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Classification of harmonic structures on graphs ☆

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ABSTRACT

Graphs, viewed as one-dimensional simplicial complexes, can be given harmonic structures satisfying the Brelot axioms. In this paper, we describe all possible harmonic structures on graphs. We determine those harmonic structures which induce discrete harmonic structures when restricted to the set of vertices. Conversely, given a discrete harmonic structure on the set of vertices and an arbitrarily prescribed harmonic structure on each edge, we determine when these structures yield a harmonic structure on the graph. In addition, we provide a variety of interesting examples.

1. Introduction

Classical potential theory is concerned with the study of harmonic functions, and more generally, of the solutions of elliptic and parabolic partial differential equations on open subsets of $\mathbb{R}^n$. From the analysis of these different environments, it was realized that these theories could be unified, and this led to the development of axiomatic potential theory in which the underlying space $\mathbb{R}^n$ was replaced by an abstract topological space. Many axiomatic treatments of potential theory were formulated in the last century. For a survey of the different developments of the theory and a historical context, see [4].

Parallel to this approach, is the subject of Markov chain theory which can be formulated in terms of discrete potential theory on the vertices of graphs ([5] for trees, [9] for Markov chains, [14] for

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lattices, [10,11,15] for random walks on groups). In this work, we attempt to bridge the two approaches by studying harmonic structures on graphs. A graph may be considered to be a discrete object by looking only at its vertices, or may be considered as a topological space by viewing it as a one-dimensional simplicial complex (or polyhedron). Each interpretation yields a notion of harmonic structure. Other treatments involving potential theory on one (respectively, higher) dimensional polyhedra include [6] and [8], respectively. The harmonic structures in those two settings involve only the classical Laplacian. By contrast, in the present work we consider all possible harmonic structures on graphs in the sense of the theory of harmonic functions developed by Brelot (see [3]). Though this seems very natural, to the best of our knowledge, such a study has not been made.

Throughout this paper, when we use the terminology *harmonic structure* in a nondiscrete setting, we shall always mean a harmonic structure in the sense of Brelot.

Our goals in the present work are to understand the possible harmonic structures on a graph and their relations to discrete harmonic structures.

To understand this paper, the needed background on graphs and Brelot spaces is fairly minimal. We give this background in Section 2.

In Section 3, we focus our attention on the study of harmonic structures on the edges, that is, on intervals. Specifically, we divide the harmonic structures on an interval into three types: *quasi-linear* (the constants are harmonic), *quasi-hyperbolic* (the nonzero constants are not harmonic but there exists a positive harmonic function), and *quasi-trigonometric* (there are no positive harmonic functions). We then give a complete classification of each of these.

We call two harmonic structures *Brelot isomorphic* if there is a homeomorphism between the spaces which carries one harmonic structure onto the other. In Theorem 3.3 we show that there are three non-Brelot isomorphic quasi-linear structures which correspond to the linear structures on $(0,1)$, $(0,\infty)$, and $\mathbb{R}$. We then show that after a suitable normalization, to each positive continuous function there corresponds a quasi-hyperbolic structure which is unique up to a one-parameter family of linear fractional transformations (see Theorem 3.5).

A different classification of Brelot structures on intervals was provided in [12], which was based on an earlier work on one-dimensional harmonic spaces (see [13]). However, the authors considered structures which they call equivalent that are non-Brelot isomorphic under our definition of Brelot isomorphism. In [12], two harmonic structures are called *equivalent* if, up to a Brelot isomorphism, one harmonic structure is obtained by multiplying each member of the other structure by some fixed positive continuous function. In particular, in [12] all quasi-linear and quasi-hyperbolic structures are in a single equivalence class, while they are very different in terms of Brelot isomorphism, as observed above. Similarly, in the quasi-trigonometric case, it is easy to find non-Brelot isomorphic structures which are equivalent in the sense of [12]. For example, the structure induced by $\{\sin x, \cos x\}$ on $\mathbb{R}$ is equivalent to the structure induced by $\{(2+\sin 4x)\sin x, (2+\sin 4x)\cos x\}$, while these structures are not Brelot isomorphic because a linear combination of $\sin x$ and $\cos x$ is at most 2 to 1 between two consecutive zeros, while $(2+\sin 4x)\cos x$ can be as much as 4 to 1 between consecutive zeros, so no homeomorphism linking these functions can exist on $\mathbb{R}$. Thus, the classification of Brelot structures on intervals presented in Section 3 is much finer than the classification in [12].

In Section 4, we develop the concept of extendibility of a harmonic structure on an edge as a first step towards producing a harmonic structure on the graph.

In Section 5, we show that, given a graph and an arbitrary extendible harmonic structure on each edge, it is possible to construct harmonic structures on the graph which restrict to the given structure.

In Section 6, we introduce concepts such as Dirichlet domain, positive Dirichlet domain, as well as the ball and weak ball regularity axioms. These all relate to various aspects of solving the Dirichlet problem. We illustrate such concepts with several examples. In particular, we give an example of a harmonic structure on a graph whose restriction to the vertices yields only the constant functions.

In Section 7, we explore the connection between harmonic structures on a graph and discrete harmonic structures on the vertices. The weak ball regularity axiom is key to this connection. We conclude the section by showing that a relatively compact domain in a tree is regular if and only if there exists a harmonic function on it which is positive and continuous on its closure.

The principal objectives of this paper are summarized as follows:

- Describe all possible harmonic structures on a graph in Section 3 (where we characterize the harmonic structures on the edges) and in Theorems 5.1 and 7.4.
- Determine those harmonic structures which induce discrete harmonic structures when restricted to the set of vertices in Theorem 7.1.
- Given a discrete harmonic structure on the set of vertices and an arbitrarily prescribed harmonic structure on each edge, we determine in Theorems 5.2 and 7.3 when these structures yield a harmonic structure on the graph.

2. Terminology and notation

2.1. Graphs

Two vertices v and w of a graph G that have an edge connecting them are called **neighbors**, in which case we use the notation $v \sim w$. A vertex with a single neighbor is called **terminal**. The **degree** of a vertex v is the number of neighbors of v. A **path** is a finite or infinite sequence of edges $[v_0, v_1], [v_1, v_2], \ldots,$ such that $v_k \sim v_{k+1}$. A **geodesic path** is a path $[v_0, v_1, \ldots]$ such that $v_{k-1} \neq v_{k+1}$ for all k. The **length** of a finite path $[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_n]$ is n.

To avoid a compatibility issue, in this paper we shall be considering only infinite graphs G with no parallel edges or loops, that is, graphs with infinitely many vertices such that for each pair of neighboring vertices there exists a unique edge between them and for which a vertex is never a neighbor of itself. In any case parallel edges and loops can be eliminated by adding new vertices to the graph. We shall further assume that G is locally-finite and connected: each vertex has a finite number of neighbors and for any pair of distinct vertices v and w there is a path from v to w. The distance $d(v, w)$ between two vertices v and w is the length of the shortest path from v to w.

A **tree** is a simply-connected graph, that is, a graph for which between any two vertices there is a unique geodesic path. A tree is said to be **homogeneous of degree *d*** if each vertex has exactly d neighbors.

The **interior** of a subset K of a graph G is the set of vertices of K all of whose neighbors are also in K.

By a function on a graph G we mean a function on its set of vertices.

A **discrete harmonic structure** on a graph G is given by a function $P : G \times G \to [0, \infty)$ such that $P(u, v) > 0$ if and only if $u \sim v$. We define a function f on G to be **discrete harmonic** at a nonterminal vertex u, if the Laplacian of f at u defined by

$$\Delta_P f(u) = \sum_{v \sim u} P(u, v) f(v) - f(u)$$

is equal to 0. Given an open set U in $\tilde{G}$ (see Section 2.3), we define f to be discrete harmonic on $U \cap G$ if it is discrete harmonic at each vertex of the interior of $U \cap G$. Denote by $\mathcal{H}_P(U)$ the set of all such functions.

A **stochastic structure** on G is a discrete harmonic structure whose corresponding function P satisfies the condition $\sum_{v \sim u} P(u, v) = 1$ for all $u \in G$, i.e. the constant functions are discrete harmonic. A stochastic structure is also called a **nearest-neighbor transition probability on *G***.

If P is a discrete harmonic structure on G and $\lambda > -1$, we may define a new discrete harmonic structure on G by $P_1 = \frac{P}{\lambda+1}$. Then a function f on G is harmonic with respect to P_1 if and only if $\Delta_P f = \lambda f$, that is, f is an eigenfunction of the Laplacian Δ_P.

2.2. Brelot harmonic spaces

Definition 2.1. A **Brelot space** is a connected, locally connected, locally compact but not compact Hausdorff space Ω together with a harmonic structure $\mathcal{H}$ in the following sense. For each open set $U \subset \Omega$ there is an associated vector space $\mathcal{H}(U)$ of real-valued continuous functions on U (which are called **harmonic functions on** U) satisfying the following three axioms.

Axiom 1 *(Sheaf Property).*

(i) If U_0 is an open subset of U, the restriction to U_0 of any function harmonic on U is harmonic on U_0.
(ii) A function defined on an open set U which is harmonic on an open neighborhood of each point of U is harmonic on U.

Definition 2.2. Let U be an open relatively compact subset of Ω. If for a real-valued continuous function f on ∂U, there exists a unique harmonic function h_f^U on U approaching f at each point of ∂U, we say that h_f^U solves the **Dirichlet problem on U with boundary values** f. If for any such f, the Dirichlet problem with boundary values f can be solved, and if, in addition, $f \geqslant 0$ implies $h_f^U \geqslant 0$, then we say that U is **regular**.

Axiom 2 *(Regularity Axiom).* There exists a base of regular domains for the open sets of Ω.

Axiom 3 *(Harnack's Property).* Any increasing directed family of harmonic functions defined on a domain U has an upper envelope (pointwise supremum) which is either identically $+\infty$ or is harmonic on U.

Remark 2.1. If Ω is second countable, which it is for all the spaces considered in this paper, Axiom 3 is equivalent to the corresponding statement for increasing sequences rather than directed families.

Observation 2.1. If U is a domain and f is harmonic on U, and $f \geqslant 0$, then either f is identically zero on U or f is strictly positive on U. For, if f were positive somewhere and $f(x_0) = 0$ for some $x_0 \in U$, then the increasing sequence $\{nf\}_{n=1}^{\infty}$ would have limit zero at x_0 and limit $+\infty$ at other points, contradicting Axiom 3.

Observation 2.2. Let Ω and Ω' be Brelot spaces such that $\Omega \cap \Omega'$ is open in each, and the corresponding harmonic structures are the same on the intersection. Then there is a unique harmonic structure on $\Omega \cup \Omega'$ such that the restrictions to Ω and Ω' yield the original harmonic structures on Ω and Ω'.

Observation 2.3. Let $(\Omega', \mathcal{H}')$ be a Brelot space and let Ω be a space homeomorphic to Ω'. A homeomorphism $\varphi : \Omega \to \Omega'$ induces a harmonic structure $\mathcal{H}$ on Ω as follows: if U is an open set in Ω, $h \in \mathcal{H}(U)$ if and only if $h \circ \varphi^{-1} \in \mathcal{H}'(\varphi(U))$. We call $\mathcal{H}$ the **harmonic structure induced by** φ.

Definition 2.3. Two Brelot spaces $(\Omega, \mathcal{H})$ and $(\Omega', \mathcal{H}')$ are **Brelot isomorphic** if there exists a homeomorphism $\varphi : \Omega \to \Omega'$ such that for every U open in Ω a function f is $\mathcal{H}'$-harmonic on $\varphi(U)$ if and only if $f \circ \varphi$ is $\mathcal{H}$-harmonic on U. In particular, $\mathcal{H}$ is the harmonic structure induced by φ.

