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CARLESON MEASURES FOR NON-NEGATIVE SUBHARMONIC FUNCTIONS ON HOMOGENEOUS TREES

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Abstract. In [3], we introduced several classes of Carleson-type measures with respect to a radial reference measure σ on a homogeneous tree T, equipped with the nearest-neighbor transition operator and studied their relationships under certain assumptions on σ. We defined two classes of measures σ we called *good* and *optimal* and showed that if σ is optimal and μ is a σ-Carleson measure on T in the sense that there is a constant C such that the μ measure of every sector is bounded by C times the σ measure of the sector, then there exists $C_\mu > 0$ such that $\sum f(v)\,\mu(v) \leqslant C_\mu \sum f(v)\,\sigma(v)$ for every non-negative subharmonic function f on T, and we conjectured that this holds if and only if σ is good.

In this paper we develop tools for studying the above conjecture and identify conditions on a class of non-negative subharmonic functions for which we can prove the conjecture for all functions in such a class. We show that these conditions hold for the set of all non-negative subharmonic functions which are generated by eigenfunctions of the Laplacian on T.

1. Introduction and preliminary results

Let T be a homogeneous tree rooted at a vertex o equipped with the isotropic nearest neighbor transition probability $1/(q+1)$ where $q+1$ is the *degree* of T. As a set, we identify T as its collection of vertices. Two vertices v and w are called *neighbors*, in which case we write $v \sim w$, if there is an edge connecting them. We denote by $[v,w]$ the unique geodesic path joining v to w and by $|v-w|$ the number of edges in $[v,w]$. We use the notation $|v|$ for the length of the path $[o,v]$, that is, the number of edges in $[o,v]$, which we call the *length of* v. In this paper, all paths will be geodesic paths. An infinite path will be called a *ray*. If $v \in [o,w] \setminus \{w\}$, we call v an *ancestor* of w and w a *descendant* of v. If, in addition, $v \sim w$, we call v the *parent* of w and w a *child* of v and use the notation w^- for the parent of w. For every vertex v, the *sector determined by* v is the set $S(v)$ consisting of v and all its descendants, that is, all vertices u such that $u \geqslant v$, in the sense that $v \in [o,u]$. Note that $S(o) = T$.

Following the guidelines of Hastings, Cima & Wogen and Luecking [4, 2, 5] for Bergman spaces on the disk, the polydisk or the ball in $\mathbb{C}^n$, in [3], we studied Carleson measures with respect to a reference measure σ on a homogeneous tree T, namely:

Definition 1.1. A *reference measure* is a radial positive decreasing function σ on T, such that $\|\sigma\| < \infty$, where

$$\|\sigma\| = \|\sigma\|_{\ell^1(T)} = \sigma_0 + (q+1)\sum_{k=1}^{\infty} q^{k-1}\sigma_k,$$

having denoted by σ_k the value of σ at each of the vertices of length k.

Given a reference measure σ, a positive measure μ on T is said to be σ-*Carleson* if there is a positive constant C such that $\mu(S(v)) \leqslant C\sigma(S(v))$ for each $v \in T$.

Carleson measures on disks, balls or polydisks are defined analogously with respect to the Lebesgue measure, and the problems that were considered therein correspond, in the environment of a tree, to *optimal* reference measures, namely:

Definition 1.2. A reference measure σ is *optimal* if, up to a constant factor, $\sigma(S(v))$ is bounded by $\sigma(v)$ for every vertex v.

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We call a reference measure σ *good* if

$$\sum_{v\in T}\sigma(S(v))<\infty.$$

In particular, every optimal measure is good.

If σ is optimal and μ is σ-Carleson, then it was proved in [3] that there exists a constant $C>0$ such that

$$\sum_{v\in T} f(v)\,\mu(v)\leqslant C\sum_{v\in T} f(v)\,\sigma(v) \text{ for every } \ f\geqslant 0 \text{ subharmonic,} \tag{1.1}$$

and our goal was to show that (1.1) holds for a more general class of reference measures σ. It was conjectured in [3] that (1.1) holds if and only if σ is good. This conjecture was shown to be valid for radial Carleson measures, and also for all Carleson measures when restricting our attention to non-negative subharmonic functions supported on finitely many geodesic rays.

This paper is a follow-up research on the work done in [3]. While the conjecture remains an open problem in the tree setting, as well as in the continuous environment, the analysis provided in this paper highlights the technical difficulties associated with this problem and gives alternative objectives by focusing on large classes of non-negative subharmonic functions. Before summarizing our main results, we give all needed background.

Denote by ∂T the *boundary* of T, that is the set of all rays $\omega=[o,\omega_1,\omega_2,\dots)$, where $|\omega_k|=k$ for each $k\in\mathbb{N}$. For $\omega\in\partial T$, denote by $K_\omega(v):=K(v,\omega)$ the Poisson kernel normalized to have the value 1 at the root o. Recall that [1, 6, 7]

$$K(v,\omega)=q^{2|v\wedge\omega|-|v|},$$

where $v\wedge\omega$ denotes the vertex of maximum length on ω that belongs to the path $[o,v]$.

For any vertex v, denote by $I(v)\subset\partial T$ the set of all rays starting at o and containing v (if $v=o$, let $I(o)=\partial T$). The open subsets of the boundary of T are generated by $\{I(v):v\in T\}$.

Definition 1.3. A function $f:T\to\mathbb{R}$ is called *subharmonic*, (respectively, *harmonic*) if the average value of f at the neighbors of each vertex v is at least (respectively, equal to) $f(v)$. Equivalently, f is subharmonic at $v\in T$ if the Laplacian at v, $\Delta f(v)$, is nonnegative, where for each $v\in T$

$$\Delta f(v):=\frac{1}{q+1}\sum_{w\sim v}f(w)-f(v).$$

We denote by $\mathcal{F}_+$ be the set of all non-negative functions on T, and by $\mathcal{S}_+$ (respectively, $\mathcal{H}_+$) the set of non-negative subharmonic (respectively, harmonic) functions on T.

Definition 1.4. Let $\mathcal{G}$ denote any subset of the set of non-negative functions on T. A finite measure μ on T is called a $(\mathcal{G},\sigma)$-*Carleson measure* if there exists a positive constant $C=C_{\mu,\mathcal{G}}$ such that for all $f\in\mathcal{G}$,

$$\sum_{v\in T} f(v)\mu(v)\leqslant C\sum_{v\in T} f(v)\sigma(v). \tag{1.2}$$

Let $\mathcal{M}_{(\mathcal{G},\sigma)}$ denote the set of $(\mathcal{G},\sigma)$-Carleson measures on T.

Examples of sets of $(\mathcal{G},\sigma)$-Carleson measures are

$$\begin{aligned}\mathcal{M}_{(\mathcal{F}_+,\sigma)}&=\{\mu:\mu \text{ is } (\mathcal{F}_+,\,\sigma)\text{-Carleson}\}\\ \mathcal{M}_{(\mathcal{S}_+,\sigma)}&=\{\mu:\mu \text{ is } (\mathcal{S}_+,\,\sigma)\text{-Carleson}\}\\ \mathcal{M}_{(\mathcal{H}_+,\sigma)}&=\{\mu:\mu \text{ is } (\mathcal{H}_+,\,\sigma)\text{-Carleson}\}\end{aligned}$$

It is evident that $\mathcal{M}_{(\mathcal{F}_+,\sigma)}\subset\mathcal{M}_{(\mathcal{S}_+,\sigma)}\subset\mathcal{M}_{(\mathcal{H}_+,\sigma)}$.

Definition 1.5. Let μ be any positive measure. We associate to μ the *μ-sectorial measure* τ^μ defined by $\tau^\mu(v) = \mu(S_v)$. In case μ is a fixed reference measure σ, we denote τ^σ by τ and write τ_n for $\tau(v)$ where $|v|$ is any vertex of length n. A reference measure σ is called *optimal* if $\sup_n \frac{\tau_n}{\sigma_n} < \infty$.

We say that μ is *good* if τ^μ is a finite measure. In particular, σ is good if and only if $\sum_{n=0}^\infty q^n \tau_n$ is finite. But note that the finiteness of $\|\sigma\|$ implies that $q^n \tau_n \to 0$ whether or not σ is good.

Let σ be a reference measure and let $\mathcal{M}_\sigma$ denote the set of σ-Carleson measures on T. In [3], we proved the following result.

Theorem 1.1. [3] (i) $\mathcal{M}_{(\mathcal{S}_+,\sigma)} \subset \mathcal{M}_\sigma$.

(ii) $\mathcal{M}_{(\mathcal{H}_+,\sigma)} \not\subset \mathcal{M}_\sigma$.

(iii) $\mathcal{M}_\sigma = \mathcal{M}_{(\mathcal{F}_+,\sigma)}$ *if and only if σ is optimal.*

(iv) *If σ is optimal, then* $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{S}_+,\sigma)}$.

(v) $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{H}_+,\sigma)}$ *if and only if σ is good.*

The following conjecture was stated in [3].

Conjecture: $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{S}_+,\sigma)}$ for every good measure σ.

In that work, we proved that each radial measure $\mu \in \mathcal{M}_\sigma$ is in $\mathcal{M}_{(\mathcal{S}_+,\sigma)}$. In this paper, rather than focusing on all of $\mathcal{S}_+$, we consider sets $\mathcal{G}$ of non-negative subharmonic functions and seek conditions on $\mathcal{G}$ for which the conjecture holds with $\mathcal{S}_+$ replaced by $\mathcal{G}$. This involves producing sets $\mathcal{G}$ of non-negative subharmonic functions for which we can prove $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{G},\sigma)}$ for every good measure σ. That means we wish to see when we can prove inequalities of the form (1.2) for all f in some set $\mathcal{G}$ of non-negative subharmonic functions, where σ is a given reference measure, μ is a σ-Carleson measure and C is a constant independent of f.

In Definition 5.1 we give a general condition on $\mathcal{G}$ and in Theorem 5.1 prove that for such a set $\mathcal{G}$, $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{G},\sigma)}$ for every good measure σ. As an application of Theorem 5.1, we prove in Theorem 7.1 that the theory works for the class of non-negative subharmonic functions generated by powers of the Poisson kernel, which includes all of the non-negative subharmonic eigenfunctions of the Laplacian.

The paper is structured as follows. In Section 2, we analyze the properties of good and optimal measures and introduce two related classes of measures, the *suboptimal* and the *mesa* measures. In Section 3, we provide recipes for constructing examples of σ-Carleson measures. In section 4, we study non-negative subharmonic functions and the role that their (discrete) derivative plays to construct them. As described above, in Section 5 we prove our main result, Theorem 5.1. In section 6, motivated by the main result in section 7.1, we provide a succinct characterization of the non-negative subharmonic eigenfunctions of the Laplacian. Finally, in Section 7, we apply Theorem 5.1 to the classes of non-negative subharmonic eigenfunctions of the Laplacian and of the radial non-negative subharmonic functions. We conclude the paper with two additional examples.

2. Reference measures: good, optimal, suboptimal, and mesa

In this section we discuss general properties of good and optimal reference measures, and we introduce and study suboptimal reference measures and mesa reference measures.

2.1. General comments on good and optimal measures.

We begin by giving a useful characterization of good measures.

Theorem 2.1. *A finite measure μ on T is good if and only if $\sum_{v\in T} |v|\mu(v) < \infty$. A reference measure σ is good if and only if $\sum_{m\geqslant 0} m\,\sigma_m\, q^m < \infty$.*

Proof. The first part follows at once from the following chain of equalities:

$$\sum_{w\neq o} \tau^\mu(w) = \sum_{w\neq o} \mu(S(w)) = \sum_{w\neq o}\sum_{v\geqslant w} \mu(v) = \sum_{v\in T}\sum_{o\neq w\leqslant v} \mu(v) = \sum_{v\in T} |v|\mu(v). \tag{2.1}$$

In particular, taking μ to be reference measure σ, σ is good if and only if

$$\infty > \sum_{v\in T} |v|\sigma(v) = \sum_{m=1}^{\infty} m(q+1)q^{m-1}\sigma_m = \frac{q+1}{q}\sum_{m=0}^{\infty} m\, q^m\, \sigma_m. \qquad \square$$

Theorem 2.1 allows us to give concrete examples of reference measures of various types.

