1)c_n h(e) &= (q+1) \sum_{|v|=n} h(v) = \sum_{|v|=n} \sum_{w^-=v} h(w) + \sum_{|v|=n} h(v^-) \\ &= \sum_{|w|=n+1} h(w) + q \sum_{|w|=n-1} h(w) = \sum_{|w|=n+1} h(w) + qc_{n-1}h(e),\end{aligned}$$

which implies

$$\sum_{|w|=n+1} h(w) = (q+1)c_n h(e) - qc_{n-1}h(e) = qc_n h(e) = c_{n+1}h(e),$$

as desired.

For a general reference on trees, see [5].

Throughout the paper, we shall use C to denote a constant which may differ from one occurrence to the next, but which at most depends on q and the given $p \geq 1$.

2. The harmonic Hardy space $\mathcal{H}^p$

For $1 \leq p < \infty$ we let $L^p(\partial T)$ denote the functions $f : \partial T \to [-\infty, \infty]$ such that $|f|^p$ is m-integrable.

For $p \geq 1$ and a function f on T, let $M_p(f,n)$ be the average value of $|f|^p$ over the vertices of length n, namely,

$$M_p(f,n) = \frac{\sum_{|v|=n} |f(v)|^p}{(q+1)q^{n-1}}.$$

We now define the harmonic Hardy space $\mathcal{H}^p$ on T for $p \geq 1$.

Definition 2.1. Let $1 \leq p < \infty$ and let h be harmonic on T. Then $h \in \mathcal{H}^p$ provided that

$$\|h\|^p_{\mathcal{H}^p} := \sup_{n\in\mathbb{N}} M_p(h,n) < \infty.$$

Definition 2.2. Let $f \in L^1(\partial T)$. The *Hardy-Littlewood maximal function* of f is the function Mf on ∂T defined as

$$Mf(\omega) = \sup_{\{v\in T:\, \omega\in I_v\}} \frac{\int_{I_v} |f(\tau)|\, dm(\tau)}{m(I_v)} = \sup_{\{v\in T:\, \omega\in I_v\}} (q+1)q^{|v|-1} \int_{I_v} |f(\tau)|\, dm(\tau).$$

The following result will be needed to prove Theorem 2.1 below and Theorem 3.2.

Lemma 2.1. (Covering Lemma) *Let $A \subseteq T$. Then there exists $\widehat{A} \subseteq A$ such that $\bigcup_{v\in\widehat{A}} S_v = \bigcup_{v\in A} S_v$, $\bigcup_{v\in\widehat{A}} I_v = \bigcup_{v\in A} I_v$, and for each pair of distinct vertices $v, w \in \widehat{A}$, $S_v \cap S_w = \emptyset$ and $I_v \cap I_w = \emptyset$.*

Proof. For $v \in T$, denote by v_k the ancestor of v of length k, for $k = 0, \dots, n = |v|$. Let $\widehat{A}$ be the set of vertices $v \in A$ none of whose ancestors are in A. Since $\widehat{A}$ is a subset of A, it is clear that

$$\bigcup_{v\in\widehat{A}} S_v \subseteq \bigcup_{v\in A} S_v.$$

To prove the opposite inclusion, fix $v \in A$ and let k be the minimum integer i such that $v_i \in A$. Then $S_v \subseteq S_{v_k}$ and, by definition of $\widehat{A}$, $v_k \in \widehat{A}$. Therefore,

$$\bigcup_{v\in\widehat{A}} S_v = \bigcup_{v\in A} S_v.$$

Now let $v, w \in \widehat{A}$ be such that $S_v \cap S_w \neq \emptyset$. Then either $S_v \subseteq S_w$ or $S_w \subseteq S_v$. By the defining property of $\widehat{A}$, we deduce that $v = w$. Consequently, $\bigcup_{v\in\widehat{A}} S_v$ is a disjoint union. The assertions concerning I_v follows from those for S_v. $\square$

Theorem 2.1. *Let $1 \leq p \leq \infty$ and $f \in L^p(\partial T)$. Then $Mf < \infty$ m-a.e. and the following inequalities hold:*

