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Bergman Spaces and Carleson Measures on Homogeneous Isotropic Trees

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Abstract Hastings studied Carleson measures for non-negative subharmonic functions on the polydisk and characterized them by a certain geometric condition relative to Lebesgue measure σ. Cima & Wogen and Luecking proved analogous results for weighted Bergman spaces on the unit ball and other open subsets of $\mathbb{C}^n$. We consider a similar problem on a homogeneous tree, and study how the characterization and properties of Carleson measures for various function spaces depend on the choice of reference measure σ.

Keywords Carleson measure · Bergman space · Isotropic homogeneous trees · Very regular transition operators · Green kernel · Poisson kernel

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $H(\mathbb{D})$ be the class of analytic functions on $\mathbb{D}$. A positive measure μ on $\mathbb{D}$ is called a *Carleson measure* if there is a $C > 0$ such that for each $\theta_0 \in \mathbb{R}$ and $h \in (0, 1)$,

$$\mu(S_{\theta_0,h}) \leqslant C\,h, \tag{1.1}$$

where

$$S_{\theta_0,h} := \{re^{i\theta} : 1 - h \leqslant r < 1, \ |\theta - \theta_0| \leqslant h/2\}.$$

Sets of the form $S_{\theta_0,h}$ are commonly referred to as *Carleson boxes*. If we consider as reference measure the normalized two-dimensional Lebesgue measure m on $\mathbb{D}$, we define a positive measure μ on $\mathbb{D}$ to be an *m-Carleson measure*, if the ratio of the μ measure of a Carleson box to the area of the box is bounded. This geometric condition, which has been easily extended to higher dimensions, has been shown to be equivalent to a corresponding condition, where in the case of the disk the Carleson box has been replaced by the lens-shaped domain $S(\zeta, r) = \{z \in \mathbb{D} : |1 - \overline{z}\zeta| < r\}$, where $|\zeta| = 1$ and $0 < r < 1$ [6].

In [4], Carleson proved that, for $p > 1$, a positive measure μ is a Carleson measure if and only if it is a *Carleson measure for H^p*, that is, if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leqslant C\|f\|_p^p \ \text{ for all } f \in H^p,$$

where H^p is the classical Hardy space of analytic functions on $\mathbb{D}$ (note that this integral inequality coincides with (1.1) if f is the characteristic function of the Carleson box $S_{\theta_0,h}$, but of course this characteristic function does not belong to H^p).

This theorem has led to various generalizations, including a similar result for the Bergman space $A^p = L^p(m) \cap H(\mathbb{D})$, with norm

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dm(z)\right)^{1/p}.$$

Let X be a Banach space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|$. A positive measure μ is said to be a Carleson measure for X (or an X-Carleson measure) if there exists a positive constant C such that

$$\int_{\mathbb{D}} |f|^p \, d\mu \leqslant C\|f\|^p, \quad \text{for all } f \in X.$$

In [15], Hastings showed that in the polydisk setting if m is the Lebesgue volume, a positive measure μ is an m-Carleson measure if and only if μ is a Carleson measure for A^p. In fact, rather than focusing on just integrands of the form $|f|^p$ for $f \in A^p$, he obtained the above equivalence by considering all positive subharmonic functions. In [6], in the unit ball setting, using the higher-dimensional analogs of the above sets $S(\zeta, r)$, Cima and Wogen proved that if m_α denotes the weighted area measure with Bergman weight $(1 - |z|^2)^\alpha$ (where $\alpha > -1$), then μ is an m_α-Carleson measure if and only if μ is a Carleson measure for A^p_α, where A^p_α is the space of analytic functions f such that $|f|^p$ is integrable with respect to the measure m_α. A similar result in a broader context was shown by Oleinik in [19] and Luecking in [18]. In their work, in place of the Lebesgue volume, they considered a more general class of measures on a domain in $\mathbb{C}^n$ and proved Carleson-type theorems for Bergman spaces by making use of another condition equivalent to the Carleson condition in terms of pseudo-hyperbolic balls.

The common thread of these works is the attempt to determine appropriate conditions that yield the equivalence between the class of positive measures μ on the underlying domain Ω in $\mathbb{C}^n$ which satisfy a geometric Carleson-type condition relative to a fixed reference measure and the class of Carleson measures for X. Given a positive measure σ on Ω, we call a positive measure μ on Ω a σ-Carleson measure if for some constant $C > 0$, $\mu(S) \leqslant C\,\sigma(S)$, for all S belonging to a specified family $\mathcal{F}$ of subsets of Ω.

In the classical setting of the unit disk $\mathbb{D}$, in light of the results outlined above, taking $\mathcal{F}$ to be any of (i) the collection of Carleson boxes (with respect to area measure), (ii) the collection of sets of the form $S(\zeta, r)$, for $\zeta \in \partial\mathbb{D}, r > 0$, or (iii) the collection of pseudo-hyperbolic balls with a fixed pseudo-hyperbolic radius leads to the same notion of σ-Carleson measure. For details, see the proof of Theorem 14 in [11].

Given a positive measure σ on a domain Ω in $\mathbb{C}^n$ and a Banach space X of analytic functions on Ω, an interesting question is to determine under what conditions on σ, a positive measure μ on Ω is σ-Carleson if and only if μ is a Carleson measure for X.

In this paper, we study this problem in the setting of a homogeneous tree. Comparing the discrete analogs of the above three families, while the equivalence between the families (i) and (ii) is immediate, some differences between these two and the discrete version of (iii) emerge. These differences are presented at the end of Section 3. This leads us to study the above problem using the Carleson boxes, thus developing methods that differ from those pursued by Luecking.

After giving in Section 2 some background on trees, in Section 3, we shall introduce the Bergman spaces of harmonic functions on a homogeneous tree and define the measure classes under consideration in our work in terms of certain Carleson-type conditions. We shall then give the statements of our results concerning the relations among such classes. In particular, we consider two classes of reference measures, we call respectively *optimal* and *good*. For optimal measures σ, we establish a full equivalence between the class of σ-Carleson measures and the class of measures μ on the tree that satisfy the Carleson-type condition $\langle f, \mu\rangle \leqslant C\langle f, \sigma\rangle$, for all non-negative subharmonic functions f, for some constant C independent of f. (The notation $\langle f, \mu\rangle$ is used for the integral of f over the tree with respect to the measure μ.)

We present some interesting constructions of non-optimal measures. For good (non-optimal) measures only some of these results are accessible, exactly as for the continuous case. We conjecture that even for such measures, for which we can provide the sufficiency, the above equivalence holds. We prove this conjecture for the special case of radial Carleson measures. Furthermore, we show that the equivalence also holds for all Carleson measures if we restrict our attention to non-negative subharmonic functions supported on finitely many geodesic rays.