Corollary 2.2. (i) *Let $\varepsilon > 0$. Then σ defined by $\sigma_n = q^{-(1+\varepsilon)n}$ or $\sigma_n = (q+\varepsilon)^{-n}$ is optimal.*

(ii) *Let $\varepsilon \in \mathbb{R}$. Then σ defined by $\sigma_n = q^{-n}n^{-(1+\varepsilon)}$ is a reference measure if $\varepsilon > 0$, is not good if $0 < \varepsilon \leqslant 1$, and is good but not optimal if $\varepsilon > 1$.*

Proof. Part (i) follows from

$$\tau_n = \sum_{k=0}^{\infty} \sigma_{n+k}q^k = \sum_{m=n}^{\infty} \sigma_m q^{m-n} \tag{2.2}$$

and summing the appropriate geometric series. For $\sigma_n = q^{-n}n^{-(1+\varepsilon)}$, we see that σ is good if and only if $\varepsilon > 1$ using Theorem 2.1. If $\varepsilon > 1$, then it follows from (2.2) and the integral test of infinite series that τ_n is of the order $q^{-n}n^{-\varepsilon}$, and so τ_n/σ_n is of the order n, proving that σ is not optimal. $\square$

We refer the reader to [3], Theorems 3.1 and 3.2, to see how to construct many examples of reference measures which are optimal, or good but not optimal.

Let τ be the sectorial measure associated with σ as in Definition 1.5. A non-optimal reference measure σ has the property that if a_n is defined by $\tau_n = a_n\sigma_n$, then $\limsup_{n\to\infty} a_n = \infty$. The next result and its corollary deal with the relation between σ_n, a_n, and τ_n. We only consider the particular case that $a_n \to \infty$.

Theorem 2.3. *Let σ be a reference measure. If $\tau_n = a_n\sigma_n$, where $a_n \to \infty$, then*

$$\sigma_n = \frac{a_1\sigma_1}{q^{n-1}a_n}\prod_{j=1}^{n-1}\Big(1-\frac{1}{a_j}\Big),$$

where, as customary, the value of an empty product is 1. Consequently, for all $n \geqslant 1$,

$$\sum_{k=0}^{\infty}\frac{1}{a_{n+k}}\prod_{j=n}^{n+k-1}\Big(1-\frac{1}{a_j}\Big) = 1. \tag{2.3}$$

Proof. We have $a_n\sigma_n = \tau_n = \sigma_n + q\tau_{n+1} = \sigma_n + qa_{n+1}\sigma_{n+1}$, from which we get $\sigma_{n+1} = \frac{(a_n-1)\sigma_n}{qa_{n+1}}$. The first formula of the statement now follows by induction. Thus

$$\frac{a_1\sigma_1}{q^{n-1}}\prod_{j=1}^{n-1}\Big(1-\frac{1}{a_j}\Big) = a_n\sigma_n = \tau_n = \sum_{k=0}^{\infty}\sigma_{n+k}q^k = \sum_{k=0}^{\infty}\frac{a_1\sigma_1 q^k}{q^{n+k-1}a_{n+k}}\prod_{j=1}^{n+k-1}\Big(1-\frac{1}{a_j}\Big).$$

Simplifying the right-hand side and cancelling common factors from this and the left-hand side gives the desired formula. $\square$

Corollary 2.4. *If $\sum_{n=1}^{\infty}\frac{1}{a_n} < \infty$, then it is not the case that $\tau_n = a_n\sigma_n$. In particular, there is no reference measure σ such that $\tau_n = b_n n^{1+\alpha}\sigma_n$, where $\alpha > 0$ and $\{b_n\}$ is a positive sequence bounded away from 0.*

Proof. Assume that $\tau_n = a_n\sigma_n$, where $\sum_{k=0}^{\infty}\frac{1}{a_k} < \infty$. Then (2.3) holds but $\sum_{k=0}^{\infty}\frac{1}{a_{n+k}} \to 0$ as $n \to \infty$. A contradiction results from (2.3), since each factor $1-1/a_j$ is less than 1, however, for sufficiently large n, $\sum_{k=0}^{\infty}\frac{1}{a_{n+k}} < 1$. $\square$

2.2. **Suboptimal reference measures and associated power series.**

Definition 2.1. A reference measure σ is said to be *suboptimal* if there exists an optimal reference measure σ' such that $\sup_n \frac{\sigma_n}{\sigma'_n} < \infty$.

To any reference measure σ we can associate the power series $\sum_{n=0}^{\infty} \sigma_n z^n$. In the theorem below, we show that suboptimality is characterized by a condition on the radius of convergence of this power series. To this end, we first give a result for optimal reference measures.

Theorem 2.5. *Let σ be an optimal reference measure with non-negative non-increasing coefficients σ_n, let $C = \sup_n \frac{\tau_n}{\sigma_n}$, where τ is the associated sectorial measure. Then the radius of convergence of the series $\sum \sigma_n z^n$ is at least $\alpha := qC/(C-1)$.*

Proof. First observe that $C = \alpha/(\alpha - q)$. Moreover, $C\sigma_n \geqslant \tau_n = \sigma_n + q\tau_{n+1}$. Therefore, $q\sigma_n/(\alpha - q) = (C-1)\sigma_n \geqslant q\tau_{n+1}$. Hence

$$\sigma_n \geqslant (\alpha - q)\tau_{n+1}. \tag{2.4}$$

Now, $\tau_{n+1} = \sigma_{n+1} + q\,\tau_{n+2}$. Therefore, applying (2.4) with n replaced by $n+1$, we obtain

$$\sigma_n \geqslant (\alpha - q)(\sigma_{n+1} + q\,\tau_{n+2}) \geqslant (\alpha - q)((\alpha - q)\tau_{n+2} + q\,\tau_{n+2}) = \alpha(\alpha - q)\tau_{n+2}\,.$$

Arguing inductively, for $m \in \mathbb{N}$,

$$\sigma_n \geqslant \alpha^m(\alpha - q)\tau_{n+m+1} \geqslant \alpha^m(\alpha - q)\sigma_{n+m+1}\,.$$

In particular, for $n = 0$, we obtain

$$\sigma_{m+1} \leqslant \frac{\alpha^{-m}\sigma_0}{\alpha - q}.$$

Hence, the sequence $\{\sigma_m\}$ is bounded by a multiple of α^{-m}, proving the result. $\square$

Theorem 2.6. *Let σ be a reference measure with non-negative non-increasing coefficients σ_n, and let τ be the associated sectorial measure. Then σ is suboptimal if and only if the series $\sum \sigma_n z^n$ has radius of convergence $R > q$.*

Proof. Suppose first that $\sum \sigma_n z^n$ has radius of convergence $R > q$ and let $r \in (q, R)$. Then $\sum \sigma_n r^n$ converges, so for all n sufficiently large, $\sigma_n r^n < 1$. Thus, we can find $C > 0$ such that $\sigma_n \leqslant C/r^n$ for all $n \in \mathbb{N}$. Since the reference measure σ' defined by $\sigma'_n := 1/r^n$ is optimal, it follows that the measure σ is suboptimal.

The converse follows immediately from Theorem 2.5 since the radius of convergence of the associated series of a suboptimal measure σ must be no smaller than that of any optimal measure σ' satisfying $\sigma_n \leqslant C\sigma'_n$ for all $n \in \mathbb{N}$. $\square$

Corollary 2.7. *If σ is suboptimal, then σ is good.*

Proof. It is evident that a measure that is less than a constant multiple of a good (respectively, suboptimal) measure is good (respectively, suboptimal). So it follows that if a measure is suboptimal, it is less than a constant multiple of some optimal measure, and since optimal measures are good, the original measure must also be good. $\square$

2.3. **Mesa measures.** Mesa measures are constructed from a given reference measure. A step function is created out of the reference measure by sampling it a certain jump points, with a flat "mesa" between successive jumps, and the mesa measure is determined by this step function. We show below that it is never optimal if the lengths of successive "mesas" diverge to ∞.

Definition 2.2. Let n_j be an increasing sequence of integers such that $n_0 = 0$ and $n_{j+1} - n_j > 1$, for $j \geqslant 0$, and let ρ be a reference measure. A *mesa measure on T generated by ρ* is a radial function σ defined by $\sigma_0 = \rho_0$ and $\sigma_m = \rho_{n_j}$ for all integers m with $n_{j-1} < m \leqslant n_j$.

If ρ is optimal, then any mesa measure generated by ρ is necessarily suboptimal.

Theorem 2.8. *The mesa measure generated by a reference measure ρ and a sequence n_j such that $n_{j+1} - n_j \to \infty$ is not optimal.*

Proof. Fix $j \in \mathbb{N}$ and integer l such that $1 \leqslant l \leqslant n_{j+1} - n_j$. Let v_{n_j+1} denote a vertex of length $n_j + 1$. The number of vertices v in $S(v_{n_j+l})$ with $n_j + l \leqslant |v| \leqslant n_{j+1}$ is

$$1 + q + q^2 + \cdots + q^{n_{j+1}-(n_j+l)} = \frac{q^{n_{j+1}-n_j-l+1} - 1}{q-1},$$

which is bounded above by $\frac{q^{n_{j+1}-n_j-l+1}}{q-1}$ and bounded below by $\frac{q^{n_{j+1}-n_j-l+1}}{q}$, and so it is of the order $q^{n_{j+1}-n_j-l+1}$, by which we mean the ratio is bounded above and below by positive constants depending only on q. There are $q^{n_{j+1}-n_j-l+1}$ vertices in $S(v_{n_j+l})$ of length $n_{j+1}+1$, so the number of vertices v in $S(v_{n_j+l})$ with $n_{j+1} < |v| \leqslant n_{j+2}$ is of the order $q^{n_{j+1}-n_j-l+1} \cdot q^{n_{j+2}-n_{j+1}} = q^{n_{j+2}-n_j-l+1}$.

The number of vertices v in $S(v_{n_j+l})$ of length $n_{j+2}+1$ is $q^{n_{j+2}-n_j-l+1}$, so the number of vertices v in $S(v_{n_j+l})$ with $n_{j+2} < |v| \leqslant n_{j+3}$ is of the order $q^{n_{j+2}-n_j-l+1} \cdot q^{n_{j+3}-n_{j+2}} = q^{n_{j+3}-n_j-l+1}$. More generally, the number of vertices v in $S(v_{n_j+l})$ with $n_{j+k} < |v| \leqslant n_{j+k+1}$ is of the order $q^{n_{j+k+1}-n_j-l+1}$. Thus

$$\tau_{n_j+l} \approx \sum_{k=1}^{\infty} \rho_{n_{j+k}} q^{n_{j+k}-n_j-l+1}. \tag{2.5}$$

Taking $l = 1$ and just the first term in the sum, we deduce that $\tau_{n_j+1}/\rho_{n_j+1} \gtrsim q^{n_{j+1}-n_j} \to \infty$ as $j \to \infty$. This proves that the mesa measure is not optimal. □

Example 2.1. *By Corollary 2.2, if $\beta > 1$, then the measure $\sigma_n = q^{-\beta n}$ is an optimal measure. More generally let $\{\beta_j\}$ be any sequence converging to $\beta > 1$. Let $\{n_j\}$ be an increasing sequence of integers with $n_0 = 0$. Let σ be the mesa measure such that $\sigma(v) = q^{-\beta_j|v|}$ for $n_{j-1} < |v| \leqslant n_j$. We can choose β' such that $1 < \beta' < \beta$ and $\beta' < \beta_j$ for all sufficiently large j. For such j, $q^{-\beta' n_j} > q^{-\beta_j n_j}$, so $\sigma_n < q^{-\beta' n}$ for sufficiently large n. Thus, since $\beta' > 1$, σ is suboptimal. If $n_j - n_{j-1} \to \infty$ as $j \to \infty$, then by Theorem 2.8, σ is not optimal.*

In the next theorem, we show that it is possible to obtain optimal measures of arbitrarily rapid decay, and in the corollary we make use of mesa measures to show the same holds for good non-optimal reference measures.

Theorem 2.9. *Given any sequence of positive numbers $\{\varepsilon_n\}$ converging to 0, there exists an optimal reference measure σ such that $\sigma_n \leqslant \varepsilon_n$ for all $n \in \mathbb{N}$.*

Proof. Let $\sigma_0 = \varepsilon_0$ and for $n \geqslant 0$, define $\sigma_{n+1} = \min\left\{\varepsilon_{n+1}, \sigma_n/q^2\right\}$. Then, $\{\sigma_n\}$ is decreasing and, arguing inductively, for each $n, k \geqslant 0$, $\sigma_{n+k} \leqslant \sigma_n/q^{2k}$. Thus

$$\tau_n = \sum_{k=0}^{\infty} \sigma_{n+k} q^k \leqslant \sigma_n \sum_{k=0}^{\infty} \frac{1}{q^k} = \frac{\sigma_n}{1 - \frac{1}{q}},$$

proving that σ is optimal. Finally, by construction, $\sigma_n \leqslant \varepsilon_n$. □

Corollary 2.10. *For any function $f : T \to [0, \infty)$, there exists an optimal measure σ such that*

$$\sum_{v \in T} \tau(v) f(v) < \infty, \tag{2.6}$$

where τ is the sectorial measure associated with σ. There also exists a good non-optimal measure σ satisfying (2.6).