(a) *If $p = 1$, then for every $\lambda > 0$,*

$$m\{\omega \in \partial T :\, Mf(\omega) > \lambda\} \leq \frac{1}{\lambda}\|f\|_{L^1(\partial T)}.$$

(b) *If* $1 < p \leq \infty$, *there exists a constant* $C > 0$ *such that for each* $f \in L^p(\partial T)$,

$$\|Mf\|_{L^p(\partial T)} \leq C\|f\|_{L^p(\partial T)}.$$

Proof. Let $f \in L^1(\partial T)$ and $\lambda > 0$. Let $U = \{\omega \in \partial T : Mf(\omega) > \lambda\}$. If $\omega \in U$, then there exists $v = v_\omega \in T$ with $\omega \in I_v$ and

$$\int_{I_v} |f(\tau)|dm(\tau) > \lambda\, m(I_v).$$

It follows that $\omega \in I_v \subseteq U$, so U is open and $U = \bigcup_{\omega \in U} I_{v_\omega}$. Let $A = \{v_\omega : \omega \in U\}$. Define $\widehat{A}$ as in Lemma 2.1. Then

$$m(U) = \sum_{v \in \widehat{A}} m(I_v) \leq \frac{1}{\lambda} \sum_{v \in \widehat{A}} \int_{I_v} |f|dm \leq \frac{1}{\lambda}\|f\|_{L^1},$$

since $\{I_v : v \in \widehat{A}\}$ is a disjoint union. The finiteness assertion concerning Mf follows from the inequality in (a).

Using the fact that (a) is true and the obvious fact that (b) holds in case $p = \infty$, the general result in (b) follows from the Marcinkiewicz interpolation theorem [9]. □

Definition 2.3. Let h be harmonic on T. The *radial maximal function* of h is the function h^* defined on ∂T by $h^*(\omega) = \sup_n |h(\omega_n)|$.

Theorem 2.2. *For every* $f \in L^1(\partial T)$ *and* $\omega \in \partial T$,

$$(Pf)^*(\omega) \leq \frac{q^2}{q^2-1}\, Mf(\omega).$$

Proof. Let $f \in L^1(\partial T)$ and $\omega \in \partial T$. Let $v = \omega_n$ for some $n \geq 0$. Let $v_0, v_1, \ldots, v_n$ be the vertices of $[e, v]$. Then

$$\begin{aligned}
|Pf(v)| &\leq \int q^{2|v\wedge\eta|-|v|}|f(\eta)|\, dm(\eta) \\
&\leq \sum_{k=0}^{n} q^{2k-n} \int_{I_{v_k}} |f(\eta)|dm(\eta) \\
&= \sum_{k=0}^{n} q\, \frac{q^{k-n}}{q+1}(q+1)q^{k-1} \int_{I_{v_k}} |f(\eta)|\, dm(\eta) \\
&\leq q\, Mf(\omega) \sum_{k=0}^{n} \frac{q^{k-n}}{q+1} \\
&\leq \frac{q^2}{q^2-1} Mf(\omega).
\end{aligned}$$

Taking the supremum over all such v yields the result. □

Our next aim is to characterize the elements of $\mathcal{H}^p$. We first prove two lemmas.

Lemma 2.2. *For a fixed $n \geq 1$, let h be harmonic on $\{v \in T : |v| \leq n\}$ and let f_n be the function defined on ∂T by*

$$f_n = \sum_{|v|=n} \left(\frac{qh(v) - h(v^-)}{q-1} \right) \chi_{I_v}, \tag{3}$$

where χ_E denotes the characteristic function of the set E. Then $Pf_n = h$ on $|v| \leq n$.

Proof. Extend h harmonically to T so that for each v with $|v| = n$, h is radial on S_v. Fix v such that $|v| = n$. Then for $w \in S_v$ we have

$$h(w) = \frac{qh(v) - h(v^-)}{q-1} - \left(\frac{h(v) - h(v^-)}{q-1} \right) q^{-(|w|-|v|)}.$$

Indeed, it is straightforward to verify that the above function is harmonic and radial on S_v. It is clear that h is bounded and its representing measure is absolutely continuous with respect to m. By the Fatou radial limit theorem, the density of the representing measure is m-a.e. equal to the radial limit function of h. From the above formula this is just $\dfrac{qh(v) - h(v^-)}{q-1}$ on I_v for $|v| = n$. The result follows at once. $\square$