While the main focus of the paper is on Carleson-measure-type theorems for spaces of harmonic and subharmonic functions on homogeneous trees, we also present, in Section 4, some results on radialization and harmonic radialization of a function on a tree, and in Section 5, we discuss the Poisson transform and its interplay with radialization, and use some of these results to prove that the Hardy space $\mathcal{H}^p(T)$ of harmonic functions on the tree T is contained but not closed in the Bergman space $A^p(\sigma)$ if the reference measure σ is good.

Finally, in Section 6, we give the proofs of the results presented in Section 3.

For references, definitions and related results on continuous environments, we refer the reader to [3, 4, 6, 10, 15, 18]; on trees, to [5, 8, 9, 12, 13, 17, 20]. For general properties of transient random walks, the Poisson boundary and Poisson representations, and boundary martingales, see [2, 16].

2 Preliminaries on Trees

We denote by T a homogeneous tree where every vertex has the same number $q+1 \geqslant 3$ of neighbors, equipped with the isotropic nearest neighbor transition operator P. We write $u \sim v$ if the vertices u and v are neighbors. A reference vertex o is fixed once for all, and for every vertex $v \neq o$ we denote by $S(v)$ the sector of all vertices u such that $u \geqslant v$, in the sense that v belongs to the geodesic path from o to u. A ray, or geodesic path, is a path $v_0 < v_1 < v_2 \dots$ with $v_{n+1} \sim v_n$ for every n. We let $|v|$ denote the number of edges in the ray from o to v, and let $d(u, v)$ denote the number of edges in the geodesic path joining vertices u and v. If $v \neq o$, we let v^- be the neighbor of v which lies in the geodesic path from o to v.

Denote by Ω the *boundary* of T, that is the set of all infinite geodesic rays $\omega = [\omega_0 = o, \omega_1, \omega_2, \dots]$ and by $v \wedge \omega$ the *join* of v and ω, that is, the last vertex in common between the finite path from o to v and the geodesic ray ω starting at o. For any vertex v, denote by $I(v) \subset \Omega$ the set of all rays starting at o and containing v (if $v = o$, let $I(o) = \Omega$).

T satisfies the isoperimetric inequality: the cardinality of a ball is approximatively the same as the cardinality of its bounding circle. Indeed, for $k \geqslant 1$, the cardinality of the k-ball is $1 + \frac{q+1}{q-1}(q^k - 1)$ and the cardinality of the k-sphere is $(q+1)q^{k-1}$.

The Laplace operator Δ on functions defined on a homogeneous tree T of degree $q+1$ is defined by

$$\Delta f(v) = \frac{1}{q+1} \sum_{w \sim v} f(w) - f(v), \;\; v \in T,$$

where f is a function on T. A function f on T is called *harmonic* (respectively, *subharmonic*, *superharmonic*) if $\Delta f = 0$ (respectively, $\Delta f \geqslant 0$, $\Delta f \leqslant 0$).

For a boundary point $\omega \in \Omega$, denote by $K_\omega(v) := K(v, \omega)$ the Poisson kernel normalized to have the value 1 at o. Recall that (e.g. [5, 12, 13])

$$K(v, \omega) = q^{2|v \wedge \omega| - |v|}. \tag{2.1}$$

For a finite Borel measure μ on Ω, define the Poisson transform of μ as

$$\mathcal{K}\mu(v) = \int_\Omega K(v, \omega)\, d\mu(\omega).$$

Let ν be the equidistributed measure on Ω: $\nu(I(v)) = 1/|\{w : |w| = |v|\}| = 1/((q+1)q^{|v|-1})$. For a Borel measurable function F on Ω, if μ is the ν-absolutely continuous measure on Ω having density F, we denote $\mathcal{K}\mu$ by $\mathcal{K}F$. For every positive harmonic function h on T, there exists a unique Borel measure μ_h such that $h = \mathcal{K}\mu_h$ on T (see [5]). By a calculation it is possible to show directly that the above measure ν is the associated representing measure for the unit constant harmonic function.

Let F_+ be the set of all non-negative functions on T and denote by S_+ (respectively, H_+) the set of functions in F_+ which are subharmonic (respectively, harmonic) on T.

3 Measure Classes, Bergman Spaces and Outline of Results

We now give a broad definition of a measure σ that will play the role of the volume measure in the setting of homogeneous isotropic trees. We shall then consider different classes of measures μ and study the analogue of Hastings Theorem in the tree setting for the special case when $p = q = 1$.

Definition 1 A *reference measure* on T is a radial positive decreasing function σ on T such that $\|\sigma\| = 1$, where

$$\|\sigma\| = \|\sigma\|_{\ell^1(T)} = \sigma_0 + (q+1)\sum_{k=1}^{\infty} q^{k-1}\sigma_k,$$

having denoted by σ_k the value of σ at each of the vertices of length k.

Definition 2 Let T be a homogeneous isotropic tree of degree $q+1$ and σ a reference measure. For $1 \leqslant p < \infty$, let $L^p(T, \sigma)$ denote the space consisting of the functions f on T such that

$$\|f\|_{L^p}^p := \sum_{v \in T} |f(v)|^p \sigma(v) < \infty.$$

We define the *Bergman space* $A^p(\sigma)$ of T to be the subset of $L^p(T, \sigma)$ whose elements are harmonic functions on T. Whenever $f \in A^p(\sigma)$, the above norm shall be denoted by $\|f\|_{A^p}$.

Note that A^p is a closed subspace of $L^p(T, \sigma)$. Indeed, since convergence in norm of a sequence in $L^p(T, \sigma)$ implies pointwise convergence of a suitable subsequence, if $\{f_n\}$ is a sequence in A^p converging in $L^p(T, \sigma)$ to a function f, then f is the pointwise limit of some subsequence $\{f_{n_k}\}$, so f itself is harmonic and hence in A^p.

Notation 1 For $f \geqslant 0$, we let

$$\langle f, \sigma\rangle = \sum_{v \in T} f(v)\sigma(v).$$

In what follows, we shall assume that σ is a fixed reference measure and denote by $\mathcal{R}$ the class of such measures. For any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and any vertex v with $|v| = n$, set

$$\tau_n = \sigma(S(v)).$$

In the following definition, we introduce two classes of reference measures.

Definition 3 Let $\sigma \in \mathcal{R}$.

(1) We say that σ is *optimal* if

$$\sup_{n \in \mathbb{N}_0} \frac{\tau_n}{\sigma_n} < \infty.$$

(2) We say that σ is *good* if

$$\sum_{n=0}^{\infty} q^n \tau_n < \infty,$$

that is, the measure μ defined by $\mu(v) = \sigma(S(v))$ is finite.

The following theorem tells us how to generate examples of such reference measures using certain sequence of positive reals. In the theorem we consider the transition operator P on $\mathbb{Z}$ defined by

$$p(n, n-1) = \frac{q}{q+1}, \ p(n, n+1) = \frac{1}{q+1},$$

and zero otherwise.