Proof. Let $f_n := \sum_{|v|=n} f(v)$ and for $m \geqslant 0$, define $s_m := q^m \sum_{n=0}^m f_n q^{-n}$. Let $\{\varepsilon_m\}$ be a sequence converging to 0 such that $\sum_m \varepsilon_m s_m < \infty$. Choose an optimal measure σ as in Theorem 2.9 corresponding to $\{\varepsilon_m\}$. Then

$$\sum_{v\in T} \tau(v) f(v) = \sum_{n=0}^{\infty} \tau_n f_n = \sum_{n=0}^{\infty} \left(\sum_{m=n}^{\infty} \sigma_m q^{m-n} \right) f_n = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} f_n q^{-n} \right) q^m \sigma_m = \sum_{m=0}^{\infty} s_m \sigma_m < \infty.$$

For the last part, apply Theorem 2.8 with reference measure σ to obtain a measure which is less or equal to σ (and so satisfies (2.6) and by definition is suboptimal) and is not optimal. □

The following immediate consequence of Theorems 2.8 and 2.9 shows that the property of optimality of reference measures is not preserved by order.

Corollary 2.11. *If $\sigma_1 \leqslant \sigma_2$, with σ_2 optimal, then σ_1 need not be optimal, but there is some optimal measure σ_1 such that $0 < \sigma_1 \leqslant \sigma_2$.*

Our next result also makes use of mesa measures.

Theorem 2.12. *Let σ be a reference measure. If σ is suboptimal, then*

$$\liminf_{n\to\infty} \frac{\tau_n}{\sigma_n} < \infty, \tag{2.7}$$

but the converse is false. Moreover, there are good reference measures σ that fail to satisfy (2.7) (and hence are not suboptimal).

Proof. Assume first that σ is suboptimal. Thus, there exists an optimal measure $\hat{\sigma}$ and and a sequence $\{\varepsilon_n\}$ in $(0,1]$ such that $\sigma_n = \varepsilon_n \hat{\sigma}_n$, for all $n \in \mathbb{N}$. Let $\hat{\tau}$ be the associated sectorial measure. Thus, $\hat{\tau}_n = \sum_{k=0}^{\infty} \hat{\sigma}_{n+k} q^k$, for $n \in \mathbb{N}$. By the optimality of $\hat{\sigma}$, $\frac{\hat{\tau}_n}{\hat{\sigma}_n} = \sum_{k=0}^{\infty} \frac{\hat{\sigma}_{n+k} q^k}{\hat{\sigma}_n}$ is bounded, and

$$\frac{\tau_n}{\sigma_n} = \sum_{k=0}^{\infty} \frac{\varepsilon_{n+k} \hat{\sigma}_{n+k} q^k}{\varepsilon_n \hat{\sigma}_n}. \tag{2.8}$$

Suppose there is a subsequence $\{\varepsilon_{n_j}\}$ converging to some $\varepsilon > 0$. Then ε_{n_j} is bounded away from 0. Since $\varepsilon_{n_j+k} \leqslant 1$, the ratio $\varepsilon_{n_j+k}/\varepsilon_{n_j}$ is bounded above by some $C > 0$, and so

$$\frac{\tau_{n_j}}{\sigma_{n_j}} \leqslant C \sum_{k=0}^{\infty} \frac{\hat{\sigma}_{n_j+k} q^k}{\hat{\sigma}_{n_j}},$$

which is bounded. Therefore, (2.7) holds in this case.

On the other hand, if there is no subsequence of $\{\varepsilon_n\}$ converging inside $(0,1]$, then $\{\varepsilon_n\}$ must converge to 0. Thus, for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$, $i_n \geqslant n$, such that $\varepsilon_{i_n} = \max_{j\geqslant n} \varepsilon_j$. Then $\varepsilon_{i_n+k}/\varepsilon_{i_n} \leqslant 1$, so from (2.8) with n replaced by i_n, we see that $\frac{\tau_{i_n}}{\sigma_{i_n}} \leqslant \sum_{k=0}^{\infty} \frac{\hat{\sigma}_{i_n+k} q^k}{\hat{\sigma}_{i_n}}$, proving that (2.7) holds also in this case.

We next provide an example of a non-suboptimal reference measure σ for which (2.7) holds. For $n \in \mathbb{N}$, let $\rho_n = q^{-n} e^{-\sqrt{n}}$, and let $n_j = j^2$. Let σ be the associated mesa measure. Then

$$\sum_n n \rho_n q^n = \sum_n n e^{-\sqrt{n}} < \infty,$$

so by Theorem 2.1, ρ, and hence σ, is a good measure. For any $\varepsilon > 0$,

$$\sum_n \sigma_n q^n (1+\varepsilon)^n \geqslant \sum_j \rho_{n_j} q^{n_j} (1+\varepsilon)^{n_j} = \sum_j e^{n_j \ln(1+\varepsilon) - \sqrt{n_j}} = \infty,$$

so, by Theorem 2.6, σ is not suboptimal. We have $\tau_{n_j} = \sigma_{n_j} + q\tau_{n_j+1} = \rho_{n_j} + q\tau_{n_j+1}$, so using (2.5) with $n_j = j^2$, we obtain

$$\frac{\tau_{n_j}}{\sigma_{n_j}} = \frac{\tau_{n_j}}{\rho_{n_j}} \approx 1 + \sum_{k=1}^{\infty} \frac{\rho_{n_{j+k}} q^{n_{j+k}-n_j}}{\rho_{n_j}} = 1 + \sum_{k=1}^{\infty} \frac{q^{-(j+k)^2} e^{-(j+k)} q^{(j+k)^2-j^2}}{q^{-j^2} e^{-j}} = 1 + \sum_{k=1}^{\infty} e^{-k} = \frac{e}{e-1}.$$

which proves that (2.7) holds.

Finally, we show that there are good reference measures σ for which (2.7) fails. Let $\sigma_n = q^{-n} n^{-3}$ for $n \in \mathbb{N}$. It follows by Corollary 2.2(ii) that σ is good. Then

$$\frac{\tau_n}{\sigma_n} = \sum_{k=0}^{\infty} \frac{\sigma_{n+k} q^k}{\sigma_n} = \sum_{k=0}^{\infty} \frac{1}{\left(1+\frac{k}{n}\right)^3} \geqslant \sum_{k=0}^{n} \frac{1}{\left(1+\frac{k}{n}\right)^3} \geqslant \sum_{k=0}^{n} \frac{1}{2^3} = \frac{n+1}{8} \to \infty.$$

Therefore, (2.7) fails. □

3. Methods for generating σ-Carleson measures

Theorem 3.1. *Let σ be a reference measure and τ the associated σ-sectorial measure.*

(i) Fix a ray $\omega = [\omega_0 = o, \omega_1 \dots, \omega_n \dots]$. Denote by μ the function supported on ω defined by $\mu_{\omega_n} = \tau_n$. Then μ is a σ-Carleson measure.

(ii) Let ν be any finite Borel measure on the boundary of T. Then, the measure μ on T defined by $\mu(v) = \left(\tau_{|v|} - \tau_{|v|+1}\right) \nu(I(v))$ is σ-Carleson.

Proof. (i) It was shown in Lemma 6.1 of [3] that if in the definition of μ we replace τ_n by $\tau_n - \tau_{n+1}$, we obtain a σ-Carleson measure. The result then follows from the fact that τ_n and $\tau_n - \tau_{n+1}$ are comparable in size. Indeed, since $\tau_n = \sigma_n + q\tau_{n+1}$, it follows

$$\tau_n \geqslant \tau_n - \tau_{n+1} = \tau_n - \frac{\tau_n - \sigma_n}{q} = \tau_n\Big(1 - \frac{1}{q}\Big) + \frac{\sigma_n}{q} \geqslant \tau_n\Big(1 - \frac{1}{q}\Big).$$

(ii) Let $v \in T$ with $|v| = n$. Then

$$\begin{aligned}\tau^{\mu}(v) &= \sum_{w \geqslant v} \mu(w) = \sum_{k=0}^{\infty} (\tau_{n+k} - \tau_{n+k+1}) \sum_{w \geqslant v, |w|=n+k} \nu(I(w)) \\ &= \sum_{k=0}^{\infty} (\tau_{n+k} - \tau_{n+k+1})\, \nu(I(v)) = \tau_n \nu(I(v)) = \nu(I(v))\tau(v),\end{aligned}$$

proving that μ is σ-Carleson. □

Theorem 3.2. *Let σ be any good reference measure and let $\{a_n\}_{n=0}^{\infty}$ be any non-negative summable sequence. Then the radial measure μ defined by $\mu_n = a_n \tau_n$ is a σ-Carleson measure.*

Proof. We must show that $\frac{\tau^{\mu}_n}{\tau_n}$ is a bounded sequence. We have

$$\frac{\tau^{\mu}_n}{\tau_n} = \frac{1}{\tau_n} \sum_{k=0}^{\infty} \mu_{n+k} q^k = \frac{1}{\tau_n} \sum_{k=0}^{\infty} a_{n+k}\, \tau_{n+k}\, q^k = a_n + a_{n+1} \frac{\tau_{n+1}}{\tau_n}\, q + a_{n+2} \frac{\tau_{n+2}}{\tau_n}\, q^2 + a_{n+3} \frac{\tau_{n+3}}{\tau_n}\, q^3 + \dots$$

For any $n \geqslant 1$ we have $\tau_n = \sigma_n + q\tau_{n+1} > q\tau_{n+1}$, so $\tau_{n+1}/\tau_n < 1/q$. It follows by induction that for any $k \geqslant 0$, $\tau_{n+k}/\tau_n < q^{-k}$. Thus, $\tau^{\mu}_n/\tau_n \leqslant a_n + a_{n+1} + \dots \leqslant \sum_k a_k$, proving that μ is σ-Carleson. □

In the remainder of this section, we consider reference measures σ which are mesa measures as in Example 2.1 and show how to associate to them a measure μ whose property of being σ-Carleson or not depends on the parameters of the underlying mesa measure σ.

Let $p, \beta > 1$, and let $\{\beta_j\}_j$ be a sequence converging to β. Let $\{n_j\}_j$ be a sequence of integers converging to ∞ such that $n_j - n_{j-1} \to \infty$. Let μ be the measure $\mu(v) = q^{-p|v|}$ and let σ be the reference measure associated with $\{\beta_j\}_j$ as in Example 2.1. Since μ and τ are both radial, we can refer to $\mu(v)$ by μ_j, $\sigma(v)$ by σ_j, and $\tau(v)$ by τ_j for $|v| = j$.

Theorem 3.3. (i) *If $\beta_j \searrow \beta \geqslant p$, then μ is not σ-Carleson.*

(ii) *Suppose $\beta_j = \beta$ for all j and $\beta < p$. Then μ is σ-Carleson if $n_j/n_{j+1} \to 1$ and μ is not σ-Carleson if $n_j/n_{j+1} \to 0$.*

Proof. If in (i) we replace all the β_j with β, then the resulting mesa measure σ' is bigger than σ, so if we could show that μ is not σ'-Carleson, it would follow that μ is not σ-Carleson. Thus we may assume without loss of generality that $\beta_j = \beta$ for all j. To prove (i), we must show that $\limsup_{j\to\infty} \tau_j^{\mu}/\tau_j = \infty$. But by Corollary 2.2 μ is optimal, so it is enough to show $\limsup_{j\to\infty} \mu_j/\tau_j = \infty$. Applying (2.5) with $l = 1$, we have

$$\frac{\mu_{n_j+1}}{\tau_{n_j+1}} \approx \frac{q^{-p(n_j+1)}}{\sum\limits_{k\geqslant 1} q^{-\beta n_{j+k}} q^{n_{j+k}-n_j}} \approx \frac{q^{-(p-1)n_j}}{\sum\limits_{k\geqslant 1} q^{-(\beta-1)n_{j+k}}} \geqslant \frac{q^{-(p-1)n_j}}{\sum\limits_{m\geqslant n_{j+1}} q^{-(\beta-1)m}} \gtrsim \frac{q^{-(p-1)n_j}}{q^{-(\beta-1)n_{j+1}}} \geqslant q^{(p-1)(n_{j+1}-n_j)}$$

which diverges as $j \to \infty$.

To prove (ii), suppose first that $n_j/n_{j+1} \to 0$. Arguing as above, we have

$$\frac{\mu_{n_j+1}}{\tau_{n_j+1}} \gtrsim \frac{q^{-(p-1)n_j}}{q^{-(\beta-1)n_{j+1}}} = q^{n_{j+1}\left[(\beta-1)-(p-1)\frac{n_j}{n_{j+1}}\right]} \to \infty \text{ as } j \to \infty$$

since for large j, the bracketed part of the exponent is negative and $n_{j+1} \to \infty$.

