FRACTAL FUNCTIONS WITH NO RADIAL LIMITS IN BERGMAN SPACES ON TREES

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Abstract. For each $p > 0$ we provide the construction of a harmonic function on a homogeneous isotropic tree T in the Bergman space $A^p(\sigma)$ with no finite radial limits anywhere. Here, σ is an analogue of the Lebesgue measure on the tree. With the appropriate modifications, the construction yields a function in $A^1(\sigma)$ when T is a rooted radial tree such that the number of forward neighbors increases so slowly that their reciprocals are not summable.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in $\{z \in \mathbb{C} : |z| < 1\}$. For $p \geq 1$, let H^p be the Hardy space of analytic functions on $\mathbb{D}$, namely the analytic functions f such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

It is well known that the functions in the Hardy space have radial limits almost everywhere on the unit circle. The Bergman space A^p is defined as the space of analytic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where dA denotes Lebesgue area measure. Then $H^p \subset A^p$ and by contrast, using Theorem 2 in [6], it is easy to see that there exist functions in A^p that have no radial limits anywhere on the circle. For details on the Hardy and the Bergman spaces, see [3] and [4].

In recent years classical problems have been analyzed in various discrete settings, and in particular, in the setting of infinite trees where the class of analytic functions has been substituted by the class of harmonic functions, i.e. the functions on the vertices on a tree satisfying the mean value property at each vertex. The harmonic Hardy spaces on trees have been studied in [7] and [5].

Motivated by the study of Carleson measures on trees, in [2], we introduced the harmonic Bergman space with reference measure σ (an analogue of the Lebesgue area measure on the unit disk) to study similar questions in the tree setting.

In this article, given a positive number p we provide the construction of a harmonic function on a homogeneous isotropic tree T in the Bergman space $A^p(\sigma)$ with no finite radial limits anywhere. The construction involves combining two algorithms. The first algorithm, introduced here, is called *fractal harmonic extension* and the second algorithm, called *harmonic radialization*, was introduced in

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[2]. In addition, we show that, with the appropriate modifications, the first algorithm yields a function in $A^1(\sigma)$ when T is a general radial tree with degrees at each vertex satisfying a certain summability condition that in particular holds if the degrees are bounded throughout the tree.

2. Preliminaries on trees

By a *tree* T we mean a locally finite connected and simply-connected graph, which, as a set, we identify with the collection of its vertices. Two vertices v and w are called *neighbors* if there is an edge $[v, w]$ connecting them, in which case we use the notation $v \sim w$. The *degree* of a vertex v is the number of neighbors of v. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \dots]$ such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$, for all k. An infinite path is called a *ray*. Given a tree T rooted at o and a vertex $v \in T$, a vertex w is called a *descendant* of v if v lies in the unique path from o to w. The vertex v is then called an *ancestor* of w. We call the *parent* of a vertex $v \neq o$ the unique neighbor v^- of v which is an ancestor of v. The vertex v is called a *child* of v^-. For $v \in T$, the set $S(v)$ consisting of v and all its descendants is called the *sector* determined by v. The *distance*, $d(v, w)$, between vertices v and w is the number of edges in the path connecting v to w. Fixing a root o, the *length* of a vertex v is defined as $|v| = d(o, v)$.

A tree is called *homogeneous* if each vertex has the same number $q + 1$ of neighbors, called the *degree* of the tree. A homogeneous tree is *isotropic* if the nearest neighbor transition probabilities on the tree are equal.

A rooted tree is called *radial* if for every non-negative integer n, all vertices of length n have the same number of forward neighbors, and the (forward) conditional transition probabilities are the reciprocal of that number. Thus a radial tree is isotropic precisely when the number of forward neighbors is the same number q at each vertex $v \neq o$.

A *function on a tree* is a real-valued function on the set of its vertices. The Laplace operator Δ on functions is defined by

$$\Delta f(v) = \sum_{w \sim v} p(v, w)[f(w) - f(v)], \ v \in T.$$

A function f on T is called *harmonic* at v if $\Delta f(v) = 0$, and harmonic on a subset of T if it is harmonic at each of the vertices of that subset.

Any nonnegative function σ on T which is summable can be viewed as a finite measure on T, where the σ-measure of any subset of T is obtained by summing the values of σ over that set. In this paper, all measures are radial, that is, constant on vertices having the same length. For each $n \geq 0$, we denote by σ_n the σ-measure of any vertex of length n. In case T is isotropic, we will take $\sigma_n = q^{-2n}$. Indeed, a simple calculation shows that for any $v \in T$, the σ-measure of $S(v)$ is of the order of $q^{-2|v|}$ (so most of the mass is at v), which is also the square of the "Euclidean" distance of v to the boundary of T. We view the function theory on an isotropic tree as being closest to the function theory on $\mathbb{D}$. A point $z \in \mathbb{D}$ determines a tent having z as vertex, and Lebesgue measure of the order of $(1 - |z|)^2$, i.e. the square of the distance of z to the boundary of $\mathbb{D}$. For a vertex $v \in T$, the sector $S(v)$ is the analog of the above tent. Because of this analogy with Lebesgue area measure on $\mathbb{D}$, σ is a natural choice of measure on T.

Given a measure σ on T and $p > 0$, a function f is in the *Bergman space* $A^p(\sigma)$ provided it is a harmonic function on T belonging to $L^p(\sigma)$, that is

$$\sum_{v \in T} |f(v)|^p \sigma(v) < \infty.$$

Denote by Ω the *boundary* of T, that is the set of all rays $\omega = [\omega_0 = o, \omega_1, \omega_2, \dots]$ starting at o, and by $v \wedge \omega$ the *join* of v and ω, that is, the last vertex in common between the finite path from o to v and the ray ω starting at o. For any vertex v, denote by $I(v) \subset \Omega$ the set of all boundary points containing v (in particular, $I(o) = \Omega$). Given a homogeneous tree of degree $q+1$ with $q \geq 2$, let P be the probability measure on the Borel sets of Ω such that for every $v \neq o$, $P(I(v)) = \frac{q}{q+1} q^{-|v|}$.