Let σ be a reference measure. Define

$$a_n = \tau_n q^n. \tag{3.1}$$

Then,

$$\sigma_n = \tau_n - q\tau_{n+1} = q^{-n}(a_n - a_{n+1}). \tag{3.2}$$

Therefore σ is a non-negative measure on T if and only if $\{a_n\}$ is decreasing, and it is finite if and only if a_n tends to a finite limit, because $\sum q^n \sigma_n \approx \lim_{n\to\infty} a_n$. Moreover, it is easy to check that the sequence $\{\sigma_n\}$ is decreasing if and only if $\{a_n\}$ is subharmonic on $\mathbb{Z}^+$ with respect to the simple transition operator P. Now, Eq. 3.1 yields $\tau_n/\sigma_n = a_n/(a_n - a_{n+1})$. Thus, with a_n as above, we obtain the following result, where the proof of the second part follows at once from Eq. 3.2.

Theorem 3.1 *Let σ be a reference measure on T. Then $\{a_n\}$ is a strictly decreasing sequence on $\mathbb{N}_0$ which is P-subharmonic on $\mathbb{N}_0$. The reference measure σ is good if and only if $\{a_n\}$ is summable, and σ is optimal if and only if*

$$\sup_{n\in\mathbb{N}} \frac{a_n}{a_n - a_{n+1}} < \infty. \tag{3.3}$$

Conversely, let $\{a_n\}_{n\in\mathbb{N}_0}$ be any sequence of positive real numbers which is strictly decreasing and P-subharmonic on $\mathbb{N}_0$. Let $\sigma_n = q^{-n}(a_n - a_{n+1})$. Define $\sigma(v) = \sigma_{|v|}$. Then σ is a reference measure on T. In addition, σ is good if and only if $\{a_n\}$ is summable; σ is optimal if and only if $\{a_n\}$ satisfies (3.3).

So, non-optimal reference measures are those associated to strictly decreasing sequences $\{a_n\}$ of non-negative reals, $\left(\frac{q}{q+1}, \frac{1}{q+1}\right)$-subharmonic on $\mathbb{Z}_+$ such that (3.3) fails. The following results shows that there are many examples of such sequences.

Theorem 3.2 *Let $\{b_n\}_{n\in\mathbb{N}_0}$ be any unbounded increasing sequence of positive integers. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}_0}$ of non-negative numbers satisfying the following properties.*

- *(i)* $\{a_n\}$ *is strictly decreasing,*
- *(ii)* $\{a_n\}$ *is* $\left(\frac{q}{q+1}, \frac{1}{q+1}\right)$*-subharmonic on* $\mathbb{Z}_+$*,*
- *(iii)* $\sum_{n=0}^{\infty} a_n < \infty$*,*
- (iv) $\displaystyle\limsup_{n\to\infty} \frac{a_n}{(a_n - a_{n+1})\, b_n} = \infty.$

Definition 4 Let σ be a reference measure on T.

(1) A *σ-Carleson measure* is a positive measure μ on T such that there exists a positive constant C (which we refer to as a *Carleson constant for μ relative to σ*) such that $\mu(S(v)) \leqslant C\sigma(S(v))$ for all $v \in T$.

(2) Given a class X of non-negative functions on T, a finite measure μ on T is called an $(X,\ \sigma)$-*Carleson measure* if there exists a positive constant C such that for all $f \in X$,

$$\langle f, \mu\rangle \leqslant C\langle f, \sigma\rangle\,. \tag{3.4}$$

If $X = A^p(\sigma)$, in Eq. 3.4 we replace f by $|f|^p$.

Fixing a $\sigma \in \mathcal{R}$, we introduce the following classes of measures on T:

$$\begin{aligned}
\mathcal{M}_\sigma &= \{\mu : \mu \text{ is } \sigma - \text{Carleson}\}\\
\mathcal{M}_{(F_+,\sigma)} &= \{\mu : \mu \text{ is } (F_+,\ \sigma) - \text{Carleson}\}\\
\mathcal{M}_{(S_+,\sigma)} &= \{\mu : \mu \text{ is } (S_+,\ \sigma) - \text{Carleson}\}\\
\mathcal{M}_{(H_+,\sigma)} &= \{\mu : \mu \text{ is } (H_+,\ \sigma) - \text{Carleson}\}\\
\mathcal{M}_{(A^p(\sigma),\sigma)} &= \{\mu : \mu \text{ is } A^p(\sigma) - \text{Carleson}\}
\end{aligned}$$

Our aim is to establish inclusion relations among these classes and characterize the measures σ which, when possible, yield such inclusions.

We now state our main results.

Theorem 3.3 *Fix $\sigma \in \mathcal{R}$.*

(*a*) $\mathcal{M}_{(F_+,\sigma)} \subset \mathcal{M}_{(S_+,\sigma)} \subset \mathcal{M}_{(H_+,\sigma)}$ *and* $\mathcal{M}_{(S_+,\sigma)} \subset \mathcal{M}_{(A^p(\sigma),\sigma)}$.
(*b*) $\mathcal{M}_{(S_+,\sigma)} \subset \mathcal{M}_\sigma$*; if σ is optimal, then* $\mathcal{M}_\sigma \subset \mathcal{M}_{(S_+,\sigma)}$.
(*c*) $\mathcal{M}_{(F_+,\sigma)} \subset \mathcal{M}_\sigma$*;* $\mathcal{M}_\sigma \subset \mathcal{M}_{(F_+,\sigma)}$ *if and only if σ is optimal.*
(*d*) $\mathcal{M}_{(H_+,\sigma)} \not\subset \mathcal{M}_\sigma$*;* $\mathcal{M}_\sigma \subset \mathcal{M}_{(H_+,\sigma)}$ *if and only if σ is good.*

Concerning Theorem 3.3(b), we expect a stronger result to hold, namely, that optimality is not a necessary condition. In view of (d), the goodness of σ is necessary to ensure $\mathcal{M}_\sigma \subset \mathcal{M}_{(S_+,\sigma)}$. This leads to the following conjecture.

Conjecture $\mathcal{M}_\sigma = \mathcal{M}_{(S_+,\sigma)}$ *for every good measure σ.*

We are able to prove the conjecture for the subclass of measures μ in the above classes which are radial. Note that goodness of σ is not needed to prove that this subclass is contained in $\mathcal{M}_{(S_+,\sigma)}$.

Theorem 3.4 *Let $\sigma \in \mathcal{R}$. Then for a radial measure μ, $\mu \in \mathcal{M}_\sigma$ if and only if $\mu \in \mathcal{M}_{(S_+,\sigma)}$.*

If we restrict attention to the opposite case of σ-Carleson measures μ supported on a geodesic ray ω, and subharmonic functions supported on the same ray, we are able to provide another partial result in support of the above conjecture in the special case where the tails of the reference measure σ decay slowly enough.