Finally, suppose that $n_j/n_{j+1} \to 1$. Using (2.5) with general l and a similar argument as above, we get

$$\frac{\mu_{n_j+l}}{\tau_{n_j+l}} \approx \frac{q^{-p(n_j+l)}}{\sum\limits_{k\geqslant 1} q^{-\beta n_{j+k}} q^{n_{j+k}-n_j-l+1}} \leqslant \frac{q^{-p(n_j+l)}}{q^{-\beta n_{j+1}} q^{n_{j+1}-n_j-l+1}} \lesssim \frac{q^{-(p-1)n_j}}{q^{-(\beta-1)n_{j+1}}} = q^{n_{j+1}\left[(\beta-1)-(p-1)\frac{n_j}{n_{j+1}}\right]} \to 0$$

as $j \to \infty$ since $\beta < p$. Thus μ is σ-Carleson. □

Applying (ii) of Theorem 3.3 we get the following result.

Corollary 3.4. *Let $1 < \beta < p$. Let $\mu(v) = q^{-p|v|}$ and let $\rho(v) = q^{-\beta|v|}$. With respect to a specific choice of n_j, let σ be the mesa measure associated with ρ. If $n_j = j^2$, then μ is σ-Carleson, and if $n_j = 2^{2^j}$, then μ is not σ-Carleson.*

4. Some results on non-negative subharmonic functions

4.1. **Subharmonic functions supported on rays or unions of rays.** Define α to be the larger root of the equation $x^2 - (q+1)x + 1$. The other root is α^{-1}. Since $q^2 - (q+1)q + 1 = -q + 1 < 0$, it follows that q lies between the two roots α^{-1} and α. Thus $q < \alpha < q + 1$.

Theorem 4.1. *Let $\omega = [\omega_0, \omega_1, \dots]$ be a ray and let $A, B \in \mathbb{R}$. Then there is a unique function $h = h_{A,B}$ defined on the union of ω and the vertices a distance 1 from $\{\omega_n : n \geqslant 1\}$ with the following properties:*

(i) *$h(\omega_0) = A$, $h(\omega_1) = B$,*

(ii) *$h(v) = 0$ at each vertex v outside ω at a distance 1 from each ω_n, $n \geqslant 1$,*

(iii) *h harmonic at each ω_n, $n \geqslant 1$.*

It is given by

$$h(\omega_n) = \left(\frac{B - \frac{A}{\alpha}}{\alpha - \frac{1}{\alpha}}\right)\alpha^n + \left(\frac{A\alpha - B}{\alpha - \frac{1}{\alpha}}\right)\alpha^{-n}.$$

Proof. For each non-negative integer n, let h_n denote $h(\omega_n)$. The harmonicity condition for h at ω_n is $(q+1)h_n = h_{n+1} + h_{n-1}$, whose characteristic equation is $x^2 - (q+1)x + 1 = 0$. Thus,

$$h_n = c_1\alpha^n + c_2\alpha^{-n}, \tag{4.1}$$

for some constants c_1 and c_2, with α as above. From the initial conditions $h_0 = A$ and $h_1 = B$ we obtain $c_1 + c_2 = A$ and $c_1\alpha + c_2\alpha^{-1} = B$. Solving for c_1 and c_2 and plugging into (4.1), we obtain the desired formula. $\square$

Theorem 4.2. *Let $\omega = [\omega_0, \omega_1, \dots]$ be a ray. Let f be a function on T with the following properties:*

(i) $f(v) = 0$ at each vertex v at distance 1 from ω except for the neighbors of ω_0.
(ii) f is subharmonic at each ω_n, $n \geqslant 1$,
(iii) $0 \leqslant f(\omega_0) \leqslant f(\omega_1)$.

Then $f(\omega_n)$ increases with n and $f(\omega_n) \geqslant h_{A,B}(\omega_n)$ for each $n \geqslant 0$, where $A = f(\omega_0)$ and $B = f(\omega_1)$. If $\alpha f(\omega_1) > f(\omega_0)$, then $f(\omega_n) \to \infty$ at least as fast as a multiple of α^n as $n \to \infty$.

Proof. Let f_n denote $f(\omega_n)$. The proof that f_n is increasing is by induction on n, $n \geqslant 0$. By assumption it holds for $n = 0$. If $n \geqslant 0$ and $0 \leqslant f_n \leqslant f_{n+1}$, then from the subharmonicity condition of f at ω_{n+1}, we have

$$(q+1)f_{n+1} \leqslant f_n + f_{n+2} \leqslant f_{n+1} + f_{n+2},$$

and so $f_{n+2} \geqslant q f_{n+1} \geqslant f_{n+1}$ (with equality in the last inequality if and only if $f_{n+1} = 0$). This completes the proof that f_n increases with n.

To prove $f(\omega_n) \geqslant h_{A,B}(\omega_n)$, let $g := f - h_{A,B}$. Then g satisfies the conditions of the first part of the theorem, and $g(\omega_0) = g(\omega_1) = 0$. Thus $g(\omega_n)$ increases with n, so in particular it is non-negative. For the last assertion, if $\alpha B = \alpha f(\omega_1) > f(\omega_0) = A$, then $h_{A,B}$ is greater than some positive multiple of α^n. This completes the proof. $\square$

We next look at a few examples of non-negative subharmonic functions, at times making use of the above theorems. In the above two theorems we made use of the function α^n supported on a ray ω (i.e. taken to be 0 outside that ray), where n is the distance along the ray from ω_0. It is harmonic at each ω_n for $n \geqslant 1$, and it is clearly subharmonic at each vertex outside of ω, but it is superharmonic at ω_0 since the Laplacian there is $\alpha/(q+1) - 1 < 0$. However, if we replace α by the right constant, we do get a non-negative subharmonic function on T.

Theorem 4.3. *Let $\omega = [\omega_0, \omega_1, \dots]$ be a ray. For $\beta > 0$, the function f supported on the vertices in ω whose value at ω_n is β^n is subharmonic on T if and only if $\beta \geqslant q + 1$.*

Proof. Since $f \geqslant 0$, f is subharmonic at each vertex where it vanishes. The subharmonicity condition of f at ω_0 is $\beta \geqslant q+1$, so the condition $\beta \geqslant q+1$ is necessary for subharmonicity, while for all $n \geqslant 1$ the subharmonicity condition of f at ω_n is $\beta^{n+1} + \beta^{n-1} \geqslant (q+1)\beta^n$, which is equivalent to $\beta^2 - (q+1)\beta + 1 \geqslant 0$. This holds provided β lies outside the interval determined by the roots α and $1/\alpha$, and so in particular if $\beta \geqslant q+1$, since $q+1 > \alpha$. Therefore, the condition $\beta \geqslant q+1$ is also sufficient for f to be subharmonic. $\square$

Theorem 4.4. *Let R denote the union of m rays $(1 \leqslant m \leqslant q)$ all starting at the same vertex ω_0. If f is a non-negative subharmonic function on T supported on R whose values at any given vertex of R depend only on its distance from ω_0, then for $n \geqslant 0$*

$$f(\omega_n) \geqslant \frac{f(\omega_0)}{\alpha - \frac{1}{\alpha}} \left[\left(\frac{q+1}{m} - \frac{1}{\alpha} \right) \alpha^n + \left(\alpha - \frac{q+1}{m} \right) \alpha^{-n} \right], \tag{4.2}$$

where ω_n is a vertex in anyone of the m rays at distance n from ω_0. If $f(\omega_0) > 0$, then $f \to \infty$ along the rays of R at least as fast as a positive multiple of α^n.

Proof. Let $\omega = [\omega_0, \omega_1, \dots]$ be one of the rays of R. Let $A = f(\omega_0)$ and $B = \frac{q+1}{m} f(\omega_0)$. Consider the function $g := f - h_{A,B}$, where $h_{A,B}$ is as in Theorem 4.1. Then $g(\omega_0) = 0$. The subharmonicity condition on f at ω_0 says that $m f(\omega_1) \geqslant (q+1) f(\omega_0)$. This implies that $g(\omega_1) \geqslant 0$. Thus by Theorem 4.2, $g \geqslant 0$. This says

$$f(\omega_n) \geqslant \left(\frac{B - \frac{A}{\alpha}}{\alpha - \frac{1}{\alpha}} \right) \alpha^n + \left(\frac{A\alpha - B}{\alpha - \frac{1}{\alpha}} \right) \alpha^{-n} = \frac{f(\omega_0)}{\alpha - \frac{1}{\alpha}} \left[\left(\frac{q+1}{m} - \frac{1}{\alpha} \right) \alpha^n + \left(\alpha - \frac{q+1}{m} \right) \alpha^{-n} \right].$$

Since $m \leqslant q < (q+1)\alpha$, the coefficient of α^n in $h_{A,B}$ is positive, and so it follows from (4.2) that $f(\omega_n) \to \infty$ like a positive multiple of α^n. □

The following result follows from Theorem 4.4 for $m = 1$ and Theorem 4.3.

Theorem 4.5. *Let $\{n_j\}$ be a sequence of positive integers such that $n_{j+1} - n_j \geqslant 2$ for each j. Fix a ray $\omega = [\omega_0, \omega_1, \dots]$. For each j, choose a ray $\rho^{(j)} = [\rho_0^{(j)} = \omega_{n_j}, \rho_1^{(j)}, \rho_2^{(j)}, \dots]$, where $\rho^{(j)} \cap \omega = \{\omega_{n_j}\}$. Define a function f supported on the union of the rays $\rho^{(j)}$ in the following way. For each j, choose $b_j > 0$ and for all $k \geqslant 0$, let $f(\rho_k^{(j)}) = b_j(q+1)^k$ and let f be 0 elsewhere. Then f is non-negative subharmonic on T.*

Theorem 4.6. *Let σ be a reference measure and f a non-negative subharmonic function supported only on a single ray ω, with $f(\omega_0) > 0$. If $\sum_{v\in T} f(v)\,\sigma(v)$ is finite, then σ is suboptimal.*

Proof. By Theorem 4.4 with $m = 1$, $f(\omega_n) \geqslant c\,\alpha^n$ for some positive constant c. We have

$$\sum_{v\in T} f(v)\,\sigma(v) = \sum f(\omega_n)\sigma_n \;\geqslant\; \sum c\,\alpha^n\sigma_n,$$

so if this is finite then the radius of convergence of $\sum \sigma_n z^n$ is greater than or equal to α. Recalling that $\alpha > q$, by Theorem 2.6, we see that σ is suboptimal. □

4.2. **The discrete derivative.** In this subsection we study the discrete derivative of a function on T and use it in Theorem 4.11 to show how to generate a wide variety of non-negative subharmonic functions.

Definition 4.1. For $f : T \to \mathbb{R}$, define $f' : T \to \mathbb{R}$ by

$$f'(v) = \begin{cases} 0 & \text{if } v = o \\ f(v) - f(v^-) & \text{if } v \neq o \end{cases}$$

Theorem 4.7. *Let $f : T \to \mathbb{R}$ and let μ be a good measure. Suppose that either f' is upper or lower bounded, or satisfies*

$$\sum_{w\in T} |f'(w)|\tau^\mu(w) < \infty. \tag{4.3}$$

Then

$$\sum_{v\in T} f(v)\mu(v) = \sum_{v\in T} f'(v)\tau^\mu(v) + f(0)\|\mu\|. \tag{4.4}$$

Proof. Observe that for any $v \in T$,

$$f(v) - f(o) = \sum_{w\in[o,v]} f'(w). \tag{4.5}$$

Assume first that f' is lower bounded. Choose $M \in \mathbb{R}$ such that $f' + M \geqslant 0$ on T. Then for each $v \in T$, $f(v) - f(o) + M(|v|+1) = \sum_{w\in[o,v]}(f'(w) + M)$. Using Fubini's theorem and (2.1), we obtain

$$\begin{aligned}\sum_{v\in T}(f(v) - f(o) + M(|v|+1))\mu(v) &= \sum_{v\in T}\sum_{w\in[o,v]}(f'(w)+M)\mu(v) = \sum_{w\in T}\sum_{v\geqslant w}(f'(w)+M)\mu(v) \\ &= \sum_{w\in T}(f'(w)+M)\tau^\mu(w) = \sum_{w\in T} f'(w)\tau^\mu(w) + M\sum_{v\in T}(1+|v|)\mu(v).\end{aligned}$$

By Theorem 2.1, the quantity $\sum_{v\in T}(1+|v|)\mu(v)$ is finite, so subtracting it from the first and last terms gives the result. The case of f upper bounded is done by applying the above result to $-f$.