Definition 2.4. A **B.H.** space is a Brelot space whose sheaf of harmonic functions contains the constant functions.

Observation 2.4. By Observation 2.1, in a B.H. space, if a harmonic function has a local extremum at a point, it must be constant in a neighborhood of that point.

2.3. Graphs as Brelot spaces

Let G be an infinite graph. Consider the space $\tilde{G}$ which is the graph viewed as a 1-dimensional simplicial complex, with the addition of an open edge at each of the terminal vertices. That is, for all u, v nonterminal vertices with $u \sim v$, consider the set $[u, v] = \{(1-t)u + tv\colon\ 0 \leqslant t \leqslant 1\}$. If u is a terminal vertex and $u \sim v$, set $[u, v] = \{(1-t)u + tv\colon\ -1 < t \leqslant 1\}$, so that u is in the interior of $[u, v]$.

Then $\tilde{G} = \bigcup_{u\sim v}[u, v]$. Of course, we identify $(1-t)u + tv$ with $tv + (1-t)u$ for $0 \leqslant t \leqslant 1$, $0u + v$ with v, and we assume that different edges intersect at most at a common vertex.

Let u be a vertex and let $B_1(u)$ be the unit ball centered at u, i.e.

$$B_1(u) = \bigcup_{w\sim u}[u, w).$$

For $n > 1$, define the n-ball centered at u inductively by

$$B_n(u) = \{u\} \cup \bigcup_{v\sim u} B_{n-1}(v).$$

In [1] we showed that if T is a tree, $\tilde{T}$ may be given a structure of a Brelot space by extending ordinary harmonic functions on the vertices linearly on the edges. By lifting to the universal cover, the argument used in the tree setting can be easily adapted to graphs, which are a special case of what Eells and Fuglede call 1-dimensional Riemannian polyhedra. In [8], they show that a Riemannian polyhedron is a Brelot space. The linear harmonic structure on graphs was also studied in [6]. Furthermore, in [1] we showed that the eigenfunctions of the Laplacian relative to a positive eigenvalue r on a tree T also yield harmonic structures on $\tilde{T}$. When restricted to the edges, they are the solutions to the Helmholtz equation $y'' = ry$. Again, lifting to the universal cover of a graph G, which is a tree, it is easy to see that the Brelot axioms hold on $\tilde{G}$. Unlike the structure defined by the kernel of the Laplacian operator, these do not contain the constants. In the present paper we will consider harmonic structures on graphs which have more general structures on the edges.

3. Classification of harmonic structures on an interval

In this section we are going to give a complete classification of harmonic structures on an interval up to a Brelot isomorphism. Since endpoints of an interval cannot have a regular neighborhood, we consider only open intervals.

Definition 3.1. A harmonic structure on an interval I is called **quasi-linear** if the constant functions are harmonic, **quasi-hyperbolic** if the nonzero constants are not harmonic but there exist positive harmonic functions on I, **quasi-trigonometric** if there are no positive harmonic functions on I.

On $\mathbb{R}$ the solutions to the differential equation $u'' + \alpha u = 0$ are examples of quasi-linear ($\alpha = 0$), quasi-hyperbolic ($\alpha < 0$), and quasi-trigonometric ($\alpha > 0$) harmonic structures, respectively. Actually, these three examples yield quite literally the linear, hyperbolic, and trigonometric functions, and the fact that $\mathbb{R}$ under these structures is a Brelot space was remarked in [7, p. 70].

Without loss of generality, we shall focus our analysis on the interval $(0, 1)$. Assume $(0, 1)$ is endowed with a harmonic structure $\mathcal{H}$. If U is a domain in $(0, 1)$, let $\mathcal{H}(U)$ be the set of all harmonic functions on U. Unless specified otherwise, all intervals in this section are subintervals of $(0, 1)$.

Proposition 3.1. *If h is harmonic on (a, b) and zero on a subinterval, then h is identically 0. Two harmonic functions on (a, b) which are equal on the boundary of a regular subinterval are equal on (a, b).*

Proof. Let $(c, d) \subset (a, b)$ be a maximal subinterval on which h is 0. If $(c, d) \neq (a, b)$, then without loss of generality, we may assume $a < c$. Let (c', d') be a regular interval with $a < c' < c < d' < d$. Since h is the solution to the Dirichlet problem on (c', d') with boundary values $h(c')$ and $h(d')$, with $h(d') = 0$, it follows from Axiom 2 that h does not change sign inside (c', d'). Since h is identically 0 on (c, d'), it follows from Observation 2.1 that h is identically 0 on (c', d') and hence on (c', d), contradicting the maximality of (c, d).

If two harmonic functions agree on the boundary of a regular subinterval of (a, b), then their difference must vanish on the subinterval, hence on (a, b). □

Corollary 3.1.

(a) *Harmonic functions which are linearly independent on an interval are linearly independent on any subinterval.*
(b) *Two harmonic functions which are linearly independent on an interval I generate all the harmonic functions on I.*

The proof follows easily from Proposition 3.1.

Theorem 3.1. *Any harmonic function f on a subinterval of $(0,1)$ can be uniquely extended to a harmonic function on $(0,1)$.*

Proof. Let (a,b) be a maximal interval on which f can be extended harmonically. Assume $a>0$. Let (c,d) be a regular neighborhood of a, $0<c<a<d<b$. Let h,k be the harmonic functions on (c,d) solving the Dirichlet problem so that $h(c)=k(d)=0$, $h(d)=k(c)=1$. Thus h,k are linearly independent on the two-point set $\{c,d\}$, hence on $[c,d]$, and so by Corollary 3.1, there exist $\alpha,\beta\in\mathbb{R}$ such that $f=\alpha h+\beta k$ on (a,d). Now define

$$\tilde{f}=\begin{cases} f & \text{on } (a,b),\\ \alpha h+\beta k & \text{on } (c,d).\end{cases}$$

By Axiom 1, $\tilde{f}$ is a harmonic extension of f to (c,b), contradicting the maximality of (a,b). Thus $a=0$. Similarly we can show that $b=1$. The uniqueness follows from Proposition 3.1. $\square$

We now show that the space of harmonic functions on any subinterval (a,b) of $(0,1)$ is two-dimensional.

Theorem 3.2. *There exist linearly independent harmonic functions f,g on $(0,1)$ such that f,g restricted to any subinterval form a basis for the space of harmonic functions on that subinterval.*

Proof. Let (a,b) be a regular interval in $(0,1)$ and let f,g be the solutions to the Dirichlet problems such that $f(a)=g(b)=0$ and $f(b)=g(a)=1$. By Theorem 3.1, f and g can be extended harmonically to $(0,1)$. Since f and g are linearly independent on the two-point set $\{a,b\}$, they are also linearly independent on $(0,1)$. The result follows from Corollary 3.1. $\square$

Definition 3.2. A pair of continuous functions f and g on any interval in $\mathbb{R}$ is said to be a **Brelot basis** if f,g restricted to any subinterval form a basis for the space of harmonic functions on that subinterval.

Observation 3.1. Let I, J be any two open intervals in $\mathbb{R}$ and let $\varphi: I\to J$ be a homeomorphism. Let $\{f,g\}$ be a Brelot basis on J. Then $\{f\circ\varphi, g\circ\varphi\}$ is a Brelot basis for the induced harmonic structure on I.

Observation 3.2. If f and g are linearly independent harmonic functions on $(0,1)$, then $\{f,g\}$ is a Brelot basis by Corollary 3.1.

Proposition 3.2. *Let $x_0\in(0,1)$. If f and g are linearly independent harmonic functions on $(0,1)$ such that $f(x_0)=0$, then $g(x_0)\neq 0$.*

Proof. Let (a,b) be a regular neighborhood of x_0 in $(0,1)$. Then there exists a positive harmonic function h on (a,b). Thus $h(x_0)>0$. But h is a linear combination of f and g, so $g(x_0)\neq 0$. $\square$

Proposition 3.3. *Let (a,b) be regular in $(0,1)$ and let $c \in (a,b]$. Then there exist unique constants $\alpha > 0$ and $\beta \geqslant 0$ such that for all f harmonic on (a,b) and continuous on $[a,b]$, $f(b) = \alpha f(c) - \beta f(a)$.*

Proof. Let g, h be the functions which are harmonic on (a,b), continuous on $[a,b]$, and such that $g(a) = 1$, $g(b) = 0$, $h(a) = 0$, $h(b) = 1$. Note that $g(c) \geqslant 0$ and $h(c) > 0$, where $g(c) = 0$ only for the case $c = b$. Then for f equal to g or h, we have

$$f(c) = g(c)f(a) + h(c)f(b). \tag{1}$$

Since $\{g, h\}$ is a Brelot basis on (a,b), any harmonic function f on (a,b) is a linear combination of g and h so that (1) holds for all f. Now letting $\alpha = \frac{1}{h(c)}$ and $\beta = \frac{g(c)}{h(c)}$, we get the result. □

3.1. Quasi-linear harmonic structures

Assume throughout this section that $(0,1)$ is endowed with a quasi-linear harmonic structure.

Observation 3.3. In this setting, any nonconstant harmonic function on $(0,1)$ is 1-to-1. This follows from Observation 2.4 and Proposition 3.1.

Theorem 3.3. *Let $\mathcal{H}$ be a quasi-linear harmonic structure on $(0,1)$ and let $f \in \mathcal{H}(0,1)$ be nonconstant. Then $\{1, f\}$ is a Brelot basis. Furthermore, every relatively compact subinterval of $(0,1)$ is regular. Conversely, every continuous 1-to-1 function f of $(0,1)$ gives rise to a quasi-linear harmonic structure on $(0,1)$ such that $\{1, f\}$ is a Brelot basis. Consequently, any quasi-linear harmonic structure on an interval is Brelot isomorphic to the structure of linear functions on some interval, and hence there are three distinct classes of quasi-linear structures on intervals represented by the linear structures on $(-\infty,\infty)$, $(0,\infty)$, and $(0,1)$.*

Proof. Note that by Axiom 2 and Theorem 3.1, a nonconstant harmonic function necessarily exists. For the first part of the statement, let $f \in \mathcal{H}(0,1)$ be nonconstant, (a,b) a relatively compact subinterval of $(0,1)$, h harmonic on (a,b), and let (x_0, x_1) be regular such that $a < x_0 < x_1 < b$. By Observation 3.3 we may set

$$\alpha = \frac{h(x_0) - h(x_1)}{f(x_0) - f(x_1)}, \qquad \beta = \frac{h(x_1)f(x_0) - h(x_0)f(x_1)}{f(x_0) - f(x_1)}.$$

Then $\alpha f + \beta - h$ is a harmonic function which equals 0 at x_0 and x_1, hence, by Axiom 2 and Proposition 3.1, it is identically zero on (a,b). Thus, $\{(\alpha f + \beta)|(a,b)\colon\ \alpha, \beta \in \mathbb{R}\} = \mathcal{H}(a,b)$, i.e. $\{1, f\}$ is a Brelot basis.

By Observation 3.3, we may set

$$\alpha' = \frac{1}{f(b) - f(a)}, \qquad \beta' = -\frac{f(a)}{f(b) - f(a)}.$$

Then $\alpha' f + \beta'$ is harmonic, vanishing at a and with value 1 at b, so, by Observation 3.3, it is increasing hence positive on (a,b). Similarly, we can find constants α'' and β'' such that the harmonic function $\alpha'' f + \beta''$ has value 1 at a and vanishes at b, hence it is positive on (a,b). Thus the Dirichlet problem can be solved uniquely and nonnegative values on $\{a,b\}$ yield a nonnegative solution on (a,b), so (a,b) is regular.