Let T be a homogeneous tree of degree $q+1$, where $q \geq 2$. A function f on T is said to be *fractal* if for all $v \in T \setminus \{o\}$, there is a descendant w of v and a tree isomorphism $\varphi : S(v) \to S(w)$ (that is, a bijection which preserves the nearest neighbor structure) such that $f \circ \varphi = f$.

3. Fractal harmonic functions with no radial limits

From now on, except for Section 6, T will denote an isotropic tree of degree $q+1$, where $q \geq 2$.

For each $v_0 \neq o$, and real numbers a and b, we define the *fractal harmonic* function $H_{a,b,v_0} = H$ on $\{v_0^-\} \cup S(v_0)$ by $H(v_0^-) = a$, $H(v_0) = b$, and for any $v \in S(v_0)$ for which $H(v^-)$ and $H(v)$ are defined, define H at the children of v according to the following algorithm:

- If $H(v) \neq 0$ or $H(v^-) \neq 0$, we define f to be $(q+1)H(v) - H(v^-)$ on one child of v and 0 on the other $q-1$ children of v.
- If $H(v) = 0$ and $H(v^-) = 0$, in case q is even we define H to be 1 on half of the children of v and -1 on the remaining children of v.
- If $H(v) = 0$ and $H(v^-) = 0$, in case $q = 2k+1$ is odd, we define H to be 1 on k of the children of v, -1 on $k-1$ of the children of v, and $-1/2$ on the two remaining children of v;

For $n \geq 1$, let $H_{a,b,v_0,n}$ be the restriction of H_{a,b,v_0} to $\{v_0^-\} \cup \{v \in S(v_0) : d(v, v_0) \leq n\}$. An illustration showing the first few steps of the construction of $H_{0,0,v_0}$ for $q = 2$ is provided in Fig. 1.

It is immediate that the above algorithm defines a harmonic function at each vertex of $S(v_0)$ and any choice of a and b with $0 \leq a < b$ determines a single ray on which the function increases unboundedly according to the following recurrence relation:

$$x_{n+1} = (q+1)x_n - x_{n-1} \text{ for } n \geq 1,$$

with initial conditions $x_0 = a$ and $x_1 = b$.

Consider, for example, the case $q = 2$ where we start with the initial values $x_0 = 0$ of H at v_0^- and $x_1 = 1$ at v_0. We obtain the sequence

$$0, 1, 3, 8, 21, 55, 144, 377, \dots.$$

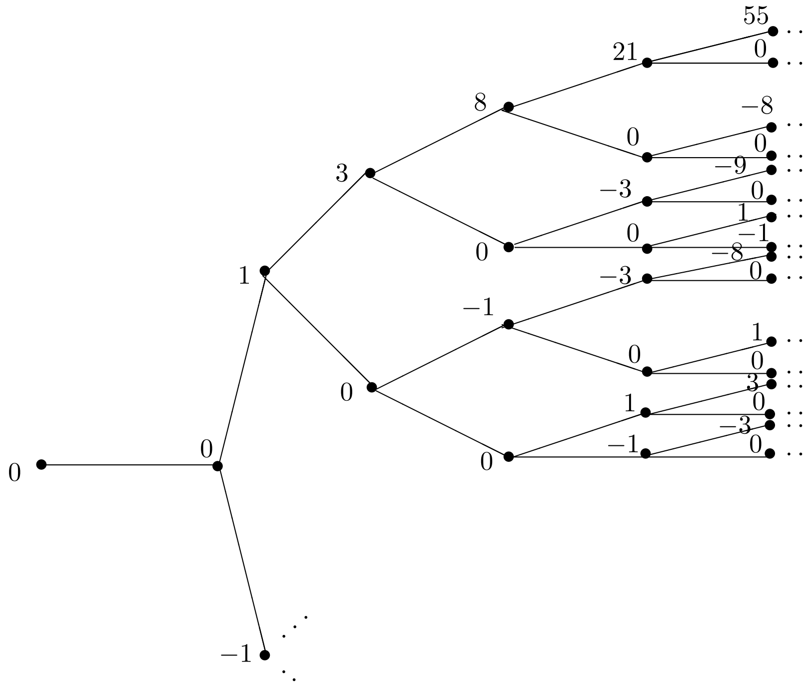

FIGURE 1. Values of $H_{0,0,v_0}$ on the vertices up to length 6 in case $q = 2$. The (omitted) values on the bottom branch of the tree are the respective negatives of those shown.

These numbers correspond to the elements of the subsequence f_{2n} of the Fibonacci sequence f_n, namely, $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. The closed-form formula is

$$x_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{3+\sqrt{5}}{2}\Big)^n - \Big(\frac{3-\sqrt{5}}{2}\Big)^n\Big]. \tag{1}$$

For a general value of q, we have

$$x_n = \frac{1}{\sqrt{q^2+2q-3}}\Big[\Big(\frac{q+1+\sqrt{q^2+2q-3}}{2}\Big)^n - \Big(\frac{q+1-\sqrt{q^2+2q-3}}{2}\Big)^n\Big]. \tag{2}$$

Therefore, x_n behaves like α^n, where

$$\alpha = \frac{q+1+\sqrt{q^2+2q-3}}{2}. \tag{3}$$

Note that when q is even, every value of H is a finite product of elements of $\{x_n\}$ or their negatives, and conversely any such product appears as a value of H at some vertex in the tree (actually, such a value will appear infinitely often). To see this, we argue as follows.