Lastly, suppose $\sum_{w\in T}|f'(w)|\tau^\mu(w)<\infty$. Since $\sum_{w\in T}|f'(w)|\tau^\mu(w)=\sum_{w\in T}\sum_{v\geqslant w}|f'(w)|\mu(v)$, by Fubini's theorem, $\sum_{w\in T}\sum_{v\geqslant w}f'(w)\mu(v)=\sum_{v\in T}\sum_{w\leqslant v}f'(w)\mu(v)$ and both sides are finite. Thus,

$$\sum_{w\in T}f'(w)\tau^\mu(w)=\sum_{w\in T}\sum_{v\geqslant w}f'(w)\mu(v)=\sum_{v\in T}\sum_{w\leqslant v}f'(w)\mu(v)=\sum_{v\in T}(f(v)-f(o))\mu(v),$$

completing the proof. □

Remark 4.1. When we formulated Theorem 4.7 we initially wanted to replace condition (4.3) with $\sum_{v\in T}|f(v)|\mu(v)<\infty$. Theorem 4.7 shows that this condition is a consequence of condition (4.3). We now show that these two conditions are not equivalent by giving an example of a good measure μ and a function f for which $\sum_{v\in T}|f(v)|\mu(v)<\infty$ but $\sum_{w\in T}|f'(w)|\tau^\mu(w)|=\infty$.

Let μ be the radial measure for which

$$\mu_n=\begin{cases}0 & \text{if } n=0 \text{ or } n \text{ is odd},\\ \frac{1}{\left(\frac{n}{2}\right)^3 c_n} & \text{if } n \text{ even and } n\geqslant 2,\end{cases}$$

where $c_n=(q+1)q^{n-1}$ is the number of vertices of length n. Since μ is dominated by the reference measure σ with $\sigma_n=q^{-n}n^{-3}$, it follows by Corollary 2.2 that μ is good. Let f be the radial function on T for which

$$f_n=\begin{cases}0 & \text{if } n \text{ is even},\\ \left(\frac{n+1}{2}\right)^2 & \text{if } n \text{ is odd}.\end{cases}$$

Then f' is also radial and

$$f'_n=\begin{cases}\left(\frac{n+1}{2}\right)^2 & \text{if } n \text{ is odd}\\ -\left(\frac{n}{2}\right)^2 & \text{if } n \text{ is even}.\end{cases}$$

We have $\sum_{v\in T}|f(v)|\mu(v)=0$, but

$$\sum_{v\in T}|f'(v)|\tau^\mu(v)\geqslant\sum_{v\in T}|f'(v)|\mu(v)=\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

Theorem 4.8. *Let $g:T\to\mathbb{R}$ with $g(o)=0$. Let $b\geqslant 0$. Then g is the derivative of a non-negative function $f:T\to[0,\infty)$ with $f(o)=b$ if and only if for all $v\in T$,*

$$\sum_{w\in[o,v]}g(w)\geqslant -b. \tag{4.6}$$

Proof. If such a function f exists, then by (4.5),

$$\sum_{w\in[o,v]}g(w)=\sum_{w\in[o,v]}f'(w)=f(v)-f(o)=f(v)-b\geqslant -b.$$

Conversely, if (4.6) holds, define $f(v)=\sum\limits_{w\in[o,v]}g(w)+b$. Then $f(o)=b$, $f\geqslant 0$, and $g=f'$. □

4.3. **Use of the derivative to construct non-negative subharmonic functions.**

Definition 4.2. Let $g:T\to\mathbb{R}$. We say g is *wealth increasing* if for all $v\in T$,

$$\sum_{w^-=v}g(w)\geqslant g(v).$$

Theorem 4.9. *Let $f:T\to\mathbb{R}$. Then f is subharmonic if and only if f' is wealth increasing.*

Proof. At the root, f is subharmonic if and only if $\sum_{|w|=1} f(w) \geqslant (q+1)f(o)$. Transposing all the terms to the left and collecting, we see this happens if and only if $\sum_{|w|=1} f'(w) \geqslant 0 = f'(o)$, and so if and only if f' is wealth increasing at the root.

At $v \neq o$, f is subharmonic at v if and only if $\sum_{w^-=v} f(w) + f(v^-) \geqslant (q+1)f(v)$. Transposing q of the $f(v^-)$ terms to the left and the $f(v^-)$ term to the right, we see this is equivalent to $\sum_{w^-=v} f'(w) \geqslant f'(v)$, and this just says that f' is wealth increasing at v. ☐

Corollary 4.10. *Let f be subharmonic on T. Then for all $n \geqslant 0$, $\sum_{|v|=n} f'(v) \geqslant 0$. Consequently if σ is a reference measure for which $\sum_{v\in T} |f'(v)|\tau(v) < \infty$, then $\sum_{v\in T} f'(v)\tau(v) \geqslant 0$.*

Proof. The proof is by induction on n. Since $f'(0) = 0$, the result holds for $n = 0$. Let $n \geqslant 1$ and suppose the result holds for n. Fix v such that $|v| = n$. Since f' is wealth increasing, $\sum_{w^-=v} f'(w) \geqslant f'(v)$. Summing on all such v, we obtain $\sum_{|w|=n+1} f'(w) \geqslant \sum_{|v|=n} f'(v) \geqslant 0$, completing the proof of the induction.

If σ is a reference measure such that $\sum_{v\in T} |f'(v)|\tau(v) < \infty$, then since τ is radial,

$$\sum_{v\in T} f'(v)\tau(v) = \sum_{n=1}^{\infty} \tau_n \sum_{|v|=n} f'(v) \geqslant 0,$$

completing the proof. ☐

From Theorems 4.8 and 4.9 we deduce

Theorem 4.11. *Let $g : T \to \mathbb{R}$ with $g(o) = 0$ and let $b \geqslant 0$. Then g is the derivative of a non-negative subharmonic function f with $f(o) = b$ if and only if g is wealth increasing and $\sum_{w\in[o,v]} g(w) \geqslant -b$ for all $v \in T$.*

4.4. **Construction of radial subharmonic functions and increasing sequences.** In the next theorem, we apply the results of the previous subsection to the construction of radial positive subharmonic functions.

If f is any radial function on T, we let f_n denote $f(v)$ for any v with $|v| = n$. If f is radial and subharmonic, we refer to $\{f_n\}$ as a subharmonic sequence. The function f' is also radial, so it has a corresponding sequence which we write as $\{f'_n\}$ and refer to it as the derivative of $\{f_n\}$. Conversely, any sequence of numbers $\{f_n\}$ represents a unique radial function f as above.

Theorem 4.12. (i) *Let f be a non-negative radial subharmonic function. Then $\{f_n\}$ is increasing, and $0 \leqslant f'_n \leqslant q\, f'_{n+1}$. Conversely, if $\{g_n\}$ is a sequence such that $g_0 = 0$ and $0 \leqslant g_n \leqslant q\, g_{n+1}$ for all n, then $\{g_n\}$ is the derivative of a non-negative subharmonic sequence $\{f_n\}$, unique if we prescribe $f_0 = 0$.*

(ii) *Let $\{g_n\}$ be the derivative of a subharmonic sequence $\{f_n\}$ such that $f_0 = 0$. Define $\gamma_n := q^n\, g_n$. Then $\{\gamma_n\}$ is an increasing sequence. Conversely, if $\{\gamma_n\}$ is any increasing sequence with $\gamma_0 = 0$, then the sequence $\{g_n\}$ defined by $g_0 = 0, g_n = q^{-n}\gamma_n$ is the derivative of a unique non-negative subharmonic sequence $\{f_n\}$ with $f_0 = 0$.*

Proof. Note that for any radial function g on T with $g(o) = 0$, the condition that g be wealth increasing is precisely that $g_n \leqslant q\, g_{n+1}$.

(i) Let f be non-negative radial subharmonic on T, and let $g = f'$. Then by Theorem 4.11, f' is wealth increasing, so $f'_n \leqslant q\, f'_{n+1}$. Since $f'_0 = 0$, it follows by induction that $f'_n \geqslant 0$, and this just says that $\{f_n\}$ is increasing. Conversely, the conditions $g(o) = 0$ and $g_n \leqslant q\, g_{n+1}$ say that g is wealth increasing, so the result follows by Theorem 4.11.

(ii) Let $\{f_n\}$, $\{g_n\}$ and $\{\gamma_n\}$ be as in the first part of the statement of (ii). Since $\{g_n\}$ is wealth increasing, we have $\gamma_{n+1} = q^{n+1} g_{n+1} = q^n q\, g_{n+1} \geqslant q^n\, g_n = \gamma_n$, proving $\{\gamma_n\}$ is increasing. Conversely, let $\{\gamma_n\}$ be an increasing sequence with $\gamma_0 = 0$. Define $g_n := q^{-n}\, \gamma_n$. Then $q^{n+1}\, g_{n+1} = \gamma_{n+1} \geqslant \gamma_n = q^n\, g_n$, so $g_n \leqslant q\, g_{n+1}$. The result then follows from part (i). ☐

From Theorem 4.12(ii) we immediately obtain

Corollary 4.13. *Non-negative radial subharmonic functions can have arbitrarily prescribed slow growth.*

Example 4.1. *The function* $f(v) = |v|^\alpha$ *is a non-negative radial subharmonic function provided* $\alpha \geqslant 1$. *To see this, consider the corresponding derivative sequence* $\{g_n\}$, *namely* $g_0 = 0$ *and* $g_n = n^\alpha - (n-1)^\alpha$. *The derivative of the function* $h(x) = x^\alpha - (x-1)^\alpha$ *satisfies* $h'(x) = \alpha x^{\alpha-1} - \alpha(x-1)^{\alpha-1} \geqslant 0$ *for* $x \geqslant 1$, *since* $\alpha \geqslant 1$. *Thus* $\{g_n\}$ *is increasing, and so* $g_n \leqslant g_{n+1} \leqslant q\, g_{n+1}$. *It follows from Theorem 4.12(i) that* f *is subharmonic.*

Example 4.2. *The function* $f_p(v) = q^{p|v|}$ *is non-negative radial subharmonic provided* $p > 0$. *To see this, consider the sequence* $\{\gamma_n\}$ *where* $\gamma_0 = 0$ *and* $\gamma_n = q^n\left(q^{pn} - q^{p(n-1)}\right)$. *Note that* $\gamma_n = q^n q^{pn}(1 - q^{-p})$. *Since* $p > 0$, *this clearly increases with* n, *so by Theorem 4.12,* f *is subharmonic.*

5. A Reformulation of the Original Problem

As stated in the introduction, the main object of the paper is to study when we can prove inequalities of the form

$$\sum_{v\in T} f(v)\mu(v) \leqslant C \sum_{v\in T} f(v)\sigma(v) \tag{5.1}$$

for all f in some set of non-negative subharmonic functions $\mathcal{G}$, where σ is a given reference measure, μ is a σ-Carleson measure, and C is a constant independent of f.

If it were the case that $f' \geqslant 0$ for all $f \in \mathcal{G}$, then we could easily prove (5.1) by using formula (4.4) of Theorem 4.7. In fact, this is what we do in Theorem 7.3 in the next section.

Theorem 5.1 below allows us to deal with the more typical case where f' can be positive or negative. It gives us a useful reformulation of how to prove inequalities of the form (5.1).

Definition 5.1. For $f : T \to \mathbb{R}$ define the sets

$$I_f = \{v \in T : f'(v) < 0\} \text{ and } I\!I_f = \{v \in T : f'(v) \geqslant 0\}.$$

For $0 < \varepsilon < 1$ and a reference measure σ, let $\mathcal{G}_{\varepsilon,\sigma}$ be the set of non-negative subharmonic functions f on T satisfying the following conditions:

(i) $\displaystyle\sum_{v\in T} |f'(v)|\tau(v) < \infty$;

(ii) $\displaystyle -\sum_{v\in I_f} f'(v)\tau(v) \leqslant (1-\varepsilon) \sum_{v\in I\!I_f} f'(v)\tau(v) + f(0)\|\sigma\|$.

Remark 5.1. Given a non-negative subharmonic function f on T satisfying (i), $f \in \mathcal{G}_{\varepsilon,\sigma}$, for any ε satisfying $0 < \varepsilon < \min\left\{1, \frac{\sum_{v\in T} f'(v)\tau(v)}{\sum_{v\in I\!I_f} f'(v)\tau(v)} + f(o)\|\sigma\|\right\}$.