The converse assertion follows from Observation 3.1 and the fact that the constant 1 and the identity is a Brelot basis which gives rise to the set of affine functions on any interval.

Finally, if $\{1, f\}$ is a Brelot basis for the harmonic structure $\mathcal{H}$ on an interval (a,b), then f is a homeomorphism from (a,b) to some interval (c,d), possibly infinite. Thus, f is itself a Brelot isomorphism from $\mathcal{H}$ to the linear structure on (c,d). Since linear structures can only be preserved by a linear function, the three different types of linear structures are determined by the cases: c and d are both finite, one endpoint is finite and the other is infinite, $c = -\infty$ and $d = \infty$. □

3.2. Quasi-hyperbolic harmonic structures

In this section, we assume the existence of a positive harmonic function.

Theorem 3.4. *Let $\mathcal{H}$ be a quasi-hyperbolic structure on $(0,1)$ and let k be a positive harmonic function on $(0,1)$. Then there exists a harmonic function h on $(0,1)$ such that $\{h,k\}$ is a Brelot basis. For any such k, h/k is 1-to-1. All relatively compact subintervals of $(0,1)$ are regular.*

Conversely, let k be any positive continuous function on $(0,1)$ and let φ be any homeomorphism from $(0,1)$ to some interval. Then the pair $\{k, k\varphi\}$ is a Brelot basis for a harmonic structure on $(0,1)$ which is either quasi-linear or quasi-hyperbolic. It is quasi-linear if and only if k is constant or φ is a linear combination of $1/k$ and 1.

Proof. Applying Theorem 3.2, we may find $h \in \mathcal{H}(0,1)$ such that $\{h,k\}$ is a Brelot basis. Let us define $\mathcal{H}_{1/k}$ on $(0,1)$ by

$$\mathcal{H}_{1/k}(a,b) = \left\{ f k^{-1}|_{(a,b)}\colon\ f \in \mathcal{H}(0,1) \right\},$$

for $0 \leqslant a < b \leqslant 1$. It follows by a straightforward argument that $\mathcal{H}_{1/k}$ is a harmonic structure on $(0,1)$ which contains the constants. Since the function $h/k \in \mathcal{H}_{1/k}$ and is nonconstant, it is 1-to-1 by Observation 3.3. By Theorem 3.3, (a,b) is regular for $\mathcal{H}_{1/k}$.

For the converse statement, observe that if φ is a homeomorphism from $(0,1)$ to some interval, then by Theorem 3.3, $\{1,\varphi\}$ is a Brelot basis for a harmonic structure $\mathcal{H}$ on $(0,1)$. Thus, using the above notation, if k is a positive continuous function on $(0,1)$, then $\{k, k\varphi\}$ forms a Brelot basis for the harmonic structure $\mathcal{H}_k$. The constant 1 is harmonic with respect to this structure if and only if k is constant or φ is a linear combination of $1/k$ and 1. □

Observation 3.4. Let $\mathcal{H}$ be a harmonic structure on $(0,1)$ for which $(0,1)$ is regular with respect to the Dirichlet problem on $[0,1]$. Let k be the solution to the Dirichlet problem such that $k(0) = 1 = k(1)$. Then k is positive on $[0,1]$, so $\mathcal{H}$ is either quasi-linear or quasi-hyperbolic.

Remark 3.1. Given a positive continuous function k on $(0,1)$, by composing it with an appropriate homeomorphism, we may assume that the pair $\{k(x), xk(x)\}$ is a Brelot basis for a quasi-linear or quasi-hyperbolic structure on $(0,1)$, which we refer to as the *harmonic structure induced by k*.

Theorem 3.5. *Let k and k' be positive continuous functions on $(0,1)$ whose induced harmonic structures $\mathcal{H}$ and $\mathcal{H}'$ are quasi-hyperbolic. Then $\mathcal{H}$ and $\mathcal{H}'$ are Brelot isomorphic if and only if k' is a positive constant multiple of either the function $x \mapsto \frac{1}{ax+1}k(\frac{(a+1)x}{ax+1})$, for some $a > -1$, or the function $x \mapsto \frac{1}{1-bx}k(\frac{1-x}{1-bx})$, for some $b > 1$.*

Proof. By Remark 3.1, the harmonic structures induced by k and k' have Brelot basis $\{k(x), xk(x)\}$ and $\{k'(x), xk'(x)\}$, respectively. Thus, these structures are Brelot isomorphic if and only if there exist a homeomorphism φ of $(0,1)$ onto itself and real constants $\alpha, \beta, \alpha', \beta'$ such that

$$k'(x) = \alpha k\big(\varphi(x)\big) + \beta k\big(\varphi(x)\big)\varphi(x) \tag{2}$$

and

$$xk'(x) = \alpha' k\big(\varphi(x)\big) + \beta' k\big(\varphi(x)\big)\varphi(x), \tag{3}$$

for each $x \in (0,1)$. Since k is a positive function, multiplying Eq. (2) by x and subtracting Eq. (3) yields $\varphi(x) = \frac{\alpha x - \alpha'}{\beta' - \beta x}$. Since the only linear fractional transformations that map $(0,1)$ onto itself are the transformations $\varphi(x) = \frac{(a+1)x}{ax+1}$ with $a > -1$, and $\varphi(x) = \frac{1-x}{1-bx}$ with $b > 1$, and observing that $\alpha + \beta\varphi(x)$ equals $\frac{\alpha}{ax+1}$ in the first case, and $\frac{\alpha(1-b)}{1-bx}$ in the second case, we obtain the result. $\square$

3.3. Zero sets of harmonic functions

Assume that $(0,1)$ is endowed with a harmonic structure.

Proposition 3.4. *Any nonzero harmonic function on* $(0,1)$ *endowed with a quasi-linear or quasi-hyperbolic structure has at most one zero.*

This follows immediately from Proposition 3.1 since by Theorems 3.3 and 3.4, every relatively compact subinterval of $(0,1)$ is regular.

The following result follows from the fact that a harmonic structure on a regular interval is quasi-linear or quasi-hyperbolic.

Theorem 3.6. *Any subinterval of a regular interval is regular.*

There are quasi-trigonometric structures on $(0,1)$ having nonregular subintervals, as the following example shows.

Example 3.1. Let $a > \pi$ and consider the differential equation $y'' + a^2 y = 0$ on $(0,1)$. The set of solutions to this equation forms a Brelot space on $(0,1)$ for which $\{\sin at, \cos at\}$ is a Brelot basis. Then every open subinterval of length greater than π/a is not regular since the harmonic function $h(t) = \sin a(t - t_0)$ vanishes at t_0 and $t_0 + \pi/a$ for any $t_0 \in (0, 1 - \pi/a)$.

Let f and g form a Brelot basis on $(0,1)$, which will remain fixed throughout the remainder of this section, except possibly for changing the sign of f or g, or interchanging f with g.

Theorem 3.7. *The zero sets of f and g are discrete, disjoint, and alternating, i.e. between any two consecutive zeros of one there is a unique zero of the other.*

Proof. Assume $f(x_0) = 0$. Then by Proposition 3.2, there is a neighborhood U of x_0 on which g is not 0. Changing the sign if necessary, we may assume that g is positive on U. By Theorem 3.4, the function f/g is 1-to-1 on U, thus f has exactly one zero on U, proving the discreteness of the zero sets. Let $x_0 < x_1$ be consecutive zeros of f. Then we may assume that f is positive on (x_0, x_1), so by the same argument, g/f is 1-to-1 on (x_0, x_1) and so g can have at most one zero on (x_0, x_1). On the other hand, if g had no zeros inside (x_0, x_1), then again by the same argument, the function f/g would be 1-to-1 on (x_0, x_1), contradicting the fact that f/g vanishes at x_0 and x_1. Thus, g must have exactly one zero inside (x_0, x_1), proving that the zeros of f and g alternate. $\square$

There are three possibilities for $Z_f = \{x_n\}$ and $Z_g = \{y_n\}$, the zero sets of f and g.

Case 1. They are both doubly infinite. By Theorem 3.7, we may assume that for all $n \in \mathbb{Z}$, $x_n < y_n < x_{n+1}$.

Case 2. They are both finite. By Theorem 3.7, we may assume that either both zero sets have the same cardinality N or that one of them, say Z_f, has one more element than the other. Specifically, we may assume that either $x_1 < y_1 < \cdots < x_N < y_N$ or $x_1 < y_1 < \cdots < x_{N-1} < y_{N-1} < x_N$.

Case 3. They are both infinite but not doubly infinite. Without loss of generality we may assume that either $x_{-N} < y_{-N} < \cdots < x_0 < y_0 < x_1 < y_1 < \cdots$ or $\cdots < x_0 < y_0 < x_1 < y_1 < \cdots < x_N < y_N$.

Consider the trigonometric structure on $\mathbb{R}$ with $\{\sin x, \cos x\}$ as a Brelot basis. This is an example of Case 1. Restricting to an open bounded interval of length greater than π yields an example of Case 2, and restricting to a proper open unbounded subinterval yields an example of Case 3.

By Observation 2.1, f must have different signs on (x_{n-1}, x_n) and (x_n, x_{n+1}), and the same is true for g on (y_{n-1}, y_n) and (y_n, y_{n+1}). Thus, without loss of generality, we may assume that $(-1)^n f$ is positive on (x_n, x_{n+1}) and $(-1)^n g$ is positive on (y_n, y_{n+1}).

3.4. Quasi-trigonometric structures

In this section, we shall show that a quasi-trigonometric harmonic structure is very closely related to a structure generated by $\sin x$ and $\cos x$ on some appropriate interval.

Definition 3.3. Let $-\infty \leqslant a < b \leqslant \infty$ with $b - a > \pi$. A **trigonometric harmonic structure** on (a, b) is a structure generated by two continuous functions $S(x)$ and $C(x)$ such that for all $x \in (a, b)$ the sign of $S(x)$ equals the sign of $\sin x$, the sign of $C(x)$ equals the sign of $\cos x$, and $S(x)/C(x) = \tan x$.

Theorem 3.8. *Any quasi-trigonometric harmonic structure on* $(0, 1)$ *is Brelot isomorphic to a trigonometric structure on some interval* (a, b)*, where* $-\infty \leqslant a < b \leqslant \infty$*,* $b - a > \pi$*.*

Proof. Using the notation, the signs, and the ordering of the zeros of f and g following Theorem 3.7, let f and g form a Brelot basis for a quasi-trigonometric structure on $(0, 1)$, which will remain fixed throughout, except possibly for changing the sign of f or g, or interchanging one for the other.

We shall construct a homeomorphism φ from $(0, 1)$ onto an interval (a, b) and show that $C = f \circ \varphi^{-1}$ and $S = g \circ \varphi^{-1}$ satisfy the hypotheses of Definition 3.3.

We shall define φ so that on each interval (x_n, x_{n+1}), φ is an increasing homeomorphism onto $((n - 1/2)\pi, (n + 1/2)\pi)$. This will extend to the closure of the intervals except in the case that $x_1 = 0$ or, for some N, $x_N = 1$ (in which cases we do not extend). Consider $\tan^{-1} : (-\infty, \infty) \to ((n - 1/2)\pi, (n + 1/2)\pi)$.