By construction, the values of H at the children of a vertex v such that $H(v) = H(v^-) = 0$ are 1 or -1, so moving forward, the nonzero values of H are the elements of the sequence $\{x_n\}$ or their negatives. If $H(v) = 0$, but $H(v^-) = x \neq 0$, then the values of H at the forward neighbors are 0 or $-x$, so the values of H along the single path forward where all values are nonzero are x times x_1, x_2, x_3, etc. Next, let v

be any vertex such that $H(v) \neq 0$. Identify the nearest ancestor v' of v such that $H(v') = 0$ and let w be the child of v' that lies in the path $[v', v]$. If $m = d(v', v)$, then $H(v) = x_m H(w)$. It follows by induction that $H(v)$ is a finite product of the x_n or their negatives.

Definition 3.1. For each vertex $v \in S(v_0)$, we refer to v as a *non-zero* if $H(v) \neq 0$, a *zero* if $H(v) = 0$, a *single zero* if $H(v) = 0$ and $H(v^-) \neq 0$, and a *double zero* if $H(v) = H(v^-) = 0$.

Note that between any consecutive zeros, the values of H are either all positive or all negative, $|H|$ strictly increases and, at the vertices where H does not vanish, $|H|$ is at least $1/2$.

Theorem 3.1. *For any vertex $v_0 \neq o$ and any constants a, b, the function $H = H_{a,b,v_0}$ converges to $\pm\infty$ on countably many paths and does not converge along any other path.*

Proof. For any vertex $v \in S(v_0)$, the set I_v contains either one or q elements ω such that there are no zeros among the descendants of v in the ray $[v, \omega)$. Furthermore, there are exactly q such paths if and only if v is a double zero.

If ω has infinitely many zeros, then H has no limit along ω, because there can be at most two consecutive zeros and any non-zero has absolute value at least $1/2$.

On the other hand, if ω has finitely many zeros, let v be the last zero. Then v is the last zero of at most q such paths, so there are countably many paths with no zeros from some point on, and the limit of H along these is $\pm\infty$. $\square$

4. Fractal harmonic functions in $A^p(\sigma)$ for small p

In this section we fix a vertex $v_0 \neq o$ and show that the fractal function $H : \{v_0^-\} \cup S(v_0) \to \mathbb{R}$ introduced in Section 3 is in the Bergman space $A^p(\sigma)$ for small values of p, including 1. In order to estimate $\sum_{v \in T} |H(v)|^p \sigma(v)$, we need to analyze the number of zeros (which we will separate into two groups, the single zeros and the double zeros) and the number of non-zeros of H.

Definition 4.1. For each $n \geq 0$, consider the set of vertices of $S(v_0)$ whose distance from v_0 is n. For those q^n vertices define

- $a_n = \#$ non-zeros
- $b_n = \#$ double zeros
- $c_n = \#$ zeros
- $d_n = \#$ single zeros

Let $A_n = a_n/q^n$ denote the proportion of vertices in $S(v_0)$ a distance n from v_0 which are non-zeros. Define B_n, C_n, and D_n similarly.

Proposition 4.1. *For $q \geq 2$, the sequence $\{b_n\}$ satisfies the recurrence relation*

$$b_{n+2} + (q-1)b_{n+1} + (q-1)^2 b_n = (q-1)^2 q^n, \ n \geq 1, \tag{4}$$

and has solution

$$b_n = c_1(q-1)^n \cos\left(\frac{2\pi}{3}n\right) + c_2(q-1)^n \sin\left(\frac{2\pi}{3}n\right) + \frac{(q-1)^2}{3q^2 - 3q + 1} q^n, \tag{5}$$

where c_1 and c_2 are determined by the initial values b_0 and b_1, which are in turn determined by $H(v_0^-)$ and $H(v_0)$.

Proof. Clearly we have

$$b_n + d_n = c_n,\ a_n + c_n = q^n,\ b_n = (q-1)d_{n-1},\ d_n = (q-1)a_{n-1}. \tag{6}$$

Thus

$$\begin{aligned} b_{n+2} = (q-1)d_{n+1} = (q-1)^2 a_n &= (q-1)^2(q^n - c_n) = (q-1)^2(q^n - b_n - d_n) \\ &= (q-1)^2 q^n - (q-1)^2 b_n - (q-1)b_{n+1}, \end{aligned}$$

which gives

$$b_{n+2} + (q-1)b_{n+1} + (q-1)^2 b_n = (q-1)^2 q^n. \tag{7}$$

Independent solutions of the associated homogeneous equation are $(q-1)^n \cos\left(\frac{2\pi}{3}n\right)$ and $(q-1)^n \sin\left(\frac{2\pi}{3}n\right)$, and the particular solution $\frac{(q-1)^2}{3q^2-3q+1}q^n$ is obtained by inspection. □

Corollary 4.1. *For $q \geq 2$, let $B(q) = \dfrac{(q-1)^2}{3q^2-3q+1}$. Then $1/7 \leq B(q) < 1/3$. Independent of the choice of initial conditions, for n sufficiently large we have $A_n \approx \left(\dfrac{q}{q-1}\right)^2 B(q)$, $B_n \approx B(q)$, $C_n \approx \dfrac{2q-1}{q-1}B(q)$, and $D_n \approx \dfrac{q}{q-1}B(q)$.*

Proof. A simple calculation or sketch shows that for $q \geq 2$, $B(q)$ is increasing, so the given inequalities hold. Considering the various choices, we see there are only the following three possible initial conditions: $b_0 = 1, b_1 = 0$; $b_0 = 0, b_1 = 1$; $b_0 = 0, b_1 = 0$. By dividing (5) by q^n, since $(q-1)^n/q^n \to 0$ as $n \to \infty$, it follows that if n is sufficiently large, B_n can be made as close to $B(q)$ as we wish, regardless of the values of $H(v_0^-)$ and $H(v_0)$. The approximations of A_n, C_n and D_n follow from this and (6). □