Theorem 2.3. *For a harmonic function h on T and $1 < p < \infty$, the following propositions are equivalent:*
(a) $h \in \mathcal{H}^p$.
(b) $h = Pf$ *for some function* $f \in L^p(\partial T)$.
(c) $\|h^*\|_{L^p(\partial T)} < \infty$.
(d) $|h|^p$ *has a harmonic majorant.*

Proof. (a) $\Longrightarrow$ (b): Suppose first that $h \in \mathcal{H}^p$. For $n \in \mathbb{N}$, let f_n be as in Lemma 2.2. Since

$$\sum_{|v|=n} |h(v^-)|^p = q \sum_{|v|=n-1} |h(v)|^p,$$

then

$$\begin{aligned} \int |f_n(\omega)|^p dm(\omega) &= \sum_{|v|=n} \int_{I_v} \left| \frac{qh(v) - h(v^-)}{q-1} \right|^p dm(\omega) \\ &\leq \sum_{|v|=n} C \frac{[|h(v)|^p + |h(v^-)|^p]}{(q+1)q^{n-1}} \leq C \, \|h\|_{H^p}^p. \end{aligned} \tag{4}$$

Thus the sequence $\{f_n\}$ is bounded in the $L^p(\partial T)$ norm, so viewing $L^p(\partial T)$ as the dual of $L^q(\partial T)$ where $\frac{1}{p} + \frac{1}{q} = 1$, it follows from the Banach-Alaoglu Theorem that there is a subsequence $\{f_{n_j}\}$ that converges in the weak-$*$ topology to some function $f \in L^p(\partial T)$.

Let $v \in T$. Since the mapping $\omega \mapsto P_\omega(v)$ is bounded, it is in $L^p(\partial T)$, so it follows $Pf_{n_j}(v) \to Pf(v)$ as $j \to \infty$. But by Lemma 2.2, $Pf_{n_j}(v) = h(v)$ for all $n_j \geq |v|$. Therefore, $h = Pf$.

(b) $\Longrightarrow$ (c): Assume $h = Pf$ for some $f \in L^p(\partial T)$. Then by Theorem 2.2 and by part (b) of Theorem 2.1, it follows that

$$\|h^*\|_{L^p(\partial T)} \leq C\|Mf\|_{L^p(\partial T)} \leq C\|f\|_{L^p(\partial T)}.$$

(c) $\Longrightarrow$ (d): Assume $h^* \in L^p(\partial T)$. Then for $n \in \mathbb{N}$,

$$\begin{aligned}\|h^*\|^p_{L^p(\partial T)} &= \int_{\partial T} |h^*(\omega)|^p \, dm(\omega) = \sum_{|v|=n} \int_{I_v} |h^*(\omega)|^p \, dm(\omega) \\ &\geq \frac{q}{q+1} \sum_{|v|=n} |h(v)|^p q^{-|v|} = M_p(h, n).\end{aligned}$$

Taking the supremum over all such n, we obtain that $\|h\|_{\mathcal{H}^p}$ is finite.

For each $n \in \mathbb{N}$, let g_n be the solution to the Dirichlet problem on $\{|v| \leq n\}$ with boundary values $|h|^p$ on $\{|v| = n\}$. Since $|h|^p$ is subharmonic, it follows from the Minimum Principle that $|h|^p \leq g_n$ on $\{|v| \leq n\}$ and g_n increases with n. On the other hand, $g_n(e)$ is the average value of $|h|^p$ on $\{|v| = n\}$, so it is bounded above by $\|h\|^p_{\mathcal{H}^p}$. By Harnack's Theorem, $\lim\limits_{n\to\infty} g_n$ is harmonic, and it is a majorant of $|h|^p$.