Theorem 3.5 *Suppose that, for some constant B and every $v \neq o$, $|v| = n$, the reference measure σ satisfies the inequality $\tau_n := \sigma(S(v)) > B\sigma(S(v_-)) = B\tau_{n-1}$, let μ be a σ-Carleson measure supported on a ray ω, and f a non-negative subharmonic function supported on ω. Then*

$$\langle f, \sigma\rangle \geqslant \frac{B}{C}\langle f, \mu\rangle\,,$$

where C is the Carleson constant for μ (relative to σ).

For each $v \neq o$, let f_v denote the harmonic function with limiting values 1 on $I(v)$ and 0 on $\Omega \setminus I(v)$. In the spirit of Theorem 3.3, we have the following result for the Bergman space.

Theorem 3.6 *Let σ be a reference measure and $1 < p < \infty$.*

(*i*) *If for all $v \neq o$,*

$$\sum_{T \setminus S(v)} |f_v(w)|^p \sigma(w) \leqslant C_{\sigma,p}\, \sigma(S(v)), \tag{3.5}$$

then $\mathcal{M}_{(A^p(\sigma),\sigma)} \subset \mathcal{M}_\sigma$.

(*ii*) *Let $\varepsilon > 0$ such that $1/q^p < \varepsilon < 1/q$, and let σ be the reference measure such that $\sigma_n = \varepsilon^n$. Then σ satisfies condition (3.5). Consequently $\mathcal{M}_{(A^p(\sigma),\sigma)} \subset \mathcal{M}_\sigma$.*

We end the section with some remarks on the natural discretization of the Carleson condition in terms of the pseudo-hyperbolic balls of Oleinik and Luecking, which lead us to instead adopt the conventional Carleson-box approach in our setting. The sets we shall consider here are truncated sectors defined in terms of a fast growing function on the non-negative integers.

Definition 5 A function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is called *admissible* if $f(n) > n$ for each n and in addition

$$\lim_{n \to \infty} (f(n) - n) = \infty.$$

Given an admissible function f, for $v \in T$, $n \in \mathbb{Z}_+$, let

$$S_{v,n}(f) = \{w \in S(v) : n \leqslant |w| < f(n)\}.$$

A nonnegative measure μ is called *$f\sigma$-trapezoidal* if there exists a positive constant C such that for all $v \in T$ and all $n \in \mathbb{N}$,

$$\mu(S_{v,n}(f)) \leqslant C\, \sigma(S_{v,n}(f)). \tag{3.6}$$

The result below highlights the non-equivalence of the Carleson condition in terms of a family of truncated sectors $S_{v,n}(f)$ and our choice of Carleson condition in terms of the family of sectors $S(v)$.

Theorem 3.7 (*i*) *If σ is a reference measure on T and f an admissible function, then any $f\sigma$-trapezoidal measure is a σ-Carleson measure.*

(*ii*) *Given any reference measure σ, there exists an admissible function f such that every σ-Carleson measure is $f\sigma$-trapezoidal.*

(*iii*) *Given any admissible function f, there exist a reference measure σ and a non-negative measure μ on T such that μ is a σ-Carleson measure but not $f\sigma$-trapezoidal.*

Proof (i) Fix a vertex v and define the sequence $\{a_n\}_{n=0}^\infty$ by $a_0 = |v|$ and for $n \geqslant 0$, let $a_{n+1} = f(a_n)$. Then the sector $S(v)$ is the disjoint union of the truncated sectors $S_{v,a_n}(f)$ for $n \geqslant 0$. Suppose μ is $f\sigma$-Carleson, so that there is a positive constant C

such that (3.6) holds. Then

$$\mu(S(v)) = \sum_{n=0}^{\infty} \mu(S_{v,n}(f)) \leqslant C \sum_{n=0}^{\infty} \sigma(S_{v,n}(f)) = C\mu(S(v)),$$

proving that μ is a σ-Carleson measure.

(ii) Let σ be a reference measure. Due to the radiality of σ, for each vertex v we may choose $f(|v|) \in \mathbb{Z}_+$ large enough so that $f(|v|) - |v| \to \infty$ as $|v| \to \infty$ and $\sigma\big(S(v)) \cap \{w \in T : |v| \leqslant |w| < f(|v|)\}\big) > \frac{1}{2}\sigma(S(v))$. For $n \in \mathbb{Z}_+$, $S_{v,n}(f)$ is the disjoint union of the sets $S_{w,n}(f)$ over all $w \in S(v)$ with $|w| = n$. Therefore,

$$\sigma(S_{v,n}(f)) = \sum_{\substack{w\in S(v)\\ |w|=n}} \sigma(S_{w,n}(f)) > \frac{1}{2} \sum_{\substack{w\in S(v)\\ |w|=n}} \sigma(S(w)) = \frac{1}{2}\sigma\Big(\bigcup_{\substack{w\in S(v)\\ |w|=n}} S(w)\Big).$$

Hence, if μ is a σ-Carleson measure, then

$$\begin{aligned} \mu(S_{v,n}(f)) &= \sum_{\substack{w\in S(v)\\ |w|=n}} \mu(S_{w,n}(f)) \leqslant \sum_{\substack{w\in S(v)\\ |w|=n}} \mu(S(w)) \leqslant \sum_{\substack{w\in S(v)\\ |w|=n}} C\,\sigma(S(w)) \\ &< 2C \sum_{\substack{w\in S(v)\\ |w|=n}} \sigma(S_{w,n}(f)) = 2C\,\sigma(S_{v,n}(f)). \end{aligned}$$

(iii) Given an admissible function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, define $a_1 = f(1)$ and for each $n \geqslant 1$, let $a_{n+1} = f(a_n)$. Moreover, let $B_n = \{v \in T : a_n \leqslant |v| < a_{n+1}\}$. Then the cardinality of B_n is given by

$$A_n = \frac{q+1}{q} \sum_{k=a_n}^{a_{n+1}-1} q^k \approx q^{a_{n+1}}.$$

Now define $\sigma(v) = 1/(n^3 A_n)$ for $a_n \leqslant |v| < a_{n+1}$, so that $\sigma(B_n) = n^{-3}$ and $\sigma(S(w) \cap B_n) = \frac{q}{q+1} q^{-|w|} n^{-3}$ for $|w| = a_n$. Thus, for $|w| = a_n$, the quantity

$$\beta_n := \sigma(S(w)) = q^{-|w|} \sum_{k=n}^{\infty} \frac{1}{k^3}$$

is approximately $q^{-|w|}n^{-2}$. Now pick a ray $\{x_0, x_1, x_2, \dots\}$ and let