Let f be the function on T equal to $H_{0,0,v_0}$ for each vertex v_0 of modulus 1. We wish to show that $f \in A^1(\sigma)$, that is,

$$\sum_{n=0}^{\infty} \sum_{|v|=n} |f(v)|\sigma_n < \infty.$$

Let $H = H_{0,0,v_0}$, where v_0 is a vertex of modulus 1. For each $n \geq 0$, let us define $S_{v_0,n} = \{v \in S(v_0) : d(v, v_0) = n\}$ and

$$s_n = \sum_{v \in S_{v_0,n}} |H(v)|.$$

For a vertex $w \in S_{v_0,n}$ with $d(w, v_0) = m$, we will refer to m as the *relative length of* w. Due to the symmetry of f, it suffices to analyze just the function H and prove that $\sum_{n=0}^{\infty} s_n\sigma_n < \infty$. Define

$$\tilde{q} = \begin{cases} q & \text{if } q \text{ is even}, \\ q-1 & \text{if } q \text{ is odd}. \end{cases}$$

Theorem 4.1. *The sequence $\{s_n\}$ satisfies the non-homogeneous recurrence relation*

$$s_{n+1} = (q+1)s_n + (q-2)s_{n-1} + \tilde{q}\, b_n. \tag{8}$$

Proof. Suppose that q is even. For each $n \geq 1$ we partition the vertices z of relative length $n-1$ according as z is a double zero, a non-zero, or a single zero. For each such z of relative length $n-1$, let $s_{n-1}(z)$ denote the contribution of z to s_{n-1}, let $s_n(z)$ denote the contribution of the children of z to s_n, and let $s_{n+1}(z)$ denote the contribution of the grandchildren of z to s_{n+1}. Let $b_n(z)$ denote the number of double zeros among the children of z. We are done if we can prove that in all cases,

$$s_{n+1}(z) = (q+1)s_n(z) + (q-2)s_{n-1}(z) + q\,b_n(z). \tag{9}$$

If z is a double zero, then each child v of z takes the value 1 or -1, one child of v takes the value $q+1$ or $-(q+1)$ and all the other children of v take the value 0. Thus the contribution of z to s_{n-1} is 0, the contribution of the children of z to s_n is q, the contribution of the grandchildren of z to s_{n+1} is $(q+1)q$, and there are no double zeros among the children of z. Thus (9) holds in this case.

If z is a non-zero, assume $H(z) > 0$ (the argument is analogous if $H(z) < 0$). Then $H(z^-) \geq 0$. There is one child v of z such that $H(v) = (q+1)H(z) - H(z^-)$ and every other child of z takes the value 0. Let w denote one of these other children of z. One child of v takes the value $(q+1)[(q+1)H(z)-H(z^-)]-H(z)$ and the other children of v take the value 0. One child of w takes the value $-H(z)$ and all other children of w take the 0. Thus the contribution of z to s_{n-1} is $s_{n-1}(z) = H(z)$, the contribution of the children of z to s_n is

$$s_n(z) = |(q+1)H(z) - H(z^-)| = (q+1)H(z) - H(z^-),$$

and the contribution of the grandchildren of z to s_{n+1} is

$$\begin{aligned} s_{n+1}(z) = |(q+1)[(q+1)H(z) - H(z^-)] - H(z)| + (q-1)|-H(z)| \\ = (q+1)[(q+1)H(z) - H(z^-)] + (q-2)H(z) \\ = (q+1)s_n(z) + (q-2)s_{n-1}(z). \end{aligned}$$

Since there are no double zeros among the children of z, (9) holds in this case.

If z is a single zero, assume $H(z^-) > 0$ (again, the argument is similar if $H(z^-) < 0$). There is one child v of z such that $H(v) = -H(z^-)$ and the other children w of v take the value 0. One child of v takes the value $-(q+1)H(z^-)$ and the other children of v take the value 0. The children of each such w all take the value 1 or -1. Thus the contribution of z to s_{n-1} is $s_{n-1}(z) = 0$, the contribution of the children of z to s_n is $s_n(z) = |-H(z^-)| = H(z^-)$, and the contribution of the grandchildren of z to s_{n+1} is $s_{n+1}(z) = |-(q+1)H(z^-)| + q(q-1) = (q+1)H(z^-) + q(q-1)$. Observe that the factor $(q-1)$ that appears in this last term is the number of double zeros $b_n(z)$ among the children of z. Again (9) holds.

This completes the proof in case q is even. We omit the proof in case q is odd. It is identical in case z is a non-zero, and only minor changes are needed in case z is a double zero or a single zero. $\square$

For $q \geq 2$, define

$$\beta = \frac{1}{2}(q + 1 + \sqrt{q^2 + 6q - 7}). \tag{10}$$

Noting that $2(q+1) \leq q + 1 + \sqrt{q^2 + 6q - 7} = q + 1 + \sqrt{(q+3)^2 - 16} < 2q + 4$, we see that $q + 1 \leq \beta < q + 2 \leq q^2$, so $1 < \frac{2 \log q}{\log \beta}$.

We are now ready to prove the first of our main results.

Theorem 4.2. *Fix $q \geq 2$. Then, for each $p \in \left(0, \frac{2\log q}{\log \beta}\right)$, the function f on T equal to $H_{0,0,v_0}$ for each vertex v_0 of modulus 1, is in $A^p(\sigma)$, that is,*

$$\sum_{n=0}^{\infty} \sum_{|v|=n} |f(v)|^p \sigma_n < \infty.$$

Proof. Let us begin by proving the result for $p = 1$. As we have already observed, it is enough to prove that $\sum_{n=0}^{\infty} s_n \sigma_n < \infty$. We can solve for s_n using Proposition 4.1 and Theorem 4.1 by multiplying (4) by q and substituting into it the expressions of b_n, b_{n+1}, b_{n+2} in terms of the s_k obtained from (8). We now separate the cases when q is even and q is odd.