(d) $\Longrightarrow$ (a): Let g be a harmonic majorant of $|h|^p$. Then, by (2), for $n \in \mathbb{N}$,

$$M_p(h, n) = \frac{\sum_{|v|=n} |h(v)|^p}{(q+1)q^{n-1}} \leq \frac{\sum_{|v|=n} g(v)}{(q+1)q^{n-1}} = g(e).$$

The result follows by taking the supremum over all $n \in \mathbb{N}$. □

The case of $p = 1$ is contained in the following

Theorem 2.4. *Let h be harmonic. Then $h \in \mathcal{H}^1$ if and only if $h = P(\mu)$ for some signed measure μ on ∂T.*

Proof. The proof is similar to the proof of the equivalence of (a) and (b) in the previous theorem. The only modification is to replace the use of the Banach-Alaoglu Theorem with the fact that a sequence of signed measures bounded in total variation has a weakly convergent subsequence. □

Theorem 2.5. *There exists $C > 0$ such that for all $f \in L^p(\partial T)$,*

$$C\|f\|_{L^p(\partial T)} \leq \|Pf\|_{\mathcal{H}^p} \leq \|f\|_{L^p(\partial T)}.$$

Proof. First observe that for $n \in \mathbb{N}$, and $f \in L^p(\partial T)$, by Jensen's Inequality and (2),

$$\begin{aligned}
\sum_{|v|=n} |Pf(v)|^p &= \sum_{|v|=n} \left| \int P_\omega(v) f(\omega) dm(\omega) \right|^p \\
&\leq \sum_{|v|=n} \int P_\omega(v) |f(\omega)|^p dm(\omega) \\
&= \int \left(\sum_{|v|=n} P_\omega(v) \right) |f(\omega)|^p dm(\omega) \\
&= (q+1)q^{n-1} \int |f(\omega)|^p dm(\omega).
\end{aligned}$$

Thus, $M_p(Pf, n) \leq \|f\|^p_{L^p(\partial T)}$. The upper estimate follows by taking the supremum over all $n \in \mathbb{N}$.

Next, note that for $n \in \mathbb{N}$ and $\omega \in \partial T$, formula (3) with $h = Pf$, can be written as

$$f_n(\omega) = \frac{q\, Pf(\omega_n) - Pf(\omega_{n-1})}{q-1}.$$

Since Pf has radial limits almost everywhere equal to f, it follows that f_n converges pointwise a.e. to f. By (4) and Fatou's lemma, we obtain

$$\|f\|^p_{L^p(\partial T)} = \int \liminf_{n\to\infty} |f_n(\omega)|^p \, dm(\omega) \leq \liminf_{n\to\infty} \int |f_n(\omega)|^p \, dm(\omega) \leq C\|Pf\|^p_{\mathcal{H}^p},$$

which yields the lower estimate. □

3. Discrete version of Duren's extension of Carleson's theorem

We begin the section by giving the notion of Carleson-type measures on the tree.

Definition 3.1. For $s \geq 1$, a measure σ on T is said to be an s-Carleson measure if there exists $C > 0$ such that for all $v \in T$,

$$\sigma(S_v) \leq C(m(I_v))^s.$$

The following lemma is the main tool in the proof of Theorem 3.1, which will be essential for the proof of our main result.

Lemma 3.1. *Given $m \in \mathbb{N}$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$, $\mu_1, \ldots, \mu_m > 0$, and $s > 0$,*

$$\sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})(\mu_k + \cdots + \mu_m)^s \leq \left(\sum_{k=1}^{m} \alpha_k^{1/s} \mu_k \right)^s, \tag{5}$$

with equality occurring only if $m = 1$.

Proof. The proof is based on the use of the multinomial formula

$$(x_1 + \cdots + x_m)^s = \sum \binom{s}{i_1, i_2, \ldots, i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},$$

where the sum is taken over all $i_1, \ldots, i_{m-1}$ non-negative integers, $i_m = s - (i_1 + \cdots + i_{m-1})$, and

$$\binom{s}{i_1, i_2, \ldots, i_m} := \frac{s(s-1)\ldots(i_m+1)}{i_1! i_2! \ldots i_{m-1}!}.$$