$$\mu(x_t) = \begin{cases} \beta_n - \beta_{n+1} & \text{if } t = a_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every v, $\mu(S_v) = 0$ if $x_t \neq v$ for all t (that is, if the ray $\{x_0, x_1, x_2, \dots\}$ does not definitely belong to $S(v)$), but, if $v = x_t$ for $a_{m-1} < t \leqslant a_m$, then $\mu(S_v) = \beta_m = \sigma(S_{x_{a_m}}) \leqslant \sigma(S(v))$. Therefore $\mu(S(v)) \leqslant \sigma(S(v))$. On the other hand, $S_{v,a_n}(f) = S(v) \cap B_n$, hence $\sigma(S_{v,a_n}(f)) \sim q^{-|v|}n^{-3}$. But, if we choose $x_{a_n} = v$, then

$$\mu(S_{v,a_n}(f)) = \beta_n - \beta_{n+1} = q^{-a_n} \sum_{k=n}^{\infty} \frac{1}{k^3} - q^{-a_{n+1}} \sum_{k=n+1}^{\infty} \frac{1}{k^3} > \frac{q^{-|v|}}{n^2},$$

hence $\mu(S_{v,a_n}(f))/\sigma(S_{v,a_n}(f))$ is unbounded.

$\square$

4 Radialization and Harmonic Radialization

Given a function f on T and $v \in T$ with $v \neq o$, we describe two ways to produce a new function on T which agrees with f on $T \setminus S(v)$ and which is radial on $S(v)$.

Definition 6 For $v \in T$ we introduce an operator R_v on functions on T, the *local radialization operator* on the sector $S(v)$ as follows:

$$R_o f(w) = \frac{1}{(q+1)q^{|w|-1}} \sum_{|u|=|w|} f(u),$$

and for each $v \in T$, $v \neq o$,

$$R_v f(w) = \begin{cases} \dfrac{1}{q^{|w|-|v|}} \displaystyle\sum_{u \in S(v), |u|=|w|} f(u) & \text{for } w \in S(v), \\ f(w) & \text{for } w \in T \setminus S(v). \end{cases}$$

For $n \geqslant 1$, the *n-th generation circular radialization operator* R_n is defined on functions f on T by

$$R_n f(w) = \begin{cases} R_v f(w) & \text{if } |w| \geqslant n,\, w \in S_v \text{ for } |v| = n, \\ f(w) & \text{if } |w| < n. \end{cases}$$

By the same method one defines a *sectorial radialization* for functions on the boundary: If $F : \Omega \to \mathbb{C}$, define $R_v F(\omega) = (1/\nu(I(v))) \int_{I(v)} F \, d\nu$ for $\omega \in I(v)$, and $R_v F(\omega) = F(\omega)$ for $\omega \notin I(v)$. The n-th generation boundary radialization is defined analogously.

The following lemma will motivate our definition of harmonic radialization given below, which is a second way of producing radial functions.

Lemma 4.1 *Let $v \in T$ with $v \neq o$. Let $g : \{v^-\} \cup S(v) \to \mathbb{R}$ be a radial function which in addition is harmonic on $S(v)$. Let $A = g(v^-)$, $B = g(v)$. Then for any $u \in \{v^-\} \cup S(v)$ and $j = d(u, v)$,*

$$g(u) = \frac{Bq - A}{q-1} + \frac{A - B}{q-1} q^{-j} = L + (A - L)q^{-(j+1)}, \tag{4.1}$$

where the radial limit of g in $S(v)$ is $L = \frac{Bq-A}{q-1}$. Conversely, for any real numbers A and B, the formulas in Eq. 4.1 *define a function on $\{v^-\} \cup S(v)$ which is radial and harmonic at each vertex of $S(v)$.*

Proof Let $x_{-1} = g(v^-)$ and for each nonnegative integer j, let $x_j = g(u)$, where $u \in S(v)$ and $d(u, v) = j$. Then the harmonicity and radiality of g imply $x_j = \frac{q}{q+1} x_{j+1} + \frac{1}{q+1} x_{j-1}$, for all $j \geqslant 0$. The solution of this recurrence relation is $x_j = c_1 + c_2 q^{-j}$, for some real numbers c_1, c_2. We also have $x_{-1} = A$ and $x_0 = B$, from which we deduce $c_1 = \frac{Bq-A}{q-1}$ and $c_2 = \frac{A-B}{q-1}$. Since $c_1 = \lim_{j \to \infty} x_j$, we have $L = \frac{Bq-A}{q-1}$. From here, a straightforward calculation yields (4.1). □

Remark 1 In Lemma 4.1, if we know any two of A, B or L, we know the other and so any two of them determine g uniquely on $\{v_-\} \cup S(v)$. Moreover, if $L = 0$, then $g(v^-) = q \cdot g(v)$ and $g(u) = g(v) q^{-d(u,v)}$.

Definition 7 For $v \in T$ with $v \neq o$, the *local harmonic radialization operator* RH_v is defined on functions f on T as follows:

$$RH_v f(w) = \begin{cases} \dfrac{f(v)\,q - f(v^-)}{q-1} + \dfrac{f(v^-) - f(v)}{q-1}\, q^{-d(w,v)} & \text{if } w \in S(v), \\ f(w) & \text{if } w \in T \setminus S(v). \end{cases} \tag{4.2}$$

It follows easily from Lemma 4.1 that for a fixed $v \in T \backslash \{o\}$, if f is harmonic on T, then $RH_v f$ is harmonic on T, and if f is subharmonic on T, then $RH_v f$ is subharmonic on T. Of course, it is always the case that $RH_v f$ is harmonic and radial on $S(v)$.

The following proposition allows us to answer analogous questions for $R_v f$.

Proposition 4.2 *Let f be any real-valued function on T. Fix $v \in T$ with $|v| = n \geqslant 1$. Let $u \in S(v)$, and let $m = |u|$. Then*

$$q^{m-n} \triangle (R_v f)(u) = \sum_{\tilde{u} \in S(v),\, |\tilde{u}|=m} \triangle f(\tilde{u}).$$

If f is harmonic (respectively, subharmonic) on T, then $R_v f$ is harmonic (respectively, subharmonic) on T.