Theorem 5.1. *Let* σ *be a good reference measure,* $0 < \varepsilon < 1$, *and* $\mathcal{G}_{\varepsilon,\sigma}$ *the set in Definition 5.1. Then* $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{G}_{\varepsilon,\sigma},\sigma)}$. *Specifically, if* μ *is* σ*-Carleson and* $C_\varepsilon = \frac{C_\mu}{\varepsilon} + \frac{\|\mu\|}{\|\sigma\|}$, *then*

$$\sum_{v\in T} f(v)\mu(v) \leqslant C_\varepsilon \sum_{v\in T} f(v)\sigma(v) \quad \textit{for all } f \in \mathcal{G}_{\varepsilon,\sigma}.$$

Proof. First observe that from condition (ii), for $f \in \mathcal{G}_{\varepsilon,\sigma}$,

$$\sum_{v\in I\!I_f} f'(v)\tau(v) \leqslant \frac{1}{\varepsilon}\Bigg(\sum_{v\in T} f'(v)\tau(v) + f(o)\|\sigma\|\Bigg). \tag{5.2}$$

Since $\tau^\mu \leqslant C_\mu \tau$, by condition (i) we can apply Theorem 4.7 to both μ and σ, so using (5.2), we get

$$\sum_{v\in T} f(v)\mu(v) = \sum_{v\in T} f'(v)\tau^\mu(v) + f(o)\|\mu\| \leqslant \sum_{v\in \mathit{\Pi}_f} f'(v)\tau^\mu(v) + f(o)\|\mu\| \leqslant C_\mu \sum_{v\in \mathit{\Pi}_f} f'(v)\tau(v) + f(o)\|\mu\|$$
$$\leqslant \frac{C_\mu}{\varepsilon}\Bigg(\sum_{v\in T} f'(v)\tau(v) + f(o)\|\sigma\|\Bigg) + f(o)\|\mu\| = \frac{C_\mu}{\varepsilon}\sum_{v\in T} f'(v)\tau(v) + C_\varepsilon f(o)\|\sigma\|$$
$$\leqslant C_\varepsilon\Bigg(\sum_{v\in T} f'(v)\tau(v) + f(o)\|\sigma\|\Bigg) = C_\varepsilon \sum_{v\in T} f(v)\sigma(v).$$

where we used Corollary 4.10 to deduce $\frac{C_\mu}{\varepsilon}\sum_{v\in T} f'(v)\tau(v) \leqslant C_\varepsilon \sum_{v\in T} f'(v)\tau(v)$ from the obvious inequality $C_\mu/\varepsilon \leqslant C_\varepsilon$. □

6. The set of non-negative subharmonic functions which are eigenfunctions of Δ

Much interesting harmonic analysis, and in particular the study of eigenfunctions of Δ, has been done on free groups. See for example [6], [7], and [8]. It is shown that on a free group with r generators, the real eigenvalues of Δ are precisely the numbers of the form $\frac{(2r-1)^p+(2r-1)^{1-p}}{2r} - 1$ for $p \in \mathbb{R}$, and Theorem A of [8] says that a function is an associated eigenfunction if and only if it is the Poisson transform of a martingale. In our terminology, since $q + 1 = 2r$, this says that if the homogeneity $q + 1$ is even, then for any $p \in \mathbb{R}$, f is an eigenfunction of the Laplacian with eigenvalue $\frac{q^p+q^{1-p}}{q+1} - 1$ if and only if there exists a finitely additive set function μ defined on finite unions of intervals $I(v)$ (i.e. a distribution) such that

$$f(\cdot) = \int_{\partial T} K^p_{\omega'}(\cdot)\, d\mu(\omega').$$

That $\frac{q^p+q^{1-p}}{q+1} - 1$ is an eigenvalue of Δ follows from the fact that for any fixed ω', $K^p_{\omega'}$ is an associated eigenfunction. To see this, note that a typical value of $K^p_{\omega'}$ is q^{jp} with one neighbor of value $q^{(j+1)p}$ and the remaining q neighbors of value $q^{(j-1)p}$.

Fix $\omega \in \partial T$ and $p \in \mathbb{R}$. The quantity $\frac{q^p+q^{1-p}}{q+1} - 1 = \frac{(1-q^{-p})q(q^{p-1}-1)}{q+1}$ is 0 for $p = 0$ or 1, is strictly positive for $p > 1$ or $p < 0$, and is strictly negative for $0 < p < 1$. Thus K^p_ω is positive subharmonic if and only if $p \geqslant 1$ or $p \leqslant 0$. Note also that the eigenfunctions corresponding to p and $1 - p$ share the same eigenvalue. Thus for $p \geqslant 1$ or $p \leqslant 0$ and any non-trivial non-negative finite regular Borel measure μ on ∂T, the function $\int K^p_{\omega'}(v) d\mu(\omega')$ is a subharmonic eigenfunction of Δ.

Our interest in this section is in showing the following theorem. We will make use of it in section 7.1.

Theorem 6.1. *The non-negative subharmonic functions which are eigenfunctions of the Laplacian are precisely the functions of the form $f(v) = \int_{\partial T} K^p_{\omega'}(v)d\mu(\omega')$, where $p \in [1, \infty)$ and μ is a non-negative finite regular Borel measure on ∂T. The associated eigenvalue is $\lambda = \frac{q^p+q^{1-p}}{1+q} - 1$. In particular the negative powers $K^{1-p}_\omega(v)$ do not contribute any additional eigenfunctions.*

Theorem 6.1 in case $q + 1$ is even can be proved using Theorem A of [8]. But instead we include the following complete proof based on the potential theory given in [1], which is more elementary.

Proof. If f is a non-negative subharmonic function which is an eigenfunction of Δ corresponding to the eigenvalue λ, then $0 \leqslant \Delta f = \lambda f$, so $\lambda \geqslant 0$.

Let us consider two different potential theories as described in general in Chapter 2 of Cartier [1]. Each of these potential theories is determined by the selection of a positive number $\rho(v, w)$ for each pair of neighboring vertices v and w. The isotropic potential theory which we have been using thus far comes from assigning $1/(q + 1)$ to each such pair. For the other potential theory, fix $\lambda \geqslant 0$ and for each pair of neighboring vertices v and w, define $\rho(v, w) = \frac{1}{(q+1)(\lambda+1)}$. We call $\lambda-$harmonic the functions f annihilated by the associated Laplacian $\Delta_\lambda f(v) := \sum_{w\sim v} \frac{1}{(q+1)(\lambda+1)} f(w) - f(v)$. Then the λ-harmonic functions are precisely the eigenfunctions of Δ with eigenvalue λ. The associated Green function $G_\lambda(v,w)$ is defined in Section 2.3 of [1] as follows: if we let $\Gamma_{v,w}$ denote the set of all paths from v to w and for each $\gamma \in \Gamma_{v,w}$ we define $\rho(\gamma) := \left(\frac{1}{(q+1)(\lambda+1)}\right)^n$ where n is the number of edges in γ, then $G_\lambda(v,w) = \sum_{\gamma\in\Gamma_{v,w}} \rho(\gamma)$. If we replace $\rho(v,w)$ by $1/(q+1)$ in the definition of G_λ, we get the Green function G for the isotropic potential theory. In Section 4.5 of [1] it is shown that $G(v,w) = \frac{q}{q-1} q^{-|v-w|}$. Since $\rho \leqslant 1/(q+1)$, it follows that $G_\lambda \leqslant G$.

It follows from the general theory that for each w, $v \mapsto G_\lambda(v,w)$ is $\lambda-$harmonic outside of $\{w\}$ and by symmetry it is radial with respect to w. This leads to the difference equation $qx_{k+1}+x_{k-1} = (q+1)(\lambda+1)x_k$. Both roots of the resulting characteristic equation are positive, so if we write one of the roots as q^{-p} for some $p \in \mathbb{R}$, then since the product of the roots must be $1/q$, the other root is q^{p-1}. Since the sum of the roots is $\frac{(q+1)(\lambda+1)}{q}$, we obtain $\lambda = \frac{q^p+q^{1-p}}{q+1} - 1$. The right side is positive for $p > 1$ or $p < 0$ and negative for $0 < p < 1$, and it is the same if we interchange p with $1-p$. Thus, since $\lambda \geqslant 0$, we may assume that $p \geqslant 1$. Since $G_\lambda(v,w) \leqslant G(v,w) \to 0$ as $|v| \to \infty$, it follows that $G_\lambda(v,w) \to 0$ as $|v| \to \infty$, and so it is the smaller root q^{-p} which is needed in representing G_λ. Thus we obtain $G_\lambda(v,w) = c\, G(v,w)^p$, for some constant c. It follows from the representation of the λ-Poisson kernel as a ratio of values of the $\lambda-$Green function as given in Section 2.5 of [1] that the λ-Poisson kernel is given by $K^p_\omega(v)$. The proof is completed by applying Theorem 2.1 on page 232 in [1] which says that all positive λ-harmonic functions can be written in the form $\int_{\partial T} K^p_\omega(v) d\mu(\omega)$ for some finite non-negative regular Borel measure μ on ∂T. □

7. Applications of Theorem 5.1

In the first two theorems of this section, we introduce two large classes of positive subharmonic functions for which we prove Theorem 5.1 can be applied. We conclude the paper by giving two nontrivial examples of functions which are in $\mathcal{G}_{\varepsilon,\sigma}$ introduced in Section 5 for some positive ε and a variety of reference measures σ.

7.1. **The set of non-negative subharmonic functions generated by eigenfunctions of Δ.** Let ν be a non-negative finite regular Borel measure on ∂T and let λ be a non-negative finite regular Borel measure on $[1,\infty)$ satisfying $\int_1^\infty q^{np} d\lambda(p) < \infty$ for all $n \in \mathbb{N}$. Then the mapping

$$v \mapsto \int\int K^p_\omega(v) d\nu(\omega) d\lambda(p) \tag{7.1}$$

is non-negative and subharmonic on T. It is the set $\mathcal{G}$ of all non-negative subharmonic functions that are represented in this manner which we study in the following theorem. By taking λ to be a point mass at a fixed $p \geqslant 1$, it follows from Theorem 6.1 that $\mathcal{G}$ includes all non-negative subharmonic functions which are eigenfunctions of Δ.

Theorem 7.1. *Let σ be a good reference measure. Then $\mathcal{M}_\sigma \subset \mathcal{M}_{(\mathcal{G},\sigma)}$. Of course the converse inequality also holds, by Theorem 1.1 (i).*

For the proof, we will make use of the following lemma.

Lemma 7.2. *For $1 < p < \infty$ and a reference measure σ, $\sum_{n=1}^\infty q^{np}\sigma_n < \infty \iff \sum_{n=1}^\infty q^{np}\tau_n < \infty$.*

Proof. It suffices to prove that if $\sum_{n=1}^\infty q^{np}\sigma_n$ is finite, then so is $\sum_{n=1}^\infty q^{np}\tau_n$, since the converse is obvious. Making a change of variables and switching the order of summation, we have

$$\begin{aligned}\sum_{n=1}^\infty q^{np}\tau_n &= \sum_{n=1}^\infty\sum_{k=0}^\infty q^{np}\sigma_{n+k}q^k = \sum_{n=1}^\infty\sum_{m=n}^\infty q^{np}\sigma_m q^{m-n} = \sum_{m=1}^\infty\sum_{n=1}^m q^{n(p-1)}\sigma_m q^m \\ &< \frac{q^{p-1}}{q^{p-1}-1}\sum_{m=1}^\infty q^{m(p-1)}\sigma_m q^m = \frac{q^{p-1}}{q^{p-1}-1}\sum_{m=1}^\infty q^{mp}\sigma_m,\end{aligned}$$

proving the required implication. □

Proof of Theorem 7.1. Let $\mu \in \mathcal{M}_\sigma$. We will be done if we show there exists $C > 0$ such that

$$\sum_{v\in T} K_\omega^p(v)\mu(v) \leqslant C \sum_{v\in T} K_\omega^p(v)\sigma(v) \text{ for all } \omega \in \partial T \text{ and } p \geqslant 1. \tag{7.2}$$

The desired result will then follow upon integrating both sides of inequality (7.2) with respect to any non-negative finite regular Borel measure ν on ∂T and any non-negative finite regular Borel measure λ on $[1, \infty)$.

Fix $\omega \in \partial T$ and $p \geqslant 1$. Assume first p satisfies $\sum_{n=1}^\infty q^{np}\tau_n = \infty$. Since σ is good, $p > 1$. Then by Lemma 7.2, $\sum_{v\in T} K_\omega^p(v)\sigma(v) \geqslant \sum_{n=1}^\infty K_\omega^p(\omega_n)\sigma(\omega_n) = \sum_{n=1}^\infty q^{np}\sigma_n = \infty$, so (7.2) holds in this case for any choice of C, in particular for $C = C_1$. Thus we may assume without loss of generality that there exists $p_0 > 1$ such that $\sum_{n=1}^\infty q^{np_0}\tau_n < \infty$.