Both sides of (5) will be infinite (formal) sums of coefficients in $\alpha_1, \ldots, \alpha_m$ times $\binom{s}{i_1, i_2, \ldots, i_m} \mu_1^{i_1} \mu_2^{i_2} \cdots \mu_m^{i_m}$. Since the sums under consideration have only nonnegative terms, it will be sufficient to show that the coefficients in $\alpha_1, \ldots, \alpha_m$ satisfy the same inequality to be shown for the respective sums. Let $i_1, \ldots, i_m$ be fixed and let $t = \min\{j : i_j > 0\}$. Thus, we are looking at the coefficient of

$$\binom{s}{i_t, i_{t+1}, \ldots, i_m} \mu_t^{i_t} \cdots \mu_m^{i_m}.$$

On the left-hand side the coefficient is $\sum\limits_{k=1}^{t} (\alpha_k - \alpha_{k-1}) = \alpha_t$. On the right-hand side the coefficient is $\alpha_t^{i_t/s} \alpha_{t+1}^{i_{(t+1)/s}} \ldots \alpha_m^{i_{m/s}}$. Since $\alpha_{t+j} \geq \alpha_t$ for $j \geq 0$ and $i_t + \cdots + i_m = s$, this product is no smaller than

$$\alpha_t^{i_t/s} \alpha_t^{i_{(t+1)/s}} \ldots \alpha_t^{i_{m/s}} = \alpha_t^{(i_t + \cdots + i_m)/s} = \alpha_t.$$

(In fact, it is greater than α_t unless $i_t = s$.) Thus, (5) is established and the inequality is an equality only if $m = 1$. □

As a consequence, we deduce the following result.

Proposition 3.1. *Let $p > 1$, $s \geq 1$, $m \in \mathbb{N}$, $0 = a_0 < a_1 < \cdots < a_m$, and $\mu_1, \ldots, \mu_m > 0$. Then*

$$\sum_{k=1}^{m} (a_k^{sp} - a_{k-1}^{sp})(\mu_k + \cdots + \mu_m)^s \leq \left(\sum_{k=1}^{m} a_k^p \mu_k \right)^s.$$

Proof. Set $\alpha_k = a_k^{sp}$, for $k = 1, \ldots, m$. Then $a_k^p = (a_k^{sp})^{1/s} = \alpha_k^{1/s}$. The result follows at once from Lemma 3.1. □

Theorem 3.1. *Let (Ω, μ) be a finite measure space, $p > 1$, $s \geq 1$, and g a nonnegative measurable function on Ω. Then*

$$\int_0^\infty sp\lambda^{sp-1} \mu\{\omega \in \Omega : g(\omega) > \lambda\}^s \, d\lambda \leq \left(\int_\Omega g^p \, d\mu \right)^s.$$

Proof. We first show the result for g a simple function. Suppose $g = \sum_{k=1}^{m} a_k \chi_{E_k}$, where $0 = a_0 < a_1 < \cdots < a_m$ and $E_1, \ldots, E_m$ are pairwise disjoint measurable sets. Set $\mu_k = \mu(E_k)$, for $k = 1, \ldots m$. Then, using Proposition 3.1, we obtain

$$\begin{aligned}
\int_0^\infty sp\lambda^{sp-1}\mu\{\omega : g(\omega) > \lambda\}^s \, d\lambda &= \sum_{k=1}^{m} \int_{a_{k-1}}^{a_k} sp\lambda^{sp-1} \left(\sum_{j=k}^{m} \mu(E_j) \right)^s d\lambda \\
&= \sum_{k=1}^{m} (a_k^{sp} - a_{k-1}^{sp})(\mu_k + \cdots + \mu_m)^s \\
&\leq \left(\sum_{k=1}^{m} a_k^p \mu_k \right)^s \\
&= \left(\int_\Omega g^p \, d\mu \right)^s,
\end{aligned}$$

proving the result in this case. The general case is proved by choosing a sequence $\{g_n\}$ of simple functions increasing pointwise to g and applying the Monotone Convergence Theorem. □

We are now ready to prove the main result of this paper.