Proof For every $u \neq o$ there are exactly q vertices w such that $w^- = u$. Thus, for each $k \geqslant 0$, let $x_k = R_v f(w)$, for any $w \in S(v)$ with $|w| = n + k$. Then

$$\begin{aligned} \sum_{\substack{\tilde{u} \in S(v) \\ |\tilde{u}|=|u|}} \triangle f(\tilde{u}) &= \sum_{\substack{\tilde{u} \in S(v) \\ |\tilde{u}|=|u|}} \left[\frac{1}{q+1} \sum_{w^- = \tilde{u}} f(w) + \frac{f(\tilde{u}^-)}{q+1} - f(\tilde{u}) \right] \\ &= \frac{1}{q+1} \sum_{\substack{w \in S(v) \\ |w|=|u|+1}} f(w) + \frac{q}{q+1} \sum_{\substack{w \in S(v) \\ |w|=|u|-1}} f(w) - \sum_{\substack{\tilde{u} \in S(v) \\ |\tilde{u}|=|u|}} f(\tilde{u}) \\ &= \frac{q^{k+1}}{q+1}\, x_{k+1} + \frac{q}{q+1}\, q^{k-1} x_{k-1} - q^k x_k \\ &= q^k \left[\frac{q}{q+1} x_{k+1} + \frac{1}{q+1}\, x_{k-1} - x_k \right] = q^k \,\triangle R_v f(u). \end{aligned}$$

The assertions about harmonicity and subharmonicity on $S(v)$ now follow. Those assertions on $T \setminus S(v)$ are obvious. □

Remark 2 Let f be defined on T and let $v \in T$ with $v \neq o$. It is not generally true that $R_v f$ and $RH_v f$ are equal; $RH_v f$ is necessarily harmonic on $S(v)$ and $R_v f$ need not be harmonic.

For example, the function $f(v) = |v|$ is radial, subharmonic on T, but not harmonic anywhere, so $R_v f \equiv f$ on T, and $R_v f \neq RH_v f$.

On the other hand, if f is harmonic on $S(v)$, then $R_v f = RH_v f$, since both sides are radial on $S(v)$ and they agree at v and v^-.

5 Some Results on the Poisson Transform and the Bergman Space on a Homogeneous Tree

For a harmonic function f on T, let us define $\mu_{n,p}(f)$ to be the average value of $|f|^p$ on the circle C_n of radius n centered at o. Thus $\mu_{0,p}(f) = |f(o)|^p$ and for $n \geqslant 1$,

$$\mu_{n,p}(f) = \frac{1}{(q+1)q^{n-1}} \sum_{|v|=n} |f(v)|^p,$$

Observe that $\|f\|_{L^p}^p$ is essentially equal to $\sum_n \mu_{n,p}(f) q^n \sigma_n$. Thus,

$$f \in A^p(\sigma) \text{ if and only if } \sum_n \mu_{n,p}(f) q^n \sigma_n < \infty.$$

For $p \geqslant 1$, the harmonic Hardy space $\mathcal{H}^p(T)$ is defined as the space of harmonic functions f on T such that $\{\mu_{n,p}(f)\} \in \ell^\infty$, with norm

$$\|f\|_{\mathcal{H}^p}^p := \sup_{n \in \mathbb{N}} \mu_{n,p}(f).$$

Equivalently, $f \in \mathcal{H}^p(T)$ if and only f is harmonic on T and f has a harmonic majorant on T. See [7], Theorem 2.3. Note that for all $f \in \mathcal{H}^p(T)$,

$$\|f\|_{A^p}^p = \sum_{v \in T} |f(v)|^p \sigma(v) = \sigma(o)\mu_{0,p}(f) + \sum_{n=1}^{\infty} \mu_{n,p}(f)(q+1)q^{n-1}\sigma_n \leqslant \|f\|_{\mathcal{H}^p}^p \|\sigma\|.$$

Thus, the inclusion operator of $\mathcal{H}^p(T)$ into $A^p(\sigma)$ is bounded.

Lemma 5.1 *Let $v \in T$, $|v| = n \geqslant 1$. Let $\chi = \chi_{I(v)}$. Then*

$$\mathcal{K}\chi(w) = \begin{cases} 1 - \dfrac{q^{-d(w,v)}}{q+1} & \text{if } w \in S(v), \\ \dfrac{q}{q+1}\, q^{-d(w,v)} & \text{if } w \in T \setminus S(v). \end{cases}$$

Proof It follows from Eq. 2.1 that, for $\omega \in I(v)$, $K(v, \omega) = K(\omega_n, \omega) = q^{|v|}$. Therefore

$$\mathcal{K}\chi(v) = \int_{I(v)} K(v, \omega)\, d\nu = q^{|v|}\, \nu(I(v)) = \frac{q}{q+1}\,.$$

Let $w \in T \setminus S(v)$. Then for $\omega \in I(v)$, $2|\omega \wedge w| - |w| = |v| - d(w, v)$. Thus

$$\mathcal{K}\chi(w) = \int_{I(v)} K(w, \omega)\, d\nu = q^{|v|-d(w,v)}\, \nu(I(v)) = \frac{q}{q+1}\, q^{-d(w,v)},$$

in agreement with the formula in the statement. In particular, $K\chi(v^-) = 1/(q+1)$. By symmetry, $\mathcal{K}\chi$ is radial on $S(v)$. Using Lemma 4.1 with $A = \mathcal{K}\chi(v^-) = 1/(q+1)$ and $B = \mathcal{K}\chi(v) = q/(q+1)$, we obtain the desired result on $S(v)$. $\square$

Remark 3 As a consequence, we see that $\lim_{m\to\infty} \mathcal{K}\chi(\omega_m) = 1$, as expected.

Note that, $\mathcal{K}\chi$ is clearly radial when restricted to the sector $S(v)$ and also radial *around* v in the complement $T \setminus S(v)$, hence it is radial in each of the other sectors S_w with $|w| = |v|$.

Corollary 5.2 *For every $v \in T$, there is a non-negative harmonic function f such that $f \approx \chi_{I(v)}$, in the sense that there exist $0 < C_1 < C_2 < 1$ independent of v such that $C_2 \leqslant f \leqslant 1$ for $v \in S(v)$, and $0 \leqslant f \leqslant C_1$ for $v \notin S(v)$. Moreover, there exists a non-negative subharmonic function g supported in $S(v)$ such that $g \approx \chi_{I(v)}$.*

Proof From Lemma 5.1 we see that an instance of f is $\mathcal{K}\chi_{I(v)}$. For g we make use of Lemma 4.1. Define g to be 0 on $T \setminus S(v)$. We wish to apply Lemma 4.1 with $A = 0$ and $L = 1$. Since $L = (Bq - A)/(q - 1)$, we get $B = (q - 1)/q$. Then define g on $S(v)$ by $g(u) = \frac{Bq-A}{q-1} + \frac{A-B}{q-1}q^{-d(u,v)}$. By that lemma, g is harmonic on $S(v)$, and it is easy to see that it is subharmonic on T. □

Proposition 5.3 *The Poisson transform intertwines the radializations on T and Ω: for every integrable F on Ω and $v \in T$, one has $\mathcal{K}R_v F = R_v\mathcal{K}F$.*

Proof Note that R_v has different meanings in the above equation; on the left side it operates on a boundary function and on the right side on a tree function. By linearity, it is enough for us to consider separately the cases that F has no support on $I(v)$ and F has all of its support on $I(v)$. So suppose first that F has all its support in $\Omega \setminus I(v)$. Then by definition, $R_v F = 0 = F$ on $I(v)$, so $R_v F = F$ on Ω. Thus $\mathcal{K}R_v F = \mathcal{K}F$. Let us now calculate $R_v\mathcal{K}F$. Since F has no support in $I(v)$, it follows that $\mathcal{K}F$ is radial on $S(v)$, so $R_v\mathcal{K}F = \mathcal{K}F$ on $S(v)$, and so by definition on T. This completes the proof in case F has all its support in $\Omega \setminus I(v)$.