In the case when q is even, we obtain

$$\begin{aligned}(q-1)^2 q^{n+1} &= q[b_{n+2} + (q-1)b_{n+1} + (q-1)^2 b_n] \\ &= s_{n+3} - (q+1)s_{n+2} - (q-2)s_{n+1} \\ &\quad + (q-1)\left[s_{n+2} - (q+1)s_{n+1} - (q-2)s_n\right] \\ &\quad + (q-1)^2\left[s_{n+1} - (q+1)s_n - (q-2)s_{n-1}\right],\end{aligned}$$

which yields the 4th order recurrence relation

$$\begin{aligned}s_{n+3} - 2s_{n+2} - (3q-4)s_{n+1} - (q-1)(q^2+q-3)s_n - (q-2)(q-1)^2 s_{n-1} \\ = (q-1)^2 q^{n+1}.\end{aligned} \tag{11}$$

The characteristic equation of the associated homogeneous recurrence relation is $r^4 - 2r^3 - (3q-4)r^2 - (q-1)(q^2+q-3)r - (q-2)(q-1)^2 = 0$, which can be factored as $[r^2 - (q+1)r - (q-2)][r^2 + (q-1)r + (q-1)^2] = 0$, leading to the roots $r = \frac{1}{2}(q+1 \pm \sqrt{q^2+6q-7})$ and $r = \frac{1}{2}(q-1)(-1 \pm i\sqrt{3})$.

A particular solution of (11) is given by

$$y_n = -\frac{(q-1)q^{n+2}}{2(3q^2-3q+1)}.$$

Thus, the general solution is given by

$$\begin{aligned}s_n \;=\; & \frac{C_1}{2^n}\left(q+1+\sqrt{q^2+6q-7}\right)^n + \frac{C_2}{2^n}\left(q+1-\sqrt{q^2+6q-7}\right)^n \\ & + \frac{C_3}{2^n}(q-1)^n\left(-1-i\sqrt{3}\right)^n + \frac{C_4}{2^n}(q-1)^n\left(-1+i\sqrt{3}\right)^n + y_n.\end{aligned}$$

Clearly $s_n > 0$ for $n \geq 1$, so necessarily $C_1 > 0$. From (3) and the remarks before the theorem, we see that the solution is of the order β^n, which is bounded above by $(q+2)^n$.

In the case when q is odd, we proceed in a similar manner, using the appropriate formula in (8). Shifting as in in the previous case, multiplying this new expression by $q-1$ and substituting into it the expressions of b_n, b_{n+1}, b_{n+2} in terms of the s_k, we obtain the 4th order recurrence relation

$$\begin{aligned}s_{n+3} - 2s_{n+2} - (3q-4)s_{n+1} - (q^3-4q+3)s_n - (q-1)(q^2-3q+2)s_{n-1} \\ = (q-1)^3 q^n.\end{aligned} \tag{12}$$

The characteristic equation of the associated homogeneous recurrence relation is the same as in the above case and a particular solution of (12) is given by

$$z_n = -\frac{(q-1)^2 q^{n+1}}{2(3q^2-3q+1)}.$$

Thus, as above, the general solution is likewise of the order β^n.

Therefore, for any $q \geq 2$, since $\frac{\beta}{q^2} < 1$, we have

$$\sum s_n \sigma_n \leq C \sum \left(\frac{\beta}{q^2}\right)^n < \infty,$$

proving that $f \in A^1(\sigma)$.

Next assume $0 < p < 1$. Then, noting that for q even, the function f is integer-valued and for q odd, the smallest non-zero value of $|f|$ is $1/2$, we see that $|2f(v)|^p \leq |2f(v)|$, so $f \in A^p(\sigma)$.

Let us now assume $1 < p < \frac{2\log q}{\log \beta}$. Then $\frac{\beta^p}{q^2} < 1$ and since s_n is of the order β^n, we obtain

$$\sum_{n=0}^{\infty} \sum_{|v|=n} |f(v)|^p \sigma_n \leq \sum_{n=0}^{\infty} \Big(\sum_{|v|=n} |f(v)| \Big)^p \sigma_n = \sum_{n=0}^{\infty} s_n^p \sigma_n \leq C \sum_{n\in\mathbb{N}} \left(\frac{\beta^p}{q^2}\right)^n < \infty.$$

This completes the proof. □

5. Fractal functions in $A^p(\sigma)$ for a general $p > 0$

Our next aim is to show that for any $p > 0$, a function $f \in A^p(\sigma)$ with no radial limits exists. The growth of the function f described by the fractal harmonic extension algorithm of the previous section is too big to allow $|f|^p$ to be summable over T with respect to σ if p is too large. Therefore, we shall alternate the fractal harmonic extension operation (which is applied for a fixed number of generations n) with a process of local radialization on truncated sectors of constant thickness k, k to be determined in terms of the given value of p and n.

We first recall from [2] the notion of harmonic radialization motivated by the following result.

Lemma 5.1. (Lemma 4.1 of [2]) *Let $v \in T$ with $v \neq o$. Let $f : \{v^-\} \cup S(v) \to \mathbb{R}$ be a radial function which in addition is harmonic on $S(v)$. Then for any $u \in \{v^-\} \cup S(v)$ and $j = d(u, v)$,*

$$(13)\ f(u) = \frac{f(v)q - f(v^-)}{q-1} + \frac{f(v^-) - f(v)}{q-1} q^{-j} = L + (f(v^-) - L)q^{-(j+1)},$$

where the radial limit of f in $S(v)$ is $L = \dfrac{f(v)q - f(v^-)}{q-1}$. Conversely, the formulas in (13) *define a function on $\{v^-\} \cup S(v)$ which is radial and harmonic at each vertex of $S(v)$, where the values at v and v^- are arbitrarily prescribed.*

Definition 5.1. (Definition 7 of [2]) For $v \in T$ with $v \neq o$, the *local harmonic radialization operator* RH_v is defined on functions f on T as follows:

$$(14)\quad RH_v f(w) = \begin{cases} \dfrac{f(v)\,q - f(v^-)}{q-1} + \dfrac{f(v^-) - f(v)}{q-1}\, q^{-d(w,v)} & \text{if } w \in S(v), \\ f(w) & \text{if } w \in T \setminus S(v). \end{cases}$$

By Lemma 5.1, $RH_v f$ is radial and harmonic at each vertex of $S(v)$.