To show (7.2) holds, we treat separately the cases $1 < p < p_0$ and $p \geqslant p_0$. Assume first $1 < p < p_0$. Let $C_p := C_\mu\big(\sum_{n=0}^\infty q^{np}\tau_n\big)/\|\sigma\|$. Note C_p increases with p. Then

$$\sum_{v\in T} K_\omega^p(v)\mu(v) = \sum_{n=0}^\infty \sum_{v\in S(\omega_n)\setminus S(\omega_{n+1})} K_\omega^p(v)\mu(v) \leqslant \sum_{n=0}^\infty q^{np}\mu(S(\omega_n)\setminus S(\omega_{n+1})) \leqslant \sum_{n=0}^\infty q^{np}\tau^\mu(\omega_n) \leqslant C_\mu \sum_{n=0}^\infty q^{np}\tau_n.$$

Also, since K_ω^p is subharmonic, by Lemma 6.2 of [3], $\sum_{|v|=n} K_\omega^p(v) \geqslant (q+1)q^{n-1}K_\omega^p(o)$ for $n \geqslant 1$. Since $K_\omega^p(o) = 1$, $\sum_{v\in T} K_\omega^p(v)\sigma(v) = \sum_{n=0}^\infty \sum_{|v|=n} K_\omega^p(v)\sigma_n \geqslant \sigma_0 + \sum_{n=1}^\infty (q+1)q^{n-1}\sigma_n = \|\sigma\|$, and so

$$\sum_{v\in T} K_\omega^p(v)\mu(v) \leqslant \left(C_\mu \frac{\sum_{n=0}^\infty q^{np}\tau_n}{\|\sigma\|}\right) \sum_{v\in T} K_\omega^p(v)\sigma(v) = C_p \sum_{v\in T} K_\omega^p(v)\sigma(v) \leqslant C_{p_0} \sum_{v\in T} K_\omega^p(v)\sigma(v).$$

Next, assume $p \geqslant p_0$. In order to handle (7.2) in this case, we will make use of Theorem 5.1. For this purpose, define $\varepsilon = \frac{q^{p_0-1}-1}{q^{p_0}-1}$ and note that $p \mapsto \frac{q^{p-1}-1}{q^p-1}$ is increasing on $(0, \infty)$. Thus

$$1 - \frac{q^{p-1}-1}{q^p-1} \leqslant 1 - \frac{q^{p_0-1}-1}{q^{p_0}-1} = 1 - \varepsilon. \tag{7.3}$$

To each $v \in T$ we can associate $n = |v \wedge \omega|$ and $j = |v - v \wedge \omega|$. With this notation, we have

$$(K_\omega^p)'(v) = \begin{cases} 0 & \text{if } n = 0, j = 0, \\ (1 - q^{-p})q^{np} & \text{if } n \geqslant 1, j = 0, \\ -(q^p - 1)q^{(n-j)p} & \text{if } n \geqslant 0, j \geqslant 1. \end{cases}$$

In the notation of Definition 5.1 (but omitting the subscript of f), $\mathit{II} = \{v \in T : (K_\omega^p)'(v) \geqslant 0\} = \{\omega_n \in T : n \geqslant 0\}$ and $I = \{v \in T : (K_\omega^p)'(v) < 0\} = T \setminus \mathit{II}$.

Let $n \geqslant 1$. From the formula, $\tau_k = \sigma_k + q\tau_{k+1}$, we deduce that $\tau_{k+1} \leqslant q^{-1}\tau_k$. By this and (7.3),

$$\begin{aligned} \sum_{v\in I\cap(S(\omega_n)\setminus S(\omega_{n+1}))} -(K_\omega^p)'(v)\tau(v) &= \sum_{j=1}^\infty \big(q^p-1\big)q^{(n-j)p}(q-1)q^{j-1}\tau_{n+j} \leqslant \big(q^p-1\big)q^{np}(q-1)\sum_{j=1}^\infty q^{-jp}q^{j-1}q^{-j}\tau_n \\ &= q^{np}(1-q^{-1})\tau_n = \left(1 - \frac{q^{p-1}-1}{q^p-1}\right)(1-q^{-p})q^{np}\tau_n \\ &= \left(1 - \frac{q^{p-1}-1}{q^p-1}\right)(K_\omega^p)'(\omega_n)\tau_n \leqslant (1-\varepsilon)\,(K_\omega^p)'(\omega_n)\,\tau_n \text{ and} \end{aligned} \tag{7.4}$$

$$\begin{aligned} \sum_{v\in I\cap(S(o)\setminus S(\omega_1))} -(K_\omega^p)'(v)\,\tau(v) &= \sum_{j=1}^\infty \big(q^p-1\big)q^{-jp}q^j\tau_j \leqslant \sum_{j=1}^\infty (q^p-1)q^{-jp}q^jq^{-j}\tau_0 \\ &= (q^p-1)\frac{q^{-p}}{1-q^{-p}}\,\tau_0 = \tau_0 = \|\sigma\| = K_\omega^p(o)\|\sigma\|. \end{aligned} \tag{7.5}$$

Summing over $n \geqslant 1$ in (7.4) and adding to it the result in (7.5), we obtain

$$-\sum_{v\in I}(K^p_\omega)'(v)\ \tau(v) \leqslant (1-\varepsilon)\sum_{n=1}^{\infty}(K^p_\omega)'(\omega_n)\ \tau_n + K^p_\omega(o)\|\sigma\| = (1-\varepsilon)\sum_{v\in II}(K^p_\omega)'(v)\ \tau(v) + K^p_\omega(o)\|\sigma\|.$$

We also get

$$\begin{aligned}\sum_{v\in T}\big|(K^p_\omega)'(v)\big|\tau(v) &= -\sum_{v\in I}(K^p_\omega)'(v)\tau(v) + \sum_{v\in II}(K^p_\omega)'(v)\tau(v) \leqslant (2-\varepsilon)\sum_{v\in II}(K^p_\omega)'(v)\tau(v) + \|\sigma\| \\ &= (2-\varepsilon)(1-q^{-p})\sum_{n=1}^{\infty} q^{np}\ \tau_n + \|\sigma\| < \infty.\end{aligned}$$

It follows that $K^p_\omega \in \mathcal{G}_{\varepsilon,\sigma}$, and so by Theorem 5.1, formula (7.2) holds for all $p \geqslant p_0$ with $C = C_\varepsilon$. Finally, (7.2) holds for all $p \geqslant 1$ with $C = \max\{C_{p_0}, C_\varepsilon\}$. $\square$

Remark 7.1. Since the set of functions $\mathcal{G}$ defined in (7.1) includes more than the set of non-negative subharmonic functions which are eigenfunctions of the Laplacian, we might expect that $\mathcal{G}$ is all of $\mathcal{S}_+(T)$. However, this is not the case. Note, first of all, that any function f of $\mathcal{G}$ is strictly positive or identically 0. Indeed, if $f(v) = 0$ for some vertex v, then

$$0 = \int\int K^p_\omega(v)d\nu(\omega)d\lambda(p) \geqslant \int\int q^{-p|v|}d\nu(\omega)d\lambda(p) = \|\nu\|\int q^{-p|v|}d\lambda(p)$$

from which we deduce ν or λ is a vanishing measure, in which case $f \equiv 0$. Furthermore, taking the Laplacian of f and using the fact that K^p_ω is an eigenfunction of Δ with a non-negative eigenvalue, a similar argument shows that if f is harmonic anywhere on T then $f \equiv 0$. Thus the nontrivial members of $\mathcal{G}$ have no zeros on T and are not harmonic anywhere on T. However, there is a very rich class of non-negative subharmonic functions which have zeros and which are harmonic at many vertices of the tree.

7.2. **The set of radial non-negative subharmonic functions.** Let $\mathcal{S}^{\mathrm{rad}}_+$ denote the set of non-negative radial subharmonic functions on T. The use of Theorem 5.1 makes the following theorem easy to prove.

Theorem 7.3. *Let σ be a good reference measure. Then $\mathcal{M}_\sigma \subset \mathcal{M}(\mathcal{S}^{\mathrm{rad}}_+, \sigma)$.*

Proof. Let $\mu \in \mathcal{M}_\sigma$ and let f be non-negative radial subharmonic on T. Let $C = \max\big\{C_\mu, \|\mu\|/\|\sigma\|\big\}$. By Theorem 4.12, $f' \geqslant 0$, so by Theorem 5.1,

$$\begin{aligned}\sum_{v\in T} f(v)\mu(v) &= \sum_{v\in T} f'(v)\tau^\mu(v) + f(0)\|\mu\| \leqslant C_\mu\sum_{v\in T} f'(v)\tau(v) + f(0)\|\mu\| \\ &\leqslant C\Bigg(\sum_{v\in T} f'(v)\tau(v) + f(0)\|\sigma\|\Bigg) = C\sum_{v\in T} f(v)\sigma(v),\end{aligned}$$ $\square$

7.3. **Two interesting examples involving Theorem 5.1.** For our first example we define $g(v)$ inductively on $|v|$ in such a way that g is wealth increasing and for all v, $\sum_{w\leqslant v} g(w) \geqslant 0$. Define $g(o) = 0$ and define g to be 1 on one vertex of length 1 and 0 on the other q vertices of length 1. Let $n \geqslant 1$ and suppose we have defined $g(v)$ for all vertices of length n or less. If $|v| = n$ and $g(v) = 0$, define g to be 0 on all children of v. If $|v| = n$ and $g(v) \neq 0$, define g on the children of v depending on whether or not $\sum_{w\leqslant v} g(w)$ is positive or 0. If the sum is 0, then define g to be 1 on two children of v and 0 on all of the other children; if the sum is positive, define g to be $-\sum_{w\leqslant v} g(w)$ on one child of v, $g(v) + \sum_{w\leqslant v} g(w)$ on another child, and 0 on the remaining children.

It follows from Theorem 4.11 that g is the derivative of a unique non-negative subharmonic function f such that $f(o) = 0$.

Theorem 7.4. *With f as above, let the reference measure σ satisfy $\sum_n \tau_n F_{2n} < \infty$, where F_k denotes the k-th Fibonacci number. Then $f \in \mathcal{G}_{1/3,\sigma}$. In particular $\sum_n \tau_n F_{2n} < \infty$ holds if $q \geqslant 3$ and $\sigma_n = q^{-n} n^{-3}$.*

To prove Theorem 7.4, we first note some well-known results concerning the Fibonacci sequence.

Lemma 7.5. *Let $\{F_n\}_{n=0}^{\infty}$ denote the Fibonacci sequence with $F_0 = 0$ and $F_1 = 1$. Let $\gamma = (1+\sqrt{5})/2$ denote the golden ratio.*

(i) *The sequences $\{F_{2n}\}_{n=0}^{\infty}$ and $\{F_{2n+1}\}_{n=0}^{\infty}$ satisfy the recurrence relation $t_n - 3t_{n-1} + t_{n-2} = 0$.*

(ii) *(Cassini's formula) $F_{n+1} \cdot F_{n-1} = F_n^2 + (-1)^n$.*

(iii) *The sequence F_{2n-1}/F_{2n} is decreasing and the sequence F_{2n}/F_{2n+1} is increasing. Consequently $1/\gamma < F_{2n-1}/F_{2n} \leqslant 2/3$ and $3/5 \leqslant F_{2n}/F_{2n+1} < 1/\gamma$ for $n \geqslant 2$.*

Proof of Theorem 7.4. For $n \geqslant 0$, let a_n and b_n be given by

$$a_n = \sum_{v \in I_f, |v|=n} -f'(v), \quad b_n = \sum_{v \in II_f, |v|=n} f'(v). \tag{7.6}$$

There is a unique ray along $[o, \omega_1, \omega_2, \omega_3, \dots]$ along which f' strictly increases. Let c_n denote $f'(\omega_n)$. For $n \geqslant 2$ we have $c_n = c_{n-1} + (c_0 + c_1 + \cdots + c_{n-1})$. Similarly $c_{n-1} = c_{n-2} + (c_0 + c_1 + \cdots + c_{n-2})$. Subtracting gives $c_n - c_{n-1} = c_{n-1} - c_{n-2} + c_{n-1}$, which says $c_n - 3c_{n-1} + c_{n-2} = 0$. In addition $c_1 = 1$ and $c_2 = 2$. Thus by Lemma 7.5, $c_n = F_{2n-1}$, $n \geqslant 1$.