Since $(-1)^n f$ is a positive harmonic function on (x_n, x_{n+1}) and $(-1)^n g$ has a negative limit at x_n and a positive limit at x_{n+1},

$$\frac{g}{f} = \frac{(-1)^n g}{(-1)^n f}$$

is a 1-to-1 map from (x_n, x_{n+1}) onto $(-\infty, \infty)$ by Theorem 3.4. Let $\varphi = \tan^{-1} \circ (g/f)$.

If necessary, we define φ on $(0, x_1)$ as follows. Since f and g are positive on $(0, x_1)$, they generate a quasi-hyperbolic structure there, so that g/f is 1-to-1. Because $g(x)/f(x) \to \infty$ as $x \uparrow x_1$, g/f is increasing on $(0, x_1)$. Let $\tan^{-1}$ be the branch of the inverse tangent with image $(-\pi/2, \pi/2)$, and let

$$a = \lim_{x \to 0^+} \tan^{-1} \frac{g(x)}{f(x)}.$$

Now define $\varphi : (0, x_1) \to (a, \pi/2)$ by $\varphi(x) = \tan^{-1} \frac{g(x)}{f(x)}$.

Similarly, if necessary, we define φ on $(x_N, 1)$ as follows. Consider the branch of the inverse tangent with image $((N - 1/2)\pi, (N + 1/2)\pi)$. Let

$$b = \lim_{x \to 1^-} \tan^{-1} \frac{g(x)}{f(x)}.$$

Now define $\varphi : (x_N, 1) \to ((N - 1/2)\pi, b)$ by $\varphi(x) = \tan^{-1} \frac{g(x)}{f(x)}$.

This now gives a homeomorphism φ from $(0,1)$ onto some open interval. Then $C = f \circ \varphi^{-1}$ and $S = g \circ \varphi^{-1}$ satisfy the hypotheses of Definition 3.3 yielding a quasi-trigonometric structure on $(0,1)$. $\square$

Corollary 3.2. *Let* $(0,1)$ *be given a quasi-trigonometric structure. Then there exists a Brelot basis consisting of harmonic functions each of which has at least two zeros.*

Proof. By Theorem 3.8, it is sufficient to prove the result for a trigonometric structure on some interval $(a, a+\pi+3\epsilon)$ for some $a \in \mathbb{R}$, $\epsilon > 0$ not an integer multiple of π. Let $S(x)$ and $C(x)$ be the functions above. Let $h(t) = \cos t_0 S(t) - \sin t_0 C(t)$ where $t_0 = a + \epsilon$. Then $h(t)$ equals $\frac{\sin(t-t_0)}{\cos t} C(t)$ when $\cos t \neq 0$ or $\frac{\sin(t-t_0)}{\sin t} S(t)$ when $\sin t \neq 0$. Then $h(t_0) = h(t_0+\pi) = 0$.

Similarly, let g be defined as h with $t_0 = a + 2\epsilon$. Since ϵ is not a multiple of π, $\{g, h\}$ is a Brelot basis and both g and h have at least two zeros. $\square$

Theorem 3.9. *Let f and g be continuous functions on* $(0,1)$ *whose corresponding zero sets are disjoint, discrete, alternating, and assume that f has at least two zeros. Suppose that the restriction of f/g and g/f to any interval containing no zero of g (respectively, of f) is* 1*-to-*1*. Then $\{f, g\}$ is a Brelot basis for a quasi-trigonometric harmonic structure on* $(0,1)$.

Conversely, every quasi-trigonometric harmonic structure on $(0,1)$ *has a Brelot basis of this form.*

Proof. Consider the set of all maximal intervals containing no zero of f or no zero of g. Denote this collection by $\{I_n\}$ ordered so that the left endpoint of I_n is less than the left endpoint of I_{n+1} for each n. Then $(0,1) = \bigcup_n I_n$, $I_n \cap I_{n+1} \neq \emptyset$, $I_n \cap I_{n+2} = \emptyset$ for all n. By Theorem 3.4, the pair $\{f, g\}$ is a Brelot basis for a quasi-linear or a quasi-hyperbolic harmonic structure on I_n for each n. By Observation 2.2, $\{f, g\}$ yields a Brelot basis on any finite union $I_n \cup I_{n+1} \cup \cdots \cup I_{n+k}$ ($k \in \mathbb{N}$). Because they are local in nature, the Brelot axioms hold, so $\{f, g\}$ is a Brelot basis.

For the converse, Corollary 3.2 and Theorem 3.7 imply the existence of a Brelot basis $\{f, g\}$ whose zero sets are disjoint, discrete, and alternating. Consider $k = f/g$ on an interval I containing no zero of g. Let $x_1, x_2 \in I$ be such that $k(x_1) = k(x_2) = a$. If $x_1 < x_2$, then (x_1, x_2) is regular, since g is nonzero on $[x_1, x_2]$. But $ag(x_1) = f(x_1)$ and $ag(x_2) = f(x_2)$, so using Proposition 3.1 it follows that $ag = f$, contradicting the linear independence of f and g. $\square$

Observation 3.5. Although by Corollary 3.2 it is always possible to obtain a Brelot basis both of whose elements contain at least two zeros, there may exist a Brelot basis both of whose elements have exactly one zero. Indeed, the set $\{\sin\frac{3\pi}{2}x, \cos\frac{3\pi}{2}x\}$ is a basis for a quasi-trigonometric structure on $(0,1)$, since any linear combination $h(x)$ of the two functions satisfies the condition $h(x+\frac{2}{3}) = -h(x)$ for $0 < x < 1/3$, so there are no positive harmonic functions. Both functions in this basis have exactly one zero in $(0,1)$.

4. Extendibility of harmonic functions

We now begin our study of harmonic structures on graphs.

Definition 4.1. We say that a harmonic structure on $(0,1)$ is **extendible** if it is the restriction of some harmonic structure on $(-1,1)$.

Let G be a graph and let $[u, v]$ be an edge of G. We say that a harmonic structure on (u, v) is **extendible** if the two corresponding harmonic structures on $(0,1)$ are extendible, where we identify t with either $(1-t)u + tv$ or with $tu + (1-t)v$.

We now characterize the extendible harmonic structures on an interval.

Theorem 4.1. *A harmonic structure $\mathcal{H}$ on* $(0,1)$ *is extendible if and only if for all $f \in \mathcal{H}$, $f(0) = \lim_{t\to 0^+} f(t)$ exists and for some $f \in \mathcal{H}$ this limit is nonzero.*

Proof. The necessity is an immediate consequence of Theorem 3.1 and Proposition 3.2. To prove the sufficiency, let f be a harmonic function on $(0,1)$ such that $f(0) > 0$. Then for some $\epsilon > 0$, f is positive on $(0,\epsilon)$. Thus $\mathcal{H}|(0,\epsilon)$ is either quasi-linear or quasi-hyperbolic, so there exists a homeomorphism φ from $(0,\epsilon)$ to some interval (a,b) such that $\{f, f\varphi\}$ is a Brelot basis for $\mathcal{H}|(0,\epsilon)$. Note that $a = \lim_{t\to 0^+} \varphi(t) = \lim_{t\to 0^+}(f\varphi)(t)/f(t) > -\infty$, so that φ may be extended to a homeomorphism from $(-1,\epsilon)$ to $(a-1,b)$. Choose any positive continuous extension of f to $(-1,\epsilon)$. By Theorem 3.4 applied to the interval $(-1,\epsilon)$, $\{f, f\varphi\}$ is a Brelot basis for a harmonic structure on $(-1,\epsilon)$ which agrees with $\mathcal{H}$ on $(0,\epsilon)$, therefore, by Observation 2.2, it yields a harmonic structure on $(-1,1)$ extending $\mathcal{H}$. □

Remark 4.1. In the proof of the next proposition we introduce a technique that shall be used extensively in the remainder of the paper. The purpose of this technique is to produce a basis for the space of harmonic functions near a vertex of a graph.

Proposition 4.1. *Let $\mathcal{H}$ be a harmonic structure on a graph $\tilde{G}$. Let $[u,v]$ be an edge. Then $\mathcal{H}|(u,v)$ is extendible. Furthermore, if there exists a harmonic function f such that $f|[u,v]$ is positive, then (u,v) is regular.*

Proof. Without loss of generality we may assume that u is a nonterminal vertex of G. Let v, w be distinct neighbors of u, and let O be a regular neighborhood of u contained in $B_1(u)$. For every neighbor z of u, let z' be the intersection of the boundary of O with $[u,z]$. Let f_z be the solution to the Dirichlet problem on O which is 1 at z' and 0 at the other boundary points. The set $\{f_z\colon z \sim u\}$ is a basis for the space of harmonic functions on O. Since O is regular, $f_z(u) > 0$ by Observation 2.1. By Theorem 3.1, each f_z can be extended to a harmonic function on each edge from u, which we still denote by f_z. Observe that for $v \neq w$ the functions f_v and f_w are linearly independent on (u,v) and so, by Theorem 3.2, form a Brelot basis. Therefore, every harmonic function on (u,v) is a linear combination of f_v and f_w. By Theorem 4.1, $\mathcal{H}|(u,v)$ is extendible.

Next, assume there exists a harmonic function f such that $f|[u,v]$ is positive. Viewing (u,v) as an interval, we may now extend the harmonic structure on (u,v) to a larger interval I containing its closure. Then there exists some subinterval of I containing $[u,v]$ on which f is positive. By Theorem 3.4 applied to the interval I, (u,v) is regular. □

Corollary 4.1. *Given a graph endowed with a harmonic structure, the zero set of any nonzero harmonic function on an edge whose endpoints are both nonterminal vertices is always finite.*

Proof. Let f be a nonzero harmonic function on an edge. Then there exists a harmonic function g on the edge such that $\{f,g\}$ is a Brelot basis. By Theorem 3.7, if f has infinitely many zeros, then so does g, and they will cluster at one of the endpoints v of the edge. Hence $\lim_{x\to v} f(x) = \lim_{x\to v} g(x) = 0$, contradicting Proposition 4.1 and Theorem 4.1. □

Proposition 4.2.

(a) *If $(0,1)$ has an extendible harmonic structure, then there exists some $\epsilon_0 \in (0,1)$ such that for all $\epsilon \in (0,\epsilon_0)$, the interval $(0,\epsilon)$ is regular.*
(b) *If $\mathcal{H}$ is a harmonic structure on a graph $\tilde{G}$ and $u \in G$, then for every regular neighborhood O of u in the unit ball of u and for every edge $[u,v]$, $O \cap (u,v)$ is regular.*

Proof. (a) By Theorem 3.1 and Theorem 4.1 we can find a harmonic function f on $(0,1)$ which extends harmonically to $(-1,1)$ and is positive on $(-\epsilon_0,\epsilon_0)$ for some $\epsilon_0 > 0$. Thus the structure on $(-\epsilon_0,\epsilon_0)$ is quasi-linear or quasi-hyperbolic, so by Theorem 3.6 every subinterval is regular.