Fix $n, k \in \mathbb{N}$ and define the function f dependent on n and k as follows. Consider the annuli of thickness n and k, respectively, defined as

$$(15)\qquad F_j = \{v \in T :\ j(n+k) \leq |v| \leq (j+1)n + jk\},\ j \in \mathbb{N} \cup \{0\},$$

$$(16)\qquad R_j = \{v \in T :\ jn + (j-1)k \leq |v| \leq j(n+k)\},\ j \in \mathbb{N}.$$

For $j \geq 1$, we will refer to a vertex v as lying on the *inner boundary of* F_j or equivalently on the *outer boundary* of R_j, if $|v| = j(n+k)$. We refer to v as lying on the *outer boundary* of F_j or equivalently on the *inner boundary* of R_{j+1} if $|v| = (j+1)n + jk$.

We construct a harmonic function f inductively by first applying our fractal procedure on the set F_0, then applying a modification of the local harmonic radialization procedure on the set R_1. If a vertex v of length n is not a single zero, then using as initial data the values of f at v and at its parent vertex, we apply the local harmonic radialization procedure; if vertex v of length n is a single zero, we assign value $-f(v^-)$ to one of its children and 0 to its remaining children (so we are extending the fractal procedure for one additional generation) and then continue with the local harmonic radialization procedure. We then apply our fractal procedure on the set F_1 using as initial data the values of f at the vertices of length $n+k$ and at their parent vertices. Repeat the application of the modified local harmonic radialization procedure on the set R_2 using as initial data the values of f at the vertices of length $2n+k$ and at their parent vertices. Continue in this fashion by alternating the application of the two procedures. We shall use the notation $\varphi_{n,k}$ for the resulting function.

The effect of intertwining the two procedures is to slow down the growth of the fractal harmonic function in such a way that with the appropriate choice of the numbers n and k, the resulting function is in $A^p(\sigma)$. Furthermore, the reason for modification in the application of the local radialization procedure described above is to ensure that all the nonzero values of the harmonic function obtained in this way have absolute values that are at least $1/2$ if q is odd and at least 1 if q is even.

In what follows, the terms non-zeros, double zeros, zeros, and single zeros are used as in Definition 3.1, except H in that definition is replaced by $\varphi_{n,k}$.

Theorem 5.1. *Let $p > 0$ and α as in (3). If $n, k \in \mathbb{N}$ such that*

$$k > \frac{p \log 2 + n(p \log \alpha - \log q)}{\log q},$$

then $\varphi_{n,k} \in A^p(\sigma)$.

Proof. We begin by noting that the σ measures of the annuli F_j and R_j defined in (15) and (16) used in the construction of $\varphi_{n,k}$ are bounded above by certain powers of q: $\sigma(F_j) < q^{-j(n+k)}$ for $j \geq 0$, $\sigma(R_j) < q^{-(jn+(j-1)k)}$ for $j \geq 1$. Moreover, it can be seen from our construction that upper bounds of $|\varphi_{n,k}|$ on the sets R_j are $2^j x_{jn}$, while on the sets F_j they are $2^j x_{(j+1)n}$. Therefore, splitting the sum over T into sums over the sets F_j and R_j and recalling from (3) that for $m \in \mathbb{N}$, x_m is approximately α^m, we have

$$\begin{aligned}
\sum_{v\in T}|\varphi_{n,k}(v)|^p\sigma(v) &\le \sum_{j=0}^{\infty}2^{jp}x^p_{(j+1)n}\sigma(F_j)+\sum_{j=1}^{\infty}2^{jp}x^p_{jn}\sigma(R_j)\\
&\le \sum_{j=0}^{\infty}2^{jp}x^p_{(j+1)n}q^{-j(n+k)}+\sum_{j=1}^{\infty}2^{jp}x^p_{jn}q^{-(nj+(j-1)k)}\\
&= \sum_{j=0}^{\infty}\big(2^{jp}x^p_{(j+1)n}q^{-j(n+k)}+2^{(j+1)p}x^p_{(j+1)n}q^{-(n(j+1)+jk)}\big)\\
&= (1+2^pq^{-n})\sum_{j=0}^{\infty}2^{jp}x^p_{(j+1)n}q^{-j(n+k)}\\
&\le (1+2^pq^{-n})\alpha^{pn}\sum_{j=0}^{\infty}\Big(\frac{2^p\alpha^{pn}}{q^{n+k}}\Big)^j,
\end{aligned}$$

which converges if $\frac{2^p\alpha^{pn}}{q^{n+k}}<1$. A straightforward computation shows that this holds under the assumption on k. Therefore, for this choice of k, $\varphi_{n,k}$ is in $A^p(\sigma)$. $\square$

The following result highlights how the membership of $\varphi_{n,k}$ to a Bergman space depends on the integers n and k.

Theorem 5.2. *Let n,k be positive integers.*

(a) *If* $0<p<\dfrac{(n+k)\log q}{\log 2+n\log\alpha}$, *then* $\varphi_{n,k}\in A^p(\sigma)$.

(b) *If* $p\ge\dfrac{2(n+k)\log q}{\log 2+n\log\alpha}$, *then* $\varphi_{n,k}\notin A^p(\sigma)$.