Let d_n denote the value of $-f'$ at the child of ω_{n-1} $(n \geqslant 2)$ where f' is negative. Then

$$d_n = c_0 + c_1 + \dots c_{n-1} = \sum_{k=1}^{n-1} F_{2k-1} = \sum_{k=1}^{n-1} (F_{2k} - F_{2k-2}) = \sum_{k=1}^{n-1} F_{2k} - \sum_{k=0}^{n-2} F_{2k} = F_{2n-2} - F_0 = F_{2n-2}.$$

Now that we have identified the sequences c_n and d_n, we write down recurrence relations for a_n and b_n. Note that in assigning the values of f', once we arrive at a vertex v for which the $\sum_{w \leqslant v} f'(w) = 0$, the process begins over again. This then implies

$$a_n = d_n + 2(a_2 + a_3 + \cdots + a_{n-2}) \text{ and } b_n = c_n + 2(b_1 + b_2 + b_3 + \cdots + b_{n-2}).$$

Thus $a_{n+1} = d_{n+1} + 2(a_2 + a_3 + \cdots + a_{n-1})$. Subtracting gives $a_{n+1} - a_n = d_{n+1} - d_n + 2a_{n-1} = F_{2n} - F_{2n-2} + 2a_{n-1} = F_{2n-1} + 2a_{n-1}$, and so a_n is uniquely determined by

$$a_{n+1} - a_n - 2a_{n-1} = F_{2n-1}, a_1 = 0, a_2 = 1.$$

A similar calculation shows that b_n is uniquely determined by $b_{n+1} - b_n - 2b_{n-1} = F_{2n}, b_0 = 0, b_1 = 1$.

We show next by induction that $b_n \leqslant kF_{2n}$, where $k = (1+\gamma)/(2\gamma - 1) \approx 1.17082$. As $b_n \leqslant F_{2n}$ for $0 \leqslant n \leqslant 4$, we proceed to the inductive step. Taking $n \geqslant 4$ and assuming the result true for all $k \leqslant n$, we must prove $b_{n+1} \leqslant kF_{2n+2}$, i.e. prove $b_n + 2b_{n-1} + F_{2n} \leqslant kF_{2n+2}$. Applying the inductive hypothesis, it suffices to prove that $kF_{2n} + 2kF_{2n-2} + F_{2n} \leqslant kF_{2n+2}$. Simplifying this expression after replacing F_{2n} by $F_{2n-1} + F_{2n-2}$ and F_{2n+2} by $3F_{2n-1} + 2F_{2n-2}$ reduces this to proving $(k+1)F_{2n-2} \leqslant (2k-1)F_{2n-1}$. But from Lemma 7.5, we have that $F_{2n-2}/F_{2n-1} \leqslant 1/\gamma$, so the desired inequality follows since $\gamma = (k+1)/(2k-1)$.

Next we show by induction that for all $n \geqslant 0$, $a_n \leqslant (2/3)b_n$. This is clear for $n = 0, 1, 2$. Let $n \geqslant 2$ and suppose it holds for all indices up to n. Then applying the inductive hypothesis and Lemma 7.5, $a_{n+1} = a_n + 2a_{n-1} + F_{2n-1} \leqslant \frac{2}{3}\Big(b_n + 2b_{n-1} + F_{2n}\Big) = \frac{2}{3}b_{n+1}$. Since $a_n \leqslant (2/3)b_n$ and $b_n \leqslant kF_{2n}$, it follows that $\sum_T |f'(v)|\tau(v) < \infty$. Thus,

$$\sum_{I_f} -f'(v)\tau_v = \sum_{n=0}^{\infty} a_n \tau_n \leqslant \frac{2}{3} \sum_{n=0}^{\infty} b_n \tau_n = \frac{2}{3} \sum_{II_f} f'(v)\tau_v,$$

proving that $f \in \mathcal{G}_{1/3,\sigma}$.

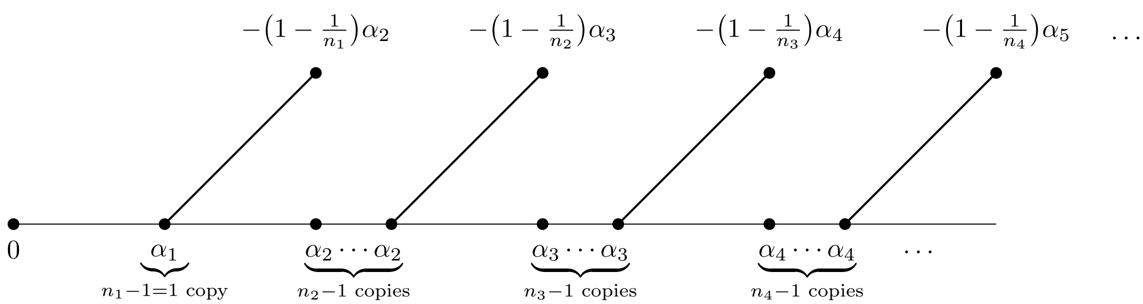
If $q \geqslant 3$ and $\sigma_n = q^{-n}n^{-3}$, then τ is of the order of $q^{-n}n^{-2}$. Since F_{2n} is of the order $\gamma^{2n} \approx 2.62^n$, it follows that $\sum_n \tau_n F_{2n} < \infty$. $\square$

Remark 7.2. In the proof of Theorem 7.4, we needed to show that f satisfies (ii) of Definition 5.1. We did this by proving the stronger result that $a_n \leqslant (2/3)b_n$ for all n, where a_n and b_n are defined in (7.6). It is easy to see that for any f satisfying the conditions of Definition 5.1, $a_n/b_n < 1$ for all n. One might wonder if it is always the case that for any such f, $a_n/b_n \leqslant 2/3$ for all n, or if not $2/3$ at least some constant strictly less than 1. We might suspect this since in constructing $g = f'$ in Theorem 7.4, at each generation we made each negative value as negative as possible as soon as possible (still satisfying (4.6) with $b = 0$ of Theorem 4.8) and the resulting positive value as small as possible (being sure to keep g wealth increasing). However, we show in the next example a function f for which $\limsup\limits_{n\to\infty} a_n/b_n = 1$, yet which satisfies $f \in \mathcal{G}_{1-1/q,\sigma}$ for any good reference measure σ.

Let $n_k = 2^k$ for $k \geqslant 1$, and let $n_0 = 0$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be the sequence for which $\alpha_1 = 1$ and $\alpha_{k+1} = n_k\alpha_k$ for each $k \geqslant 1$. It follows that $\alpha_k = n_1 n_2 \dots n_{k-1} = 2^{(k-1)k/2}$. Fix a ray $\omega = [o = \omega_0, \omega_1, \dots]$. For each $k \geqslant 1$, let $\overline{\omega}_k$ be one of the children not in ω of $\omega_{(n_1-1)+(n_2-1)+\cdots+(n_k-1)} = \omega_{n_1+\cdots+n_k-k}$. Thus $|\overline{\omega}_k| = \sum_{i=1}^{k} n_i - k + 1$. For each $k \geqslant 1$, $\sum_{i=0}^{k-1} n_i - (k-2) + (n_k - 1) = \sum_{i=0}^{k} n_i - (k-1)$. Thus we can define $g : T \to [0, \infty)$ by

$$g(v) = \begin{cases} \alpha_k & \text{if } k \geqslant 1, v = \omega_j \text{ with } \sum_{i=1}^{k-1} n_i - (k-2) \leqslant j \leqslant \sum_{i=1}^{k} n_i - k; \\ -\left(1 - \frac{1}{n_k}\right)\alpha_{k+1} & \text{if } k \geqslant 1, \text{ and } v = \overline{\omega}_k; \\ 0 & \quad \text{otherwise.} \end{cases} \tag{7.7}$$

For each k there are $n_k - 1$ consecutive vertices on which g takes the value α_k. See the figure below.



Theorem 7.6. *Let g be as in (7.7).*

(i) *There is a unique non-negative subharmonic function f with $f(o) = 0$ such that $f' = g$.*
(ii) *Define $a_n = \sum\limits_{v \in I_f, |v|=n} -g(v)$, and $b_n = \sum\limits_{v \in \mathit{II}_f, |v|=n} g(v)$. Then $\limsup_{n\to\infty} a_n/b_n = 1$.*
(iii) *Let σ be any good reference measure. Then $f \in \mathcal{G}_{1-q^{-1},\sigma}$.*

Proof. For each $k \geqslant 1$, if we sum g over the $n_k - 1$ terms on which it is α_k, we get $\alpha_k(n_k - 1) = \alpha_k n_k(1 - 1/n_k) = \alpha_{k+1}(1 - 1/n_k)$. Thus, g satisfies (4.6) with $b = 0$. Also $-(1-1/n_k)\alpha_{k+1} + \alpha_{k+1} = \alpha_{k+1}/n_k = \alpha_k$, so g is wealth preserving at some vertices and wealth increasing at all vertices. Thus (i) follows by Theorem 4.11.

Noting that for any n, a_n is either 0, or it equals $(1 - 1/n_{k-1})\alpha_k$ for some k, in which case $b_n = \alpha_k$, (ii) follows since $n_k \to \infty$ as $k \to \infty$.

Note that for $k \geqslant 1$, $n_0 + n_1 + n_2 + \cdots + n_{k-1} - (k-2) = 2^k - k$ and $n_1 + n_2 + \cdots + n_k - k = 2^{k+1} - k - 2$.

Let σ be a good reference measure. To prove (iii), we need to show $\sum_T |g(v)|\tau(v) < \infty$ and $\sum_{I_f} -g(v)\tau(v) \leqslant (1/q)\sum_{\mathit{II}_f} g(v)\tau(v)$. Clearly, $\sum_{v \in I_f} |g(v)|\tau(v) \leqslant \sum_{v \in \mathit{II}_f} g(v)\tau(v)$, so to prove the summability of $|g|\tau$ on T, it suffices to show the summability of $g\tau$ on II_f. Grouping together the terms where g is constant and using the decreasing property of $j \mapsto \tau_j$ and the assumption that σ is good, we obtain

$$\begin{aligned}\sum_{v\in \mathit{\Pi}_f} g(v)\tau(v) &= \alpha_1\tau_1+\alpha_2(\tau_2+\tau_3+\tau_4)+\alpha_3(\tau_5+\tau_6+\cdots+\tau_{11})+\alpha_3(\tau_{12}+\cdots+\tau_{26})+\dots \\ &= \sum_{k=1}^{\infty}\alpha_k \sum\Big\{\tau_j : \sum_{i=1}^{k-1} n_i-(k-2)\leqslant j\leqslant \sum_{i=1}^{k} n_i - k\Big\} = \sum_{k=1}^{\infty}\sum_{j=2^k-k}^{2^{k+1}-k-2}\alpha_k\tau_j \leqslant \sum_{k=1}^{\infty}\alpha_k(n_k-1)\tau_{2^k-k} \\ &\leqslant \sum_{k=1}^{\infty}\alpha_{k+1}\tau_{2^k-k} \leqslant \sum_{k=1}^{\infty} 2^{\frac{k(k+1)}{2}}\tau_{2^{k-1}} \leqslant 4\sum_{k=1}^{\infty} 2^{2^{k-1}}\tau_{2^{k-1}} \leqslant 4\sum_{k=1}^{\infty}2^n\tau_n \leqslant 4\sum_{k=1}^{\infty} q^n\tau_n<\infty.\end{aligned}$$

To complete the proof of (III), note that $|\overline{\omega}_k| = \sum_{i=0}^{k} n_i - k + 1 = 2^{k+1}-k-1$ and for any n, $\tau_n = \sigma_n + q\tau_{n+1} > q\tau_{n+1}$. Thus, since $j \mapsto \tau_j$ is decreasing, we have

$$\begin{aligned}\sum_{I_f} -g(v)\tau(v) &= \Big(1-\frac{1}{n_1}\Big)\alpha_2\tau_2 + \Big(1-\frac{1}{n_2}\Big)\alpha_3\tau_5 + \Big(1-\frac{1}{n_3}\Big)\alpha_4\tau_{12}+\dots \\ &= \sum_{k=1}^{\infty}\Big(1-\frac{1}{n_k}\Big)\alpha_{k+1}\tau_{2^{k+1}-k-1} = \sum_{k=1}^{\infty}\Big(n_k-1\Big)\alpha_k\;\tau_{2^{k+1}-k-1} \\ &\leqslant \frac{1}{q}\sum_{k=1}^{\infty}\Big(n_k-1\Big)\alpha_k\;\tau_{2^{k+1}-k-2} \leqslant \frac{1}{q}\sum_{k=1}^{\infty}\sum_{j=2^k-k}^{2^{k+1}-k-2}\alpha_k\;\tau_j = \frac{1}{q}\sum_{\mathit{\Pi}_f} g(v)\tau(v).\end{aligned}$$

Therefore, $f \in \mathcal{G}_{1-q^{-1},\sigma}$ and the proof is complete. □

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