Theorem 3.2. *Let $1 < p < \infty$, $1 \leq s < \infty$, and σ a finite measure on T. Then the following statements are equivalent.*

(a) *σ is an s-Carleson measure.*

(b) *There exists $C > 0$ such that for all harmonic functions h and $\lambda > 0$,*

$$\sigma\{v \in T : |h(v)| > \lambda\} \leq C \, (m\{\omega : h^*(\omega) > \lambda\})^s.$$

(c) *For all $f \in L^p(\partial T)$, $Pf \in L^{sp}(\sigma)$.*

(d) *There exists $C > 0$ such that for all $f \in L^p(\partial T)$,*

$$\|Pf\|_{L^{sp}(\sigma)} \leq C \|f\|_{L^p(\partial T)}.$$

(e) *There exists $C > 0$ such that for all $f \in L^1(\partial T)$ and $\lambda > 0$,*

$$\sigma(\{v \in T : |Pf(v)| > \lambda\}) \leq \frac{C}{\lambda^s} \|f\|_{L^1(\partial T)}^s.$$

(f) $\sup_{v \in T} \sum_{w \in T} q^{s(|w| - |v-w|)} \sigma(\{w\}) < \infty.$

Proof. We first prove (a) and (b) are equivalent. Suppose first that σ is an s-Carleson measure and let h be harmonic. For $\lambda > 0$, define $A = \{v : |h(v)| > \lambda\}$ and let $\widehat{A}$ be as in Lemma 2.1. Then, noting that for $\omega \in I_v$ with $v \in A$,

$h^*(\omega) \geq |h(v)| > \lambda$, we obtain

$$\begin{aligned}
\sigma\{v : |h(v)| > \lambda\} &\leq \sigma\left(\bigcup_{v \in \widehat{A}} S_v\right) = \sum_{v \in \widehat{A}} \sigma(S_v) \leq C \sum_{v \in \widehat{A}} (m(I_v))^s \\
&\leq C\left(\sum_{v \in \widehat{A}} m(I_v)\right)^s = C\left(m\left(\bigcup_{v \in \widehat{A}} I_v\right)\right)^s \\
&= C\left(m\left(\bigcup_{v \in A} I_v\right)\right)^s \leq C\,(m\{\omega : h^*(\omega) > \lambda\})^s,
\end{aligned}$$

proving (b).

Conversely, suppose that σ is a measure on T satisfying (b). Fix $v \in T$ and let $n = |v|$, $f = \chi_{I_v}$, and $h = Pf$. If $u \in S_v$, then

$$h(u) = \int_{I_v} q^{2|\omega \wedge u| - |u|}\, dm(\omega) \geq \int_{I_u} q^{2|\omega \wedge u| - |u|}\, dm(\omega) = q^{|u|} m(I_u) = \frac{q}{q+1};$$

if $u \notin S_v$, then $|u \wedge v| = k$ for some $k < n$, and

$$h(u) = \int_{I_v} q^{2|\omega \wedge u| - |u|}\, dm(\omega) = q^{2k - |u|} m(I_v) \leq q^{2k-k} m(I_v) = \frac{q^{k+1-n}}{q+1} \leq \frac{1}{q+1}.$$

Defining $\lambda = 1/(q+1)$, we have shown that $S_v = \{u \in T : h(u) > \lambda\}$ and $\{\omega : h^*(\omega) > \lambda\} = I_v$. Thus by the condition in (b), we obtain $\sigma(S_v) \leq C\,(m(I_v))^s$, proving that σ is an s- Carleson measure. This completes the proof of the equivalence of (a) and (b).

It is obvious that (d) implies (c). To show that (c) implies (d), we make use of the Closed Graph Theorem. By assumption P is a linear operator from $L^p(\partial T)$ to $L^{sp}(\sigma)$. Let f_n be a sequence converging in $L^p(\partial T)$ to a function f, and let Pf_n converge in $L^{sp}(\sigma)$ to a function g. We will be done if we show that $g = Pf$ σ-a.e. Fix $v \in T$. Then, by Hölder's inequality, we have

$$\begin{aligned}
|Pf_n(v) - Pf(v)| &\leq \int q^{2|\omega \wedge v| - |v|} |f_n(\omega) - f(\omega)|\, dm(\omega) \\
&\leq q^{|v|} \|f_n - f\|_{L^1(\partial T)} \\
&\leq q^{|v|} \|f_n - f\|_{L^p(\partial T)}.
\end{aligned}$$

Letting n go to ∞, we see that Pf_n converges pointwise to Pf. On the other hand, Pf_n converges in $L^{sp}(\sigma)$ to g, so some subsequence converges pointwise σ-a.e. to g. Therefore $g = Pf$ σ-a.e.