(b) Without loss of generality, by replacing O by each of its connected components, we may assume O is connected. Let f be the solution to the Dirichlet problem on O with boundary values 1, which is positive on $\overline{O}$ by the regularity of O. Let $[u,v'] = \overline{O \cap (u,v)}$. Then by Theorem 3.4 if $v' \neq v$, or Proposition 4.1 if $v' = v$, (u,v') is regular. □

Theorem 4.2. *Let $\tilde{T}$ be a tree endowed with a harmonic structure. Any function which is harmonic on a domain may be extended to a function which is harmonic on all of $\tilde{T}$. In particular, a harmonic structure on $\tilde{T}$ is given completely by describing only the space of functions which are harmonic on all of $\tilde{T}$.*

Proof. Let f be harmonic on a domain U in $\tilde{T}$ and let x be an element of the boundary of U. If $x = u$ is a vertex, let $O = B_1(u) \cap U$. Without loss of generality, we may assume that u is a nonterminal vertex. There exists a unique $v \sim u$ such that $(u, v) \cap O$ is nonempty. Call this intersection (u, y). Using the same notation as in the proof of Proposition 4.1, since $f|(u, y)$ is harmonic, it has the form $\alpha f_v + \beta f_w$ where w is another neighbor of u and α, β are constants. But $\alpha f_v + \beta f_w$ is defined on a neighborhood of u which yields a harmonic extension of f. If $x \in (u, v)$ with u, v neighboring vertices, then by Theorem 3.1, f may be extended to a harmonic function on all of (u, v).

By the above, f can be extended (if necessary) to a harmonic function on a larger domain containing a vertex v_0. By induction on the distance of vertices from v_0, f may be extended to a harmonic function on a neighborhood of every vertex. This yields a harmonic extension of f to all of $\tilde{T}$. $\square$

Observation 4.1. Theorem 4.2 cannot be extended to graphs, as the following example shows. Let G be the graph with vertices $u, v, w, 1, 2, \ldots$, with edges $[u, v], [u, w], [v, 1], [w, 1], [1, 2], [2, 3], \ldots$, and with all nearest-neighbor transition probabilities equal to 1/2, except $p(1, v) = p(1, w) = p(1, 2) = 1/3$. Consider on $\tilde{G}$ the harmonic structure which is linear on the edges. Let O be the domain which includes $[u, v]$, $[u, w]$, and the open half-edges about v and w, that is, $O = [u, v] \cup [u, w] \cup [v, \frac{1}{2}v + \frac{1}{2}1) \cup [w, \frac{1}{2}w + \frac{1}{2}1)$. Define f on $\overline{O}$ by $f(\frac{1}{2}w + \frac{1}{2}1) = -1/2$, $f(\frac{1}{2}v + \frac{1}{2}1) = 5/2$, $f(w) = 0$, $f(u) = 1$, and $f(v) = 2$. Then f has a unique linear extension to $[v, 1]$ so that $f(1) = 3$. On the other hand, f has a unique linear extension to $[w, 1]$ so that $f(1) = -1$. Thus, f is harmonic on O and cannot be extended harmonically to G.

There is, however, a local version of Theorem 4.2 that holds on graphs.

Corollary 4.2. *Let G be a graph such that $\tilde{G}$ is endowed with a harmonic structure. Then every harmonic function on a unit ball has a harmonic extension to a neighborhood of the ball in $\tilde{G}$.*

The proof of the corollary is based on the argument given in the first paragraph of the proof of Theorem 4.2.

5. Construction of harmonic structures on graphs

We now study the relationship between a given harmonic structure on a graph $\tilde{G}$ and a collection of harmonic structures on its edges. By a **directed edge** we mean an edge $\tau = [u, v]$ where u is considered as the initial vertex of τ. We use the notation $\iota(\tau) = u$.

Theorem 5.1. *Let $\tilde{G}$ be endowed with a harmonic structure. For each vertex u, let $O_u \subset B_1(u)$ be a regular neighborhood of u and for each directed edge τ with $\iota(\tau) = u$, let u_τ be the boundary point of O_u on τ. Let f_τ be the solution to the Dirichlet problem on O_u with boundary value the characteristic function of $\{u_\tau\}$. Define $P(u, u_\tau) = f_\tau(u)$. For f continuous on $\tilde{G}$ and harmonic on each edge of $\tilde{G}$, f is harmonic on $\tilde{G}$ if and only if for all $u \in G$, $f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau) f(u_\tau)$.*

Proof. Suppose first that f is harmonic on $\tilde{G}$. Thus f is harmonic on O_u and agrees with $\sum_{\iota(\tau)=u} f(u_\tau) f_\tau$ on ∂O_u, hence agrees with $\sum_{\iota(\tau)=u} f(u_\tau) f_\tau$ on O_u, and so

$$f(u) = \sum_{\iota(\tau)=u} f(u_\tau) f_\tau(u) = \sum_{\iota(\tau)=u} P(u, u_\tau) f(u_\tau).$$

Conversely, assume that f is continuous on $\tilde{G}$, harmonic on each edge and that for each vertex u, $f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau) f(u_\tau)$. We need to show that f is harmonic at u. Consider the function $\tilde{f} = \sum_{\iota(\tau)=u} f(u_\tau) f_\tau$ on O_u. Then

$$\tilde{f}(u) = \sum f(u_\tau) f_\tau(u) = \sum P(u, u_\tau) f(u_\tau) = f(u).$$

Thus on (u, u_τ), $\tilde{f}$ and f are harmonic functions which agree at the endpoints for each τ with $\iota(\tau) = u$. By part (b) of Proposition 4.2, (u, u_τ) is regular, so $\tilde{f} = f$ on (u, u_τ). Hence $\tilde{f} = f$ on O_u. Since $\tilde{f}$ is harmonic at u, so is f. $\square$

It is clear from the statement of Theorem 5.1 that for each directed edge τ we may have different choices for u_τ. The lemma below and its proof make precise the relationship among different choices.

Lemma 5.1. *Given a graph G, assume that each edge of $\tilde{G}$ has an extendible harmonic structure. For each directed edge τ, let $\iota(\tau) = u$ and pick a point $u_\tau \in \tau$, $u_\tau \neq u$, such that (u, u_τ) is regular and let $P(u, u_\tau)$ be an arbitrary positive number. Then for each $u'_\tau \in (u, u_\tau]$, there exists a positive constant $P(u, u'_\tau)$ such that for all $u \in G$ and all functions f which are continuous on $\bigcup_{v\sim u,\, \tau=[u,v]} [u, u_\tau)$ and harmonic on each (u, u_τ), the conditions*

$$f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau) f(u_\tau) \tag{4}$$

and

$$f(u) = \sum_{\iota(\tau)=u} P\big(u, u'_\tau\big) f\big(u'_\tau\big) \tag{5}$$

are equivalent.

Proof. Since (u, u_τ) is regular, applying Proposition 3.3 there exist constants $\alpha_\tau > 0$, $\beta_\tau \geqslant 0$ such that $f(u_\tau) = \alpha_\tau f(u'_\tau) - \beta_\tau f(u)$ for all f harmonic on (u, u_τ) and continuous on $[u, u_\tau]$. Thus the condition (4) is equivalent to

$$f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau)\big(\alpha_\tau f\big(u'_\tau\big) - \beta_\tau f(u)\big),$$

so that

$$\Bigg(1 + \sum_{\iota(\tau)=u} P(u, u_\tau)\beta_\tau\Bigg) f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau)\alpha_\tau f\big(u'_\tau\big).$$

Setting $\sigma = 1 + \sum P(u, u_\tau)\beta_\tau$, we can define $P(u, u'_\tau) = P(u, u_\tau)\alpha_\tau/\sigma$. Clearly (4) and (5) are now equivalent conditions. $\square$

The following result, a converse of Theorem 5.1, shows how to create a harmonic structure on $\tilde{G}$, given harmonic structures on its edges.

Theorem 5.2. *Let G be a graph with an extendible harmonic structure on each edge of $\tilde{G}$. For each directed edge τ, let $\iota(\tau) = u$ and pick a point $u_\tau \in \tau$, $u_\tau \neq u$, such that (u, u_τ) is regular and let $P(u, u_\tau)$ be an arbitrary positive number. Let $U \subset \tilde{G}$ be a connected open set. For each directed edge τ with $\iota(\tau) = u \in U$, let $u'_\tau \in (u, u_\tau] \cap U$. Let $\mathcal{H}(U)$ be the set of functions f such that for each edge $[u, v]$ intersecting U, $f|(u, v) \cap U$ is harmonic and (5) holds for each vertex u in U. Then $\mathcal{H}$ yields a harmonic structure on $\tilde{G}$.*

Observation 5.1. By Lemma 5.1, condition (5) is independent of which point u'_τ is chosen.

For the proof of Theorem 5.2 we will need the following result.

Lemma 5.2. *Under the hypotheses of Theorem* 5.2, $O = \bigcup_{\iota(\tau)=u}[u, u'_\tau)$ *is a regular open set.*

Proof. For each directed edge τ, with $\iota(\tau) = u$, let $a_\tau \in \mathbb{R}$ be given. We will show that there is a unique harmonic function f on O such that $f(u'_\tau) = a_\tau$. Let $a_0 = \sum_{\iota(\tau)=u} P(u, u'_\tau)a_\tau$ and let f be the solution to the Dirichlet problem on (u, u'_τ) with boundary values $f(u) = a_0$ and $f(u'_\tau) = a_\tau$. Then by definition, f satisfies (5), and so f is harmonic on O. This function f is unique since any harmonic extension would agree with f at each u'_τ and at u. Since each $P(u, u'_\tau)$ is positive, if $a_\tau \geqslant 0$ for all directed edges τ starting at u, then $f(u) \geqslant 0$, hence f is nonnegative on O. $\square$

Proof of Theorem 5.2. Clearly, the set of functions harmonic on any open set is a real vector space.

Axiom 1. If f is harmonic on a domain U, and U_0 is a connected open subset of U, then since Axiom 1 holds on each edge (u, v), $f|U_0 \cap (u, v)$ is harmonic.

Furthermore, by Observation 5.1, at each vertex $u \in U$ condition (5) is exactly the same for $f|U_0$ as for f. So $f|U_0$ is harmonic.

Next assume f is continuous on U and that each point in U has some connected neighborhood on which f is harmonic. Then f is harmonic on each of the edges which intersect U and (5) is satisfied at each vertex u in U. Hence f is harmonic on U. This completes the proof of the first Brelot axiom.

Axiom 2. Since each open edge is a Brelot space, each $x \in \tilde{G} \setminus G$ has a base of regular neighborhoods. Thus, we only need to show that every vertex does also. But this follows from Lemma 5.2.

Axiom 3. Let $\{f_n\}$ be an increasing sequence of harmonic functions on some domain U. On each interval contained in some edge and in U, the sequence either converges to a harmonic function or diverges to ∞ everywhere. Let O be a connected neighborhood of a vertex u contained in $U \cap \bigcup_{\iota(\tau)=u}[u, u_\tau)$, which is regular by Lemma 5.2. Let $u'_\tau \in (u, u_\tau] \cap O$ for each τ starting at u. Thus, $\{f_n\}$ is increasing on $\bigcup_{\iota(\tau)=u}[u, u'_\tau]$. By Lemma 5.1, for each edge τ there exist $P(u, u'_\tau) > 0$ such that

$$f_n(u) = \sum_{\iota(\tau)=u} P\left(u, u'_\tau\right) f_n\left(u'_\tau\right).$$

Assume that $\lim_{n\to\infty} f_n(u) = \infty$. If $x \in (u, u'_\tau)$ for some τ, then $f_n(x)$ is a positive linear combination of $f_n(u)$ and $f_n(u'_\tau)$ by Proposition 3.3, and thus $\lim_{n\to\infty} f_n(x) = \infty$. On the other hand, if for some directed edge τ_0 starting at u, f_n goes to ∞ somewhere on $(u, u_{\tau_0}) \cap O$, then $\lim_{n\to\infty} f_n(u'_{\tau_0}) = \infty$, whence $\lim_{n\to\infty} f_n(u) = \infty$. Therefore, if $\{f_n\}$ diverges anywhere, it diverges everywhere. On the other hand, if it converges, then its limit is harmonic on each edge and satisfies (5). Thus it is harmonic on O.