Proof. Part (a) follows immediately from Theorem 5.1. To prove (b), recalling (2) and noting that $x_n\ge C\alpha^n$ where $C=\frac{1}{\sqrt{q^2+2q-3}}(1-\alpha^{-2})$, a lower estimate on $|\varphi_{n,k}|^p$ can be obtained by following a single ray where the function strictly increases as follows:

$$\begin{aligned}
\sum_{v\in T}|\varphi_{n,k}(v)|^p\sigma(v) &\ge \sum_{j=0}^{\infty}2^{jp}x^p_{(j+1)n}q^{-2j(n+k)}+\sum_{j=1}^{\infty}2^{jp}x^p_{jn}q^{-2(nj+(j-1)k)}\\
&= (1+2^pq^{-2n})\sum_{j=0}^{\infty}2^{jp}x^p_{(j+1)n}q^{-2j(n+k)}\\
&\ge C(1+2^pq^{-2n})\alpha^{pn}\sum_{j=0}^{\infty}\Big(\frac{2^p\alpha^{pn}}{q^{2(n+k)}}\Big)^j.
\end{aligned}$$

Since by the assumption on p, the series on the right-hand side diverges, $\varphi_{n,k}\notin A^p(\sigma)$. $\square$

Note that the lower bound in the above estimate cannot be improved by taking into account all of the paths along which $|\varphi_{n,k}|$ increases to infinity, because for any vertex v, $\sigma(S(v))$ is in the order of $\sigma(v)$.

We end the section by explaining how to choose the integer n so that for k as in Theorem 5.1, the function $\varphi_{n,k}$ has no finite radial limits.

Given $n, k \in \mathbb{N}$, define the following sets:

$$\begin{aligned} X_{n,k} &= \{\omega \in \Omega : \exists J \in \mathbb{N},\ \forall j \geq J,\ \omega_j \text{ is a nonzero of } \varphi_{n,k}\}, \\ Y_{n,k} &= \{\omega \in \Omega : \forall J \in \mathbb{N},\ \exists j \geq J,\ \omega_j \text{ is a double zero of } \varphi_{n,k}\}. \end{aligned}$$

Thus $X_{n,k}$ consists of all the boundary points whose vertices are non-zeros of $\varphi_{n,k}$ after a while, and $Y_{n,k}$ consists of all the boundary points whose vertices have an infinite number of double zeros. Thus for $\omega \in X_{n,k}$, $|\varphi_{n,k}| \to \infty$ along the vertices of ω, and for $\omega \in Y_{n,k}$, $\varphi_{n,k}$ has no limit along the vertices of ω.

Theorem 5.3. (i) *For $k \geq 2$ and for any n, the set $X_{n,k}$ is uncountable.*
(ii) *For n sufficiently large and k as in Theorem 5.1, $P(Y_{n,k}) = 1$.*

Proof. (i) For each $j \geq 1$, starting from a non-zero vertex on the inner boundary of F_j (as defined in (15)), there is a single path consisting of all non-zeros of $\varphi_{n,k}$ joining that vertex to a vertex on the outer boundary of F_j; on the other hand, starting from a non-zero vertex on the inner boundary of R_j (as defined in (16)), there are q^k paths consisting of all non-zeros joining that vertex to a vertex on the outer boundary of R_j. Thus distinct elements of Ω can be constructed with the selection of independent choices of any of q^k vertices on the outer boundary of R_j for each j, and so $X_{n,k}$ is uncountable.

(ii) The proof is a modification of the Borel-Cantelli Lemma. For each $j \geq 1$ and each double zero v on the inner boundary of R_j, it follows from Definition 5.1 that v contributes exactly q^k double zeros on the outer boundary of R_j, and every double zero on the outer boundary of R_j arises in this way. Thus the proportion of double zeros on the inner boundary of R_j is the same as the proportion of double zeros on the outer boundary of R_j. Since $B(q) \geq 1/7$ (where $B(q)$ was defined in Corollary 4.1), by Corollary 4.1, we can choose n so that for each j, the proportion of double zeros on the outer boundary of R_j is at least $1/10$.

For each $j \geq 1$, let $Z_j = \{\omega \in \Omega : \omega_{n(j+k)} \text{ is a double zero}\}$. Let $Z_j^c := \Omega \setminus Z_j$. Then $P(Z_j^c) \leq 9/10$ and $P(Z_j^c \mid Z_{j-1}^c) \leq 9/10$. Then for $n_1 < n_2$, $P\Big(\bigcap_{j=n_1}^{n_2} Z_j^c\Big) \leq (9/10)^{n_2 - n_1 + 1}$. It follows that

$$P\Big(\bigcup_{j=n_1}^{n_2} Z_j\Big) = 1 - P\Big(\bigcap_{n_1}^{n_2} Z_j^c\Big) \to 1$$

as $n_2 \to \infty$. Thus $P\Big(\bigcup_{j=m}^{\infty} Z_j\Big) = 1$ for each m, and so $P\Big(\bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} Z_j\Big) = 1$. This says that with probability 1, ω lies in Z_j for infinitely many j, which means that it has infinitely many double zeros. $\square$

6. The general radial case

We now discuss the case when T is a general radial tree. The meaning of s_n and b_n is as in the homogeneous setting.