To show that (b) implies (d), we use the fact that for any finite Borel measure μ, $1 \leq p < \infty$ and g a μ-measurable function,

$$\|g\|^p_{L^p(\mu)} = \int_0^\infty p\lambda^{p-1} \mu\{x : |g(x)| > \lambda\}\, d\lambda,$$

which follows from Fubini's theorem. Thus, if $f \in L^p(\partial T)$ and $h = Pf$, then by the hypothesis, the appropriate change of variables and Theorems 2.2, 3.1 with $\mu = m$, and 2.1, we obtain

$$\begin{aligned}
\|h\|^{sp}_{L^{sp}(\sigma)} &= \int_0^\infty p\lambda^{p-1}\sigma\{v : |h(v)|^s > \lambda\}\, d\lambda \\
&= \int_0^\infty p\lambda^{p-1}\sigma\{v : |h(v)| > \lambda^{1/s}\}\, d\lambda \\
&\leq C\int_0^\infty p\lambda^{p-1}(m\{\omega :\, h^*(\omega) > \lambda^{1/s}\})^s\, d\lambda \\
&= C\int_0^\infty sp\lambda^{sp-1}(m\{\omega :\, h^*(\omega) > \lambda\})^s\, d\lambda \\
&\leq C\int_0^\infty sp\lambda^{sp-1}(m\{\omega : Mf(\omega) > \lambda\})^s\, d\lambda \\
&\leq C\|Mf\|^{sp}_{L^p(\partial T)} \\
&\leq C\|f\|^{sp}_{L^p(\partial T)},
\end{aligned}$$

proving (d).

To prove that (d) implies (a), fix $v \in T$ and take $f = \chi_{I_v}$ and $h = Pf$. As we showed in the proof of (b) implies (a), $S_v = \{u \in T : h(u) > 1/(q+1)\}$ and $I_v = \{\omega : h^*(\omega) > 1/(q+1)\}$, so

$$\begin{aligned}
\left(\frac{1}{q+1}\right)^{sp}\sigma(S_v) &\leq \int_{S_v} |Pf|^{sp}\, d\sigma \leq \int_T |Pf|^{sp}\, d\sigma \\
&\leq C\|f\|^{sp}_{L^p(\partial T)} = C\,(m(I_v))^s,
\end{aligned}$$

proving σ is an s - Carleson measure.

Next, (b) implies (e) follows by applying Theorem 2.2 and part (a) of Theorem 2.1. The proof of (e) implies (a) is done by applying the inequality in (e) to $f = \chi_{I_v}$.

Finally, we prove that (a) and (f) are equivalent. Assume (a) holds. Let $C > 0$ be such that $\sigma(S_v) \leq Cq^{-s|v|}$ for each $v \in T$. Fix $v \in T$ and let $n = |v|$. Let $v_0 = e, v_1, \ldots, v_n = v$ be the sequence of vertices of the path $[o, v]$ with $|v_k| = k$ for $k = 0, \ldots, n$. Then we may decompose T into the disjoint union of the sets $W_k = S_{v_k} \backslash S_{v_{k+1}}$ $(0 \leq k \leq n-1)$ and S_v. Thus,

$$\begin{aligned}
\sum_{w\in T} q^{s(|w|-|v-w|)}\sigma(\{w\}) &= \sum_{k=0}^{n-1} q^{s(2k-n)}\sigma(W_k) + q^{sn}\sigma(S_v) \\
&\leq q^{-sn}\sum_{k=0}^{n-1} q^{2sk}\sigma(S_{v_k}) + C \\
&\leq Cq^{-sn}\sum_{k=0}^{n-1} q^{sk} + C \\
&\leq C.
\end{aligned}$$

Condition (f) follows by taking the supremum over all $v \in T$.

Conversely, suppose (f) holds and let B be the supremum in (f). Let $v \in T$. Then, noting that for $w \in S_v$, $|w| - |v - w| = |v|$, we obtain

$$B \geq \sum_{w \in S_v} q^{s(|w|-|v-w|)} \sigma(\{w\}) = \sum_{w \in S_v} q^{s|v|} \sigma(\{w\}) = q^{s|v|} \sigma(S_v),$$

completing the proof. $\square$

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