Consequently, the Brelot axioms hold on $\tilde{G}$. $\square$

6. Dirichlet domains and ball regularity axioms

Let G be a graph. One of our principal aims is to use the Brelot theory on $\tilde{G}$ to understand the underlying function theory on G. Thus, it is of less interest from the discrete point of view to study a harmonic structure on $\tilde{G}$ which yields a small subspace of the harmonic functions when restricted to the vertices. In this section, we introduce certain additional axioms with the purpose of seeing when a harmonic structure on $\tilde{G}$ restricts to a discrete harmonic structure on G.

Definition 6.1. Given a Brelot space Ω, a relatively compact domain U in Ω is said to be a **Dirichlet domain** if, for any set of boundary values, there exists a unique solution to the Dirichlet problem on U, that is, for any continuous function f on ∂U there exists a unique continuous extension h_f^U of f to $\overline{U}$ which is harmonic on U.

A Dirichlet domain U is called a **positive Dirichlet domain** if the solution to the Dirichlet problem with nonnegative boundary values is nonnegative in some neighborhood of the boundary.

At the end of the section, we provide an example of a Dirichlet domain which is not a positive Dirichlet domain and of a positive Dirichlet domain which is not regular.

Definition 6.2. A Dirichlet domain U is said to be **weakly regular with respect to** $x \in U$ if for any function f defined on the boundary of U which is nonnegative and not identically zero, $h_f^U(x) > 0$.

We introduce the following axioms:

The ball regularity axiom. The unit ball centered at any (nonterminal) vertex is regular.

The weak ball regularity axiom. The unit ball centered at any (nonterminal) vertex is weakly regular with respect to its center.

Observation 6.1. The ball regularity axiom is satisfied for the harmonic structure of Theorem 5.2 in the case when each open edge is regular. This follows from Lemma 5.2. In particular, if G is a graph endowed with a nearest-neighbor transition probability, the harmonic structure obtained by extending the harmonic functions on the vertices linearly on the edges (as described in Section 2.3) satisfies the ball regularity axiom.

Example 6.5 shows a graph for which the weak ball regularity axiom fails.

Theorem 6.1. *Let $\tilde{G}$ be endowed with a harmonic structure $\mathcal{H}$.*

(a) *If $\mathcal{H}$ satisfies the weak ball regularity axiom, then each edge is a Dirichlet domain.*
(b) *$\mathcal{H}$ satisfies the ball regularity axiom if and only if it satisfies the weak ball regularity axiom and each edge is regular.*

Proof. Assume $\mathcal{H}$ satisfies the weak ball regularity axiom. Let u be a nonterminal vertex and let v, w be distinct neighbors of u. Let f_v be the solution to the Dirichlet problem on $B_1(u)$ with value 1 at v and 0 at all other neighbors of u. Define f_w similarly. Note that $f_w(u) > 0$. Then for any real numbers α and β, the function

$$g = \beta f_v + \frac{\alpha - f_v(u)\beta}{f_w(u)} f_w$$

is harmonic on (u, v), $g(u) = \alpha$, and $g(v) = \beta$. Since f_v and f_w are linearly independent on (u, v), they form a Brelot basis, so g is the unique solution to the Dirichlet problem with boundary values α and β. Thus each edge is a Dirichlet domain, proving (a).

Next assume $\mathcal{H}$ satisfies the ball regularity axiom. Then $\mathcal{H}$ obviously satisfies the weak ball regularity axiom and by Proposition 4.2(b), each edge is regular.

Conversely, assume $\mathcal{H}$ satisfies the weak ball regularity axiom and each edge is regular. We may then solve uniquely the Dirichlet problem on any unit ball $B_1(u)$. Furthermore, if the boundary values are nonnegative, the solution h is nonnegative at u. Since each edge is regular, the restriction of h to the edges containing u is unique and nonnegative. Hence h is nonnegative on $B_1(u)$. Thus $\mathcal{H}$ satisfies the ball regularity axiom. $\square$

In Example 6.6, we will show that the converse of Theorem 6.1(a) is false. In fact, in this example each edge is regular (which is stronger than being a Dirichlet domain), yet the unit ball about each vertex is not even a Dirichlet domain (which is weaker than having weak ball regularity). Example 6.6 will also show that a harmonic structure on a tree $\tilde{T}$ which is linear on the edges can fail to satisfy even the weak ball regularity axiom.

Let $\tilde{G}$ be a graph with a harmonic structure and let $v_1, \ldots, v_d$ be the neighbors of a vertex u. Assume O is a regular neighborhood of u contained in $B_1(u)$, and for each $i \in \{1, \ldots, d\}$, let v_i' be the point of ∂O on $(u, v_i]$. Let f_i be the solution to the Dirichlet problem on O such that $f_i(v_j') = \delta_{ij}$ and let $f_{ij} = f_i(v_j)$.

Proposition 6.1. *$B_1(u)$ is a Dirichlet domain if and only if the matrix $F = ((f_{ij}))$ is invertible.*

Proof. A function f on $B_1(u)$ is harmonic if and only if it has the form $f = \sum_{i=1}^{d} a_i f_i$. In this case

$$\big(f(v_1), \ldots, f(v_d)\big) = (a_1, \ldots, a_d)F \tag{6}$$

and so for an arbitrary set of values of $f|\partial B_1(u)$ there exists a unique vector $(a_1, \ldots, a_d)$ satisfying (6) if and only if F is invertible. $\square$

6.1. Examples

Example 6.1 *(Harmonic structure on $\tilde{G}$ which is constant on G).* Use Theorem 5.2 to define a harmonic structure on $\tilde{G}$ as follows. Let the structure on each directed edge $\tau = [u, v]$ be given by the Brelot basis $\{\cos 2\pi t, \sin 2\pi t\}$ and let $u_\tau = \frac{7}{8}u + \frac{1}{8}v$. Since $\cos 2\pi t$ is positive on $[0, 1/8]$, (u, u_τ) is regular. Let $P_\tau > 0$ be arbitrary. The harmonic structure on $\tilde{G}$ obtained by this construction is nontrivial, that is, on every edge the space of harmonic functions is two-dimensional. Observe, however, that if f is harmonic on $\tilde{G}$ and $u \sim v$, then $f(u) = f(v)$. Thus $f|G$ is constant. In particular, we cannot solve any Dirichlet problem with nonconstant boundary values on the unit ball centered at any vertex.

Observation 6.2. Assume $\mathbb{R}$ is endowed with a harmonic structure. Then for $a, b \in \mathbb{R}$ with $a < b$, (a, b) is a Dirichlet domain if and only if there exist harmonic functions f and g on (a, b), continuous on $[a, b]$, such that $f(a) = g(b) = 0$ and $f(b) = g(a) = 1$. In this case, $\{f, g\}$ is a Brelot basis on (a, b). Then (a, b) is regular if and only if f and g are both positive; (a, b) is weakly regular with respect to one of its points x if and only if $f(x)$ and $g(x)$ are both positive. In particular, by Theorem 4.1, if (a, b) is a Dirichlet domain, then the harmonic structure on (a, b) is extendible.

Example 6.2 *(Dirichlet domain weakly regular with respect to some points, but not all, or with respect to no points).* Let $U = (0, \frac{5}{2})$ with Brelot basis $\{\sin \pi t, \cos \pi t\}$. Using Observation 6.2, we see that U is not weakly regular with respect to any point $x \in [1/2, 2]$, but is weakly regular with respect to any other point, because the functions f and g in Observation 6.2 exist and are given by $f(t) = \sin \pi t$ and $g(t) = \cos \pi t$. On the other hand, the interval $U' = (0, \frac{3}{2})$ is a Dirichlet domain but is not weakly regular with respect to any of its points, because in this case $f(t) = -\sin \pi t$ and $g(t) = \cos \pi t$.

Example 6.3 *(Positive Dirichlet domain, but not regular).* The interval U in Example 6.2 is not regular but since it is weakly regular with respect to the points 0 and 5/2, it is a positive Dirichlet domain.

Example 6.4 *(Dirichlet, but not positive Dirichlet, domain).* On $\mathbb{R}$ consider the harmonic structure with Brelot basis $\{\cos \pi t, \sin \pi t\}$. The interval $U = (0, \frac{3}{2})$ is a Dirichlet domain because for any constants α, β, the function $h(t) = \alpha \cos \pi t - \beta \sin \pi t$ is the unique solution to the Dirichlet problem on U such that $h(0) = \alpha$, $h(\frac{3}{2}) = \beta$. On the other hand, the function $t \mapsto -\sin \pi t$ is nonnegative on ∂U but attains negative values arbitrarily close to 0. Thus U is not a positive Dirichlet domain.

Example 6.5 *(Failure of weak ball regularity).* Let G be a graph endowed with a nearest neighbor transition probability and let $\tilde{G}$ be its linear extension. Then $\tilde{G}$ is a Brelot space, and for each vertex u, $B_1(u)$ is regular. In [1] (see proof of Theorem 2.1), we showed that any ball centered at a vertex of radius $\epsilon < 1$ is regular, provided that the structure on the edges is linear. Thus, $B_{1/8}(u)$ is also regular. For each directed edge $\tau = [u, v]$, let $u_\tau = \frac{7}{8}u + \frac{1}{8}v$ and let $P(u, u_\tau)$ be the constant given by Theorem 5.1. Pick one edge $\tau_0 = [u_0, v_0]$ and replace the linear structure on τ_0 with the structure having Brelot basis $\{\cos 2\pi t, \sin 2\pi t\}$. Then (u_0, u_{τ_0}) is regular since $\cos 2\pi t$ is positive on its closure. Leaving all the constants $P(u, u_\tau)$ as before, we now get a new harmonic structure by Theorem 5.2. This structure has the property that for every f harmonic on $\tilde{G}$, $f(u_0) = f(v_0)$ and thus, by Theorem 6.1, $B_1(u_0)$ is not *weakly* regular with respect to u_0.

Example 6.6 *(Edges are regular, but $B_1(u)$ is not a Dirichlet domain, so weak ball regularity fails).* Let T be a homogeneous tree of degree 3 and for each $u \sim v$, let v' be the midpoint of the edge (u, v), and let $P(u, v') = 2/3$. We define a function f to be harmonic on $\tilde{T}$ if its restriction to each edge is linear and for each $u \in T$,

$$f(u) = \sum_{v \sim u} P(u, v') f(v') = \frac{2}{3} \sum_{v \sim u} f(v').$$

By Theorem 5.2, this defines a harmonic structure on $\tilde{T}$.

Assume f is harmonic. Since f is linear on each edge, for each $v \sim u$, we have $f(v') = (f(u) + f(v))/2$. Thus

$$f(u) = \frac{2}{3} \sum_{v \sim u} \frac{1}{2}\bigl(f(u) + f(v)\bigr) = f(u) + \frac{1}{3} \sum_{v \sim u} f(v),$$

so that $\sum_{v \sim u} f(v) = 0$. Thus, arbitrary prescribed values on the boundary of $B_1(u)$ do not yield a unique harmonic solution in $B_1(u)$. Hence, the weak ball regularity axiom is not satisfied. But since on each edge the harmonic functions are the linear functions, the open edges are regular.

Example 6.7 *(Weak ball regularity holds, but ball regularity does not).* On the interval $[0, 1]$, let $g(t) = \cos(\frac{9}{4}\pi t)$ and $h(t) = \sin(\frac{9}{4}\pi t)$, and let $\mathcal{A}$ be the space of linear combinations of g and h. Observe that $\mathcal{A}$ does not contain any positive functions. Moreover, $g(1 - t) = \frac{1}{\sqrt{2}}(h(t) + g(t))$ and $h(1 - t) = \frac{1}{\sqrt{2}}(h(t) - g(t))$, for $0 \leqslant t \leqslant 1$, so that $\{g(1 - t), h(1 - t)\}$ is a basis consisting of elements of $\mathcal{A}$. Thus, the involution $t \mapsto 1 - t$ preserves $\mathcal{A}$.