Theorem 6.1. *Let v_0 be a fixed vertex of length 1. For $n \geq 0$, let $q_n \geq 2$ be the number of children of any vertex in $S(v_0)$ at distance n from v_0. Let H be the harmonic function defined as in Section 3, where in the definition of $H(v)$, q is replaced by the number q_n of forward neighbors of v. For a nonnegative integer*

n, let s_n denote the sum of the values of $|H|$ at the vertices $v \in S(v_0)$ such that $d(v, v_0) = n$, let $\pi_0 = 1$ and, for $n \geq 1$, let $\pi_n = \prod_{k=0}^{n-1} q_k$ be the number of such vertices. For $n \in \mathbb{N}$, define

$$\tilde{q}_n = \begin{cases} q_n & \text{if } q_n \text{ is even,} \\ q_n - 1 & \text{if } q_n \text{ is odd.} \end{cases}$$

Then the sequence $\{s_n\}$ satisfies the non-homogeneous recurrence relation

$$s_{n+1} = (q_n + 1)s_n + (q_{n-1} - 2)s_{n-1} + \tilde{q}_n b_n, \tag{17}$$

with initial conditions $s_0 = 0$, $s_1 = \tilde{q}_0$. Moreover, the sequence $\{b_n\}$ satisfies the recurrence relation

$$b_{n+2} + (q_{n+1} - 1)b_{n+1} + (q_{n+1} - 1)(q_n - 1)b_n = \pi_n(q_{n+1} - 1)(q_n - 1),$$

for $n \geq 0$, with initial conditions $b_0 = 1$ and $b_1 = 0$.

The proof of the recurrence relations of s_n and b_n is carried out as in the isotropic case, where in the argument provided in the proof of Proposition 4.1, the value q is replaced by the appropriate q_n, whose index is clear from the context. We omit the details.

We now extend a special case of Theorem 4.2 to radial trees.

Theorem 6.2. *Let T be a radial tree. Under the notation and the assumptions of Theorem 6.1, suppose $\sum_{k=1}^{\infty} \frac{1}{q_k} = \infty$. Let σ be a finite radial measure satisfying the condition*

$$\sum_{n \in \mathbb{N}} 2^n \pi_n \sigma_n < \infty. \tag{18}$$

Then the function H has no finite radial limits and belongs to $A^1(\sigma)$.

Proof. Dividing (17) by π_{n+1} yields

$$\frac{s_{n+1}}{\pi_{n+1}} = \left(1 + \frac{1}{q_n}\right)\frac{s_n}{\pi_n} + \left(\frac{q_{n-1} - 2}{q_n q_{n-1}}\right)\frac{s_{n-1}}{\pi_{n-1}} + \frac{\tilde{q}_n}{q_n}\frac{b_n}{\pi_n},$$

and so after defining $\gamma_n = \frac{s_n}{\pi_n}$, we get

$$\gamma_{n+1} = \left(1 + \frac{1}{q_n}\right)\gamma_n + \left(\frac{q_{n-1} - 2}{q_n q_{n-1}}\right)\gamma_{n-1} + \frac{\tilde{q}_n}{q_n}\frac{b_n}{\pi_n}. \tag{19}$$

Noting that the last two terms on the right-hand side are nonnegative, we deduce the lower estimate $\gamma_{n+1} \geq \left(1 + \frac{1}{q_n}\right)\gamma_n$, which shows that γ_n is strictly increasing and inductively

$$\gamma_{n+1} \geq \prod_{k=1}^{n}\left(1 + \frac{1}{q_k}\right)\gamma_1. \tag{20}$$

Since $\log\left(1 + \frac{1}{q_k}\right) > \frac{2}{3}\frac{1}{q_k}$, using the assumption, we have

$$\sum_{k=1}^{n} \log\left(1 + \frac{1}{q_k}\right) > \frac{2}{3}\sum_{k=1}^{n}\frac{1}{q_k} \to \infty,$$

as $n \to \infty$. Thus, $\prod_{k=1}^{n}\left(1+\frac{1}{q_k}\right) \to \infty$, which, using (20), shows that γ_n is unbounded.

On the other hand, neglecting the term -2 in (19) and since $\tilde{q}_n \leq q_n$, we have the upper estimate

$$\gamma_{n+1} \leq \left(1 + \frac{1}{q_n}\right)\gamma_n + \frac{1}{q_n}\gamma_{n-1} + \frac{b_n}{\pi_n}. \tag{21}$$

Using the assumption $q_n \geq 2$, $b_n \leq \pi_n$, and that γ_n is increasing, from (21) we obtain

$$\gamma_{n+1} \leq \frac{3}{2}\gamma_n + \frac{1}{2}\gamma_{n-1} + 1 \leq 2\gamma_n + 1.$$

Observe that the solution to the recurrence relation $x_{n+1} = 2x_n + 1$ with initial condition $x_1 = \gamma_1$ is given by $x_n = (1+\gamma_1)\,2^{n-1} - 1$. Whence $\gamma_n \leq x_n \leq C\,2^n$ for some constant C. Thus, from the summability of the sequence $\{2^n \pi_n \sigma_n\}$, we have

$$\sum s_n\sigma_n = \sum \gamma_n \pi_n \sigma_n \leq C \sum_{n\in\mathbb{N}} 2^n \pi_n \sigma_n < \infty,$$

proving that $H \in A^1(\sigma)$. The absence of finite radial limits of H is an immediate consequence of its construction. □

Remark 6.1. Theorem 6.2 applies in case there is an upper bound on the degrees of the vertices of T, except when $q_n = 2$ for all sufficiently large n, since the sum in (18) is infinite. However, by Theorem 4.2, the conclusion of Theorem 6.2 is still true in that case.

7. Concluding remarks

Let T be any tree endowed with an infinite number of boundary points in which every vertex has degree at least 2. Suppose f is a harmonic function on T with no finite radial limits. Then there has to be at least a countable set of rays along which $|f|$ has limit ∞. This follows from the observation that such a function must be non-constant along any ray and for all $v \sim w$ with $f(v) \neq f(w)$, there is a ray containing the edge $[v, w]$ along which $|f|$ tends to ∞. Thus, in the isotropic case the result obtained in Theorem 3.1 is the best possible. In Theorem 5.3, we showed that the absolute value of the function obtained by intertwining the fractal algorithm with the local radialization procedure described in Section 5 has infinite limit along an uncountable set of paths.

Open question: Does there exist a harmonic function f in $A^p(\sigma)$ (for an arbitrary p) with no finite radial limits and such that $|f| \to \infty$ along only a countable number of paths?

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