Suppose now that F has all its support in $I(v)$. Let F_a be the number given by

$$F_a = \frac{1}{\nu(I(v))}\int_{I(v)} F(\omega)d\nu(\omega).$$

Then $R_v F = F_a\chi_{I(v)} = F_a\chi$ (notation as in Lemma 5.1) so $\mathcal{K}R_v F = F_a\mathcal{K}\chi$, where the formula for $\mathcal{K}\chi$ is given in Lemma 5.1. Now let us calculate $\mathcal{K}F(w)$ for $w \in (T\setminus S(v))\cup\{v\}$. For such w, we have $2|w\wedge\omega| - |w| = |v| - d(w, v)$, so $\mathcal{K}F(w) = q^{|v|-d(w,u)}\int F(\omega)d\nu(\omega)$. Multiplying top and bottom by $\nu(I(v))$, we get $\mathcal{K}F(w) = \frac{q}{q+1}F_a q^{-d(w,v)}$. By Lemma 5.1, this agrees with $F_a\mathcal{K}\chi(w)$ on $(T\setminus S(v))\cup\{v\}$. Now $\mathcal{K}F$ is harmonic, so $R_v\mathcal{K}F = RH_v\mathcal{K}F$ (recall Definition 7 and Remark 2), and the latter on $S(v)$ is, according to Lemma 4.1, determined by the values of $\mathcal{K}F$ at v and v^-. It follows that $R_v\mathcal{K}F = F_a\mathcal{K}\chi$ on $S(v)$, hence on T. Thus $\mathcal{K}R_v F = F_a\mathcal{K}\chi = R_v\mathcal{K}F$. □

Our next goal is to show that $\mathcal{H}^p(T)$ is not closed in $A^p(\sigma)$ for certain reference measures σ. For this purpose we first prove the following lemma. Fix a reference measure σ. Fix $\varepsilon \in (0, 1)$, $\omega^0 \in \Omega$. Define $F_\varepsilon : \Omega \to \mathbb{R}$ by $\omega \mapsto q^{\varepsilon|\omega\wedge\omega^0|}$, where $|\omega \wedge \omega^0| = \max\{k \in \mathbb{N}_0 : \omega_j = \omega^0_j$ for each $j = 0, \ldots, k\}$, and $f_\varepsilon = \mathcal{K}F_\varepsilon$.

Lemma 5.4 (*i*) *For h nonnegative harmonic on T, $\sum_{v\in T} h(v)\sigma(v) \leqslant \|\sigma\| \cdot \|\nu_h\|$, where ν_h is the representing measure of h.*
(*ii*) *$\mathcal{H}^p(T) \subseteq A^p(\sigma)$, for $p \geqslant 1$.*
(*iii*) *$c_1 \leqslant q^{-\varepsilon k}f_\varepsilon \leqslant c_2$ on $S(\omega^0_k) \setminus S(\omega^0_{k+1})$, for constants c_1, c_2 depending only on ε and q.*
(iv) *$f_\varepsilon \in \mathcal{H}^p(T)$ if and only if $\varepsilon p < 1$. If σ is good and $\varepsilon p \leqslant 1$, then $f_\varepsilon \in A^p(\sigma)$.*

Proof (i): By symmetry, since σ is radial, $\sum_{v\in T} K(v,\omega)\sigma(v)$ is independent of $\omega \in \Omega$. Integrating out ω with respect to ν gives

$$\begin{aligned}\sum_v K(v,\omega)\sigma(v) &= \int_\Omega \sum_v K(v,\omega)\sigma(v)d\nu(\omega) = \sum_v \int_\Omega K(v,\omega)d\nu(\omega)\sigma(v)\\ &= \sum_v \sigma(v) = \|\sigma\|.\end{aligned}$$

Thus, if $h \geqslant 0$ is subharmonic on T, then

$$\begin{aligned}\sum_v h(v)\sigma(v) &= \sum_v \int_\Omega K(v,\omega)d\nu_h(\omega)\sigma(v) = \int_\Omega \sum_v K(v,\omega)\sigma(v)d\nu_h(\omega)\\ &= \int_\Omega \|\sigma\| d\nu_h(\omega) = \|\sigma\| \cdot \|\nu_h\|.\end{aligned}$$

(ii): Let $g \in \mathcal{H}^p(T)$. Then $|g|^p$ has a harmonic majorant h on T. Thus

$$\sum_v |g(v)|^p\sigma(v) \leqslant \sum_v h(v)\sigma(v) = \|\nu_h\|\|\sigma\| < \infty.$$

(iii): We first calculate $f_\varepsilon(\omega_k^0)$:

$$\begin{aligned}f_\varepsilon(\omega_k^0) &= \int_\Omega q^{2|\omega\wedge\omega_k^0|-k} q^{\varepsilon|\omega\wedge\omega^0|} d\nu(\omega)\\ &= q^{-k}\frac{q}{q+1} + \sum_{j=1}^{k-1} q^{2j-k}q^{\varepsilon j}\left(\frac{q-1}{q+1}\right)q^{-j} + \sum_{j=k}^{\infty} q^{2k-k}q^{\varepsilon j}\left(\frac{q-1}{q+1}\right)q^{-j}\\ &= q^{-k}\frac{q}{q+1} + \left(\frac{q-1}{q+1}\right)q^{-k}\sum_{j=1}^{k} q^{(1+\varepsilon)j} + q^k\left(\frac{q-1}{q+1}\right)\sum_{j=k}^{\infty} q^{-(1-\varepsilon)j}\\ &= \frac{q}{q+1}q^{-k} + \left(\frac{q-1}{q+1}\right)q^{-k}\left[\frac{q^{(1+\varepsilon)k}-q^{1+\varepsilon}}{q^{1+\varepsilon}-1}\right] + q^k\left(\frac{q-1}{q+1}\right)\left[\frac{q^{-(1-\varepsilon)k}}{1-q^{-(1-\varepsilon)}}\right]\\ &= q^{\varepsilon k}\left(\frac{q-1}{q+1}\right)\left[\frac{1}{q^{1+\varepsilon}-1} + \frac{1}{1-q^{-(1-\varepsilon)}}\right] + \frac{q^{-k}}{q+1}\left[q - \frac{(q-1)q^{1+\varepsilon}}{q^{1+\varepsilon}-1}\right] \qquad (5.1)\end{aligned}$$