Let T be a tree. Identifying each edge $[u, v]$ with $[0, 1]$, let the harmonic functions on the edge be the elements of $\mathcal{A}$. Observe that it does not matter whether u or v is identified with 0. Also notice that if $f \in \mathcal{A}$, then for $0 \leqslant t \leqslant 1/9$, $f(t) = f(t + 8/9)$. Furthermore, if f is nonnegative at one endpoint, it is nonnegative at the other endpoint.

For each directed edge $\tau = [u, v]$, let $u_\tau = \frac{8}{9}u + \frac{1}{9}v$ and let $P(u, u_\tau)$ be any positive number. By Theorem 5.2, there is a unique harmonic structure $\mathcal{H}$ on $\tilde{T}$ such that a continuous function f on $\tilde{T}$ is harmonic if it is harmonic on each edge and for each $u \in T$, $f(u) = \sum_{v \sim u} P(u, u_\tau) f(u_\tau)$.

Notice that by our choice of harmonic structure on the edges, any harmonic function f has the property that for each edge $[u, v]$, $f(u_\tau) = f(v)$. Since $\bigcup_{v \sim u} [u, u_\tau)$ is regular, it follows that $B_1(u)$ is weakly regular with respect to u. Thus $\mathcal{H}$ satisfies the weak ball regularity axiom. However, since no edge is regular (because there are no positive harmonic functions), Theorem 6.1(b) implies that $\mathcal{H}$ does not satisfy the ball regularity axiom.

7. Interplay between discrete and harmonic structures

Assume that G has a discrete harmonic structure P (see Section 2.1). For each $u \in G$, let $\alpha(u) = \sum_{v \sim u} P(u, v)$ and define $P'(u, v) = P(u, v)/\alpha(u)$ for all $v \sim u$. Then P' defines a stochastic structure on G.

We now wish to characterize the harmonic structures which restrict to discrete harmonic structures on the set of vertices.

Lemma 7.1. *Let $\mathcal{H}$ be a harmonic structure on $\tilde{G}$ and let u be a vertex. Then the dimension of $\mathcal{H}|B_1(u)$ is the degree of u.*

Proof. Let $O \subset B_1(u)$ be a regular neighborhood of u. Then there is a 1-to-1 correspondence between the functions harmonic on O and the real-valued functions defined on the boundary of O. Thus the dimension of $\mathcal{H}|O$ is the cardinality of ∂O, which is the degree of u. But the restriction map $\mathcal{H}|B_1(u) \to \mathcal{H}|O$ is a monomorphism by Proposition 3.1 and an epimorphism by Theorem 3.1, and hence an isomorphism. $\square$

Theorem 7.1. *Let G be a graph. Given a harmonic structure $\mathcal{H}$ on $\tilde{G}$, the following statements are equivalent*:

(a) *$\mathcal{H}$ satisfies the weak ball regularity axiom (respectively, the ball regularity axiom).*
(b) (i) *Each edge is a Dirichlet (respectively, regular) domain, and*
   (ii) *there exists a discrete harmonic structure on G such that for every open set U in $\tilde{G}$, the restriction of $\mathcal{H}(U)$ to $U \cap G$ is equal to $\mathcal{H}_P(U)$.*

In particular, any harmonic structure on $\tilde{G}$ satisfying the weak ball regularity axiom induces a unique discrete harmonic structure on G.

Proof. (a) $\Rightarrow$ (b). Let U be an open set in $\tilde{G}$. Fix an interior vertex u in $U \cap G$ (i.e. a vertex all of whose neighbors are in U) and a neighbor v. Let f_v be the solution to the Dirichlet problem on $B_1(u)$ such that $f_v(v) = 1$ and $f_v(w) = 0$ for all the other neighbors w of u. Define $P(u, v)$ to be $f_v(u)$, which by the axiom is positive. Let f be harmonic on U. Then f and $\sum_{v \sim u} f(v) f_v$ agree on the boundary of $B_1(u)$, hence are equal on $B_1(u)$. Thus $f(u) = \sum_{v \sim u} f(v) f_v(u) = \sum_{v \sim u} P(u, v) f(v)$, so the restriction of each element of $\mathcal{H}(U)$ to $U \cap G$ yields an element of $\mathcal{H}_P(U)$.

We now show that every function $f_0 \in \mathcal{H}_P(U)$ is the restriction to $U \cap G$ of some element of $\mathcal{H}(U)$. Fixing a vertex u in the interior of $U \cap G$, observe that the function $f = \sum_{v \sim u} f_0(v) f_v$ satisfies

$$f(u) = \sum_{v \sim u} P(u, v) f_0(v) = f_0(u),$$

and f agrees with f_0 at each neighboring vertex of u. Thus, f_0 may be extended harmonically to each ball $B_1(u)$. On balls sharing a common edge, the corresponding extensions agree at the endpoints, and by Theorem 6.1, the edge is a Dirichlet domain, so the extensions agree on the entire edge. For each $v \in U \cap G$ not an interior vertex, extend f to $B_1(v)$ as follows: choose one neighbor w outside of U and prescribe its value to be $f_0(v) - (\sum_{x \sim v,\, x \in U \cap G} P(v, x) f_0(x))/P(v, w)$. For all other neighbors not in $U \cap G$ prescribe its value to be 0, and for those that are in $U \cap G$, prescribe the value to be $f_0(w)$. This yields a function harmonic on $B_1(v) \cap U$.

(b) $\Rightarrow$ (a). Let u be a nonterminal vertex of G. Let f be a function defined on the boundary of $B_1(u)$. Define

$$f(u) = \sum_{v \sim u} P(u, v) f(v). \tag{7}$$

Let U be an open set in $\tilde{G}$ containing the closure of $B_1(u)$ with no additional vertices. Then $f \in \mathcal{H}_P(U)$ so by assumption it is the restriction of an element of $\mathcal{H}(U)$. This shows that the Dirichlet problem can be solved on $B_1(u)$. By assumption, any solution of the Dirichlet problem satisfies (7). Thus, since each edge is a Dirichlet domain, the solution of the Dirichlet problem is unique. If the prescribed boundary values are nonnegative, by (7), the value at u is nonnegative. Therefore, the weak ball regularity axiom holds.

The assertions concerning ball regularity follow from part (b) of Theorem 6.1. $\square$

Theorem 7.2. *Let $\tilde{G}$ be endowed with a harmonic structure and let $u \in G$. Then $B_1(u)$ is regular if and only if there exists a positive continuous function k on $\overline{B_1(u)}$, harmonic on $B_1(u)$. Consequently, if $\tilde{G}$ is a B.H. space, then its harmonic structure satisfies the ball regularity axiom and the induced discrete harmonic structure on G is stochastic.*

Proof. Observe that if $B_1(u)$ is regular, the existence of a positive harmonic function on its closure is guaranteed by solving the Dirichlet problem with positive boundary values.

Conversely, assume there exists a positive continuous function k on $\overline{B_1(u)}$. Without loss of generality, we may assume that $k(u) = 1$. Observe that each edge in $B_1(u)$ is regular by Proposition 4.1. Let O_u be a regular neighborhood of u. For each $v \sim u$ and $\tau = [u, v]$, let u_τ and $P(u, u_\tau)$ be as in Theorem 5.1 so that f is harmonic on $B_1(u)$ if and only if f is harmonic on each of its edges and $f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau) f(u_\tau)$. For any edge τ with initial vertex u, let us denote by $\iota'(\tau)$ the terminal vertex of τ, and let h_τ be the unique continuous function on τ, harmonic in its interior, such that $h_\tau(u) = 0$ and $h_\tau(\iota'(\tau)) = 1$. Then $\{k|\tau, h_\tau\}$ is a Brelot basis on the interior of τ so that every harmonic function f inside τ has the form $f = f(u)k + [f(\iota'(\tau)) - f(u)k(\iota'(\tau))]h_\tau$. Thus

$$f(u_\tau) = f(u)k(u_\tau) + \bigl[f\bigl(\iota'(\tau)\bigr) - f(u)k\bigl(\iota'(\tau)\bigr)\bigr]h_\tau(u_\tau)$$

and the harmonicity condition of f at u becomes

$$f(u) = \sum_{\iota(\tau)=u} P(u, u_\tau)\bigl\{f(u)k(u_\tau) + \bigl[f\bigl(\iota'(\tau)\bigr) - f(u)k\bigl(\iota'(\tau)\bigr)\bigr]h_\tau(u_\tau)\bigr\}. \tag{8}$$

Since k is harmonic on $B_1(u)$, $\sum_{\iota(\tau)=u} P(u, u_\tau)k(u_\tau) = k(u) = 1$, so (8) becomes

$$f(u) \sum_{\iota(\tau)=u} P(u, u_\tau)k\bigl(\iota'(\tau)\bigr)h_\tau(u_\tau) = \sum_{\iota(\tau)=u} P(u, u_\tau)h_\tau(u_\tau) f\bigl(\iota'(\tau)\bigr).$$

For $w \sim u$ and $\tau' = [u, w]$, the quantities $P(u, u_{\tau'})$, $k(w)$, and $h_{\tau'}(u_{\tau'})$ are all positive, so

$$P(u, w) = \frac{P(u, u_{\tau'})h_{\tau'}(u_{\tau'})}{\sum_{\iota(\tau)=u} P(u, u_\tau)k(\iota'(\tau))h_\tau(u_\tau)}$$

is positive and the harmonicity condition for f at u becomes

$$f(u) = \sum_{v\sim u} P(u, v) f(v). \tag{9}$$

Now let us solve the Dirichlet problem on $B_1(u)$ with boundary values $f(v)$, for all $v \sim u$. Let $f(u)$ be given by (9). Then for each edge $[u, w]$ solve the Dirichlet problem on $[u, w]$ with boundary values $f(u)$ and $f(w)$. Thus, f is the unique solution to the Dirichlet problem on $B_1(u)$. Clearly, $f(v) \geqslant 0$ for $v \sim u$ implies $f(u) \geqslant 0$, which implies that $f \geqslant 0$ everywhere in $B_1(u)$. This completes the proof that $B_1(u)$ is regular.

Next assume $\tilde{G}$ is a B.H. space. Taking k to be the constant 1, we see that the harmonic structure satisfies the ball regularity axiom, and hence the weak ball regularity axiom. Theorem 7.1 shows that the harmonic structure induces on G a discrete harmonic structure, which is stochastic since the constants are harmonic. $\square$

In Theorem 7.5 in the special case of a tree, we generalize Theorem 7.2 to an arbitrary relatively compact subset.

Definition 7.1. Let G be a graph and let U be a relatively compact domain in $\tilde{G}$ such that $\partial U \subset G$. Then $S = \overline{U} \cap G$ is called a **finite complete subset** of G.

Let G be a graph endowed with a discrete harmonic structure P, let $\mathcal{H}$ be a harmonic structure on $\tilde{G}$, and let $u \in G$. If $f \in \mathcal{H}(B_1(u))$, then f extends to a continuous function $\hat{f}$ on $\overline{B_1(u)}$. We say that $\mathcal{H}$ **induces the discrete harmonic structure** $\mathcal{H}_P$ on G if for every vertex u, the restriction of $\hat{f}$ to the vertices of $\overline{B_1(u)}$ induces an isomorphism from $\mathcal{H}(B_1(u))$ to $\mathcal{H}_P(B_1(u) \cap G)$.

Observation 7.1. Given a graph $\tilde{G}$, assume that each edge $[u, v]$ is endowed with a harmonic structure such that (u, v) is regular and assume that there is a discrete harmonic structure P on G. The harmonic structure defined on $\tilde{G}$ by Theorem 5.2 satisfies the ball regularity axiom by Observation 6.1.

We can now state the result that sums up the relation between harmonic structures on $\tilde{G}$ and discrete harmonic structures on G.

Theorem 7.3. *Let G be a graph with a discrete harmonic structure on G. Assume that each edge of $\tilde{G}$ has a harmonic structure for which the whole edge is a positive Dirichlet domain. Then, there is a unique harmonic structure on $\tilde{G}$ which induces the given discrete structure and whose restriction to each edge is the given harmonic structure. Furthermore, it satisfies the weak ball regularity axiom.*

Proof. To define a harmonic structure $\mathcal{H}$ on $\tilde{G}$, it suffices to define for each $x \in \tilde{G}$ harmonicity in a neighborhood of x. If $x \in (u, v)$ for some neighboring vertices u, v, then a function f is harmonic at x if there is an interval O about x in (u, v) such that f is defined on O and harmonic with respect to the given structure on $[u, v]$. If x is a vertex, and f is defined in a connected neighborhood O of x in $B_1(x)$, assume that for each neighbor v of x, f is harmonic on $O \cap (x, v)$, and let $\tilde{f}$ be the unique harmonic extension of $f|O \cap (x, v)$ to $[x, v]$, so that $\tilde{f}$ is the unique extension of $f|O$ to all of $\overline{B_1(x)}$ harmonic on each segment. We say that f is harmonic at x if $f(x) = \sum_{v \sim x} P(x, v)\tilde{f}(v)$.

We claim that $\mathcal{H}$ yields a harmonic structure on $\tilde{G}$. We first show that each point has a regular neighborhood. Let $x \in \tilde{G}$. If x is not a vertex, then $x \in (u, v)$ for some neighboring vertices u, v and since $\mathcal{H}(u, v)$ is already a harmonic structure on the edge, x has a regular neighborhood in (u, v). Next assume that x is a vertex of degree d. Let $v_1, \ldots, v_d$ be the neighbors of x. Let f_i, g_i $(i = 1, \ldots, d)$ be continuous on $[x, v_i]$ and harmonic in the interior such that $f_i(x) = 0$, $f_i(v_i) = 1$, $g_i(x) = 1$, and $g_i(v_i) = 0$. For each $i \in \{1, \ldots, d\}$ choose $u_i \in (x, v_i]$ such that $f_i|(x, u_i]$ and $g_i|(x, u_i]$ are positive (we can do this because $[x, v_i]$ is a positive Dirichlet domain). Let $O = \bigcup [x, u_i)$. Let $P_i = P(x, v_i)$, $\gamma_i = g_i(u_i)$, and $\varphi_i = f_i(u_i)$. Given nonnegative numbers $\alpha_1, \ldots, \alpha_d$, we solve the Dirichlet problem on O with boundary values α_i as follows. Let h be $a_i f_i + b g_i$ on $[x, v_i]$, where $a_1, \ldots, a_d, b$ are the solutions to the system of linear equations

$$\begin{cases} a_i\varphi_i - b\gamma_i = \alpha_i, \quad i = 1, \ldots, d, \\ \displaystyle\sum_{i=1}^{d} P_i a_i - b = 0. \end{cases}$$

For $j = 1, \ldots, d$, let N_j be the determinant of the $(d + 1) \times (d + 1)$ matrix whose upper left-hand $d \times d$ block is the diagonal matrix with diagonal entries $\varphi_1, \ldots, \varphi_d$, whose bottom row is $P_1, \ldots, P_d, 0$, and whose right-hand column is the unit vector with 1 in the jth place. Then, by

Cramer's rule we can solve the above system for each a_i as long as the coefficient matrix has a nonzero determinant D. Letting $\varphi = \prod_{i=1}^{d} \varphi_i$, we see that $D = \sum_{i=1}^{d} \gamma_i N_i - \varphi$. Since $N_i = -\varphi P_i/\varphi_i$, $D = -\varphi(\sum_{i=1}^{d} \gamma_i P_i/\varphi_i + 1) < 0$. Furthermore,

$$b = \frac{\sum_{i=1}^{d} \alpha_i N_i}{D} = \frac{\sum_{i=1}^{d} \frac{\alpha_i P_i}{\varphi_i}}{\sum_{i=1}^{d} \gamma_i P_i/\varphi_i + 1} \geqslant 0.$$

By Theorem 3.4, since f_i is positive and harmonic on (x, u_i), (x, u_i) is regular. Hence, $h(x) \geqslant 0$ and $h(u_i) = \alpha_i \geqslant 0$ imply that h is nonnegative on (x, u_i). Thus, O is a regular neighborhood of x.

Now for $i = 1, \ldots, d$, let h_i be the solution to the Dirichlet problem satisfying $h_i(u_j) = \delta_{i,j}$, $j = 1, \ldots, d$. Let $p_i' = h_i(x)$. So $\{h_1, \ldots, h_d\}$ is a basis for $\mathcal{H}(O)$, and thus, a function harmonic on O is one which has the form $h = \sum_{i=1}^{d} h(u_i) h_i$. That is, a continuous function h on O is harmonic if and only if $h|(x, u_i)$ is harmonic for each i and $h(x) = \sum_{i=1}^{d} p_i' h(u_i)$. By Theorem 5.2, this yields a harmonic structure on $\tilde{G}$. $\square$

The following result follows immediately from Theorem 7.3 by considering on $\tilde{G}$ the harmonic structure obtained from the discrete harmonic structure by linear extension.

Corollary 7.1. *If G has a discrete harmonic structure, then there are harmonic structures on $\tilde{G}$ which induce P. All such structures satisfy the weak ball regularity axiom.*

Theorem 7.4. *A harmonic structure on $\tilde{G}$ satisfying the weak ball regularity axiom induces the discrete harmonic structure P defined as follows: if $u \sim v$, $P(u, v)$ is the value at u of solution $f_{[u,v]}$ to the Dirichlet problem on $B_1(u)$ with boundary values equal to the characteristic function of v.*

Proof. Let f be harmonic on $\tilde{G}$ and let u be a vertex. Then $\sum_{v \sim u} f(v) f_{[u,v]}$ is harmonic on the closure of $B_1(u)$ and agrees with f on $\partial B_1(u)$, so $f = \sum_{v \sim u} f(v) f_{[u,v]}$ and thus $f(u) = \sum_{v \sim u} f(v) f_{[u,v]}(u) = \sum_{v \sim u} P(u, v) f(v)$. Conversely, assume that $f(u) = \sum_{v \sim u} P(u, v) f(v)$ for each vertex u. Let $\tilde{f}$ be the harmonic function $\sum_{v \sim u} f(v) f_{[u,v]}$. Then for each edge $[u, v]$, $\tilde{f}(v) = f(v)$ and $\tilde{f}(u) = f(u)$. Thus, since both f and $\tilde{f}$ are harmonic on $[u, v]$, $\tilde{f} = f$ on each edge, so f is harmonic on $\tilde{G}$. $\square$

Let $\tilde{G}$ be endowed with a harmonic structure satisfying the ball regularity axiom, and for each directed edge $\tau = [u, v]$, let f_τ and P be as in the statement of Theorem 7.4 and $g_\tau = \sum_{w \sim u,\, w \neq v} f_{[u,w]}$. Thus, a function f on $\tilde{G}$ is harmonic if and only if it satisfies the following three properties:

(1) f is continuous on $\tilde{G}$;
(2) For all $u \in G$, $f(u) = \sum_{v \sim u} P(u, v) f(v)$;
(3) For any directed edge τ, $f|\tau$ is a linear combination of f_τ and g_τ.

Proposition 7.1. *Let $\tilde{G}$ be endowed with a harmonic structure satisfying the ball regularity axiom. With the above notation, $\tilde{G}$ is a B.H. space if and only if for all directed edges τ, $f_\tau + g_\tau$ is the constant 1, in which case, P is stochastic.*

Proof. For all directed edges τ, the function $(f_\tau + g_\tau)|\partial B_1(u)$ is the constant 1. Thus, $f_\tau + g_\tau$ is the constant 1 for each τ if and only if constants are harmonic, i.e. if and only if $\tilde{G}$ is a B.H. space. $\square$

The regularity of the unit ball of each vertex does not guarantee the regularity of a larger ball as the following example shows.

Example 7.1. Let T be a homogeneous tree of degree d and for each vertex u let $P(u,v) = 1/\sqrt{d}$ for all $v \sim u$ and extend P-harmonic structure on T linearly to a harmonic structure on $\tilde{T}$. The unit ball centered at any vertex u is regular, but the function f on $\overline{B_2(u)}$ defined by $f(u) = 1$, $f(v) = 1/\sqrt{d}$ for $v \sim u$, $f(w) = 0$ for $d(u,w) = 2$, and extended linearly on the edges is positive and harmonic on $B_2(u)$, yet $f|\partial B_2(u)$ is identically 0. Consequently, $B_2(u)$ is not regular, because f and the 0 function satisfy the same boundary conditions.

Let T be a tree. In the following theorem we characterize the relatively compact domains in $\tilde{T}$ which are regular, in analogy with Theorem 3.4.

Theorem 7.5. *Let $\tilde{T}$ be endowed with a harmonic structure $\mathcal{H}$ and let U be a relatively compact domain in $\tilde{T}$. Then U is regular if and only if there exists a harmonic function k such that $k|\overline{U}$ is positive.*

Proof. If U is regular, then the solution to the Dirichlet problem on U with boundary values 1 is a positive continuous function on $\overline{U}$ which can be extended to a global harmonic function.

Conversely, let U be a relatively compact domain in $\tilde{T}$ and assume there exists a harmonic function k that is positive on $\overline{U}$. Let O be a neighborhood of $\overline{U}$ such that k restricted to O is positive. Let $\mathcal{H}'$ be the harmonic structure on O defined by $\frac{1}{k}\mathcal{H}$, so that constants are harmonic on O. Clearly, it is enough to show that U is regular with respect to $\mathcal{H}'$.

Let T_0 be the tree whose vertices are the vertices in U together with ∂U. Then $\tilde{T_0} = \overline{U}$ which we take with the B.H. harmonic structure $\mathcal{H}'$. By Theorem 7.2, the ball regularity axiom is satisfied in $\tilde{T_0}$ and the induced discrete harmonic structure is stochastic. Then T_0 is a finite complete subtree of a tree endowed with a nearest-neighbor transition probability. It is well known (see [5,2]) that the Dirichlet problem can be solved uniquely on T_0 with positive boundary data yielding a positive solution. By Theorem 7.4, this solution can be extended harmonically to U. $\square$

Remark 7.1. Theorem 7.5 may be true also on graphs, but our technique works only on trees. If any relatively compact subset of a graph carried a positive potential, then the theorem would hold for graphs because in this case regularity is a local condition [3] and a graph is locally a tree. However, on a general harmonic space, the existence of a positive harmonic function does not imply the existence of a positive potential. Example 7.1 illustrates this fact.

We now state a result that follows immediately from Theorem 7.5 and is an extension to trees of Theorem 3.6.

Corollary 7.2. *Let T be a tree and suppose $\tilde{T}$ is endowed with a harmonic structure. Then any open subset of a regular set is regular.*

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