By the Fatou Theorem (see [5], Theorem 3.3), $\lim_{n\to\infty} f_\varepsilon(\omega_n) = F_\varepsilon(\omega)$ for ν−a.e. $\omega \in \Omega$. Since f_ε is radial on $S(\omega_k^0) \setminus S\left(\omega_{k+1}^0\right)$, it follows that $\lim_{|u|\to\infty} f_\varepsilon(u) = q^{\varepsilon k}$, $k \geqslant 1$, where the limit is taken in $S(\omega_k^0) \setminus S\left(\omega_{k+1}^0\right)$. Let $L_k = q^{\varepsilon k}$ and $A_k = f_\varepsilon(\omega_k^0)$. Then by Lemma 4.1, $f_\varepsilon(u) = L_k + (A_k - L_k)q^{-d(u,\omega_k^0)}$ for $u \in S(\omega_k^0) \setminus S\left(\omega_{k+1}^0\right)$. The result then follows by applying the above formula (5.1) for $f_\varepsilon(\omega_k^0)$.

(iv): To prove the result concerning $\mathcal{H}^p(T)$, it is enough to prove that $F_\varepsilon \in L^p(\Omega, \nu)$ if and only if $\varepsilon p < 1$. We have

$$\int_\Omega q^{\varepsilon|\omega\wedge\omega^0|p} d\nu(\omega) = \sum_{k=0}^{\infty} \int_{I(\omega_k^0)\setminus I(\omega_{k+1}^0)} q^{\varepsilon|\omega\wedge\omega^0|p} d\nu(\omega)$$
$$= \frac{q}{q+1} + \sum_{k=1}^{\infty} q^{\varepsilon kp} \left(\frac{q-1}{q+1}\right) q^{-k} = \frac{q}{q+1} + \left(\frac{q-1}{q+1}\right) \sum_{k=1}^{\infty} q^{-(1-\varepsilon p)k}$$

and this is finite if and only if $\varepsilon p < 1$.

Suppose now that $\varepsilon p \leqslant 1$. By part (iii), to show $\sum_{u\in T} f_\varepsilon^p(u)\sigma(u) < \infty$, we may replace f_ε on each $S(\omega_k^0) \setminus S(\omega_{k+1}^0)$ by $q^{\varepsilon k}$. This gives

$$\sum_{k=0}^{\infty} q^{\varepsilon kp} \sigma\left(S(\omega_k^0) \setminus S\left(\omega_{k+1}^0\right)\right) \leqslant \sum_{k=0}^{\infty} q^k \sigma(S(\omega_k^0) = \sum_{k=0}^{\infty} q^k \tau_k < \infty,$$

since σ is good. ☐

We are now ready to prove the following theorem.

Theorem 5.5 *Let σ be a good reference measure on T. Then $\mathcal{H}^p(T)$ is not closed in $A^p(\sigma)$.*

Proof First assume $p > 1$. Let $\varepsilon = 1/p$, ω^0, $f = f_\varepsilon$ and $F = F_\varepsilon$ as defined just before Lemma 5.4, With notation as in Eq. 4.2, for each $k \geqslant 1$, let $f_k = RH_{\omega_k^0} F$, $A_k = f(\omega_{k-1}^0)$ and $B_k = f(\omega_k^0)$. Then

$$f_k(u) = \begin{cases} f(u) & \text{if } u \notin S(\omega_k^0), \\ \dfrac{B_k q - A_k}{q-1} + \dfrac{A_k - B_k}{q-1} q^{-d(u,\omega_k^0)} & \text{if } u \in S(\omega_k^0). \end{cases}$$

For the integral estimates we wish to obtain for f_k, by Lemma 5.4(iii) we may replace f_k on $S(\omega_k^0)$ with the constant $q^{\varepsilon k}$.

Each f_k is bounded, hence in $\mathcal{H}^p(T)$. The sequence converges pointwise to f, but $f \notin \mathcal{H}^p(T)$ by Lemma 5.4(iv), so the convergence is not in $\mathcal{H}^p(T)$. Thus we will be done with this case if we show that f_k converges to f in $A^p(\sigma)$. But $f_k = f$ outside of $S(\omega_k^0)$, so we just need to check that $\sum_{S(\omega_k^0)} |f_k(u) - f(u)|^p \sigma(u) \to 0$ as $k \to \infty$, and this will hold provided $\sum_{S(\omega_k^0)} |f_k(u)|^p \sigma(u) \to 0$ and $\sum_{S(\omega_k^0)} |f(u)|^p \sigma(u) \to 0$ as $k \to \infty$. The first of these follows from $q^{\varepsilon kp}\ \sigma\left(S(\omega_k^0)\right) = q^k \tau_k \to 0$ as $k \to \infty$, and the second by applying the Dominated Convergence Theorem to the sequence $\chi_{S(\omega_k^0)} \cdot f^p$. This completes the proof in case $p > 1$.

For $p = 1$, instead of f we use $K(\cdot, \omega^0)$ for a fixed $\omega^0 \in \Omega$. This is in $A^1(\sigma)$ (by Lemma 5.4(i)) but not in $\mathcal{H}^1(T)$ (since $\delta_{\omega^0} \notin L^1(\nu)$). The proof is carried out as before by harmonically radializing $K(\cdot, \omega_k^0)$. ☐

6 Proofs of the Results in Section 3

Recall that σ is a reference measure, $\sigma_n = \sigma(\{v\})$ where $|v| = n$, and $\tau_n = \sigma(S(v))$. We begin with the following lemma.

Lemma 6.1 *Choose a geodesic ray $\omega = [\omega_0 = o, \omega_1, \dots, \omega_n, \dots]$. Denote by μ the function supported on this geodesic ray defined by $\mu_{\omega_n} = \tau_n - \tau_{n+1}$. Then μ is a σ-Carleson measure.*

Proof If $v \notin \omega$ then $\mu(S(v)) = 0$. On the other hand,

$$\mu(S(\omega_n)) = \sum_{k=n}^{\infty} \mu(\omega_k) = \sum_{k=n}^{\infty} (\tau_k - \tau_{k+1}) = \tau_n = \sigma(S(\omega_n)) .$$

So μ is σ-Carleson. $\square$

In the following proof, C denotes a number depending at most on parameters of T, but possibly varying in different instances (even in the same string of inequalities).

Proof of Theorem 3.3 The first string of inequalities of (a) are evidently true, and the last follows since $|f|^p$ is subharmonic whenever f is harmonic.

To prove (3.3), suppose first that μ is an (S_+, σ)-Carleson measure. Let $v \neq o$ and let g be the non-negative subharmonic function of Corollary 5.2: $g \approx \chi_{S(v)}$. Then $\mu(S(v)) \approx \langle \mu, g \rangle \leqslant C \langle \sigma, g \rangle \approx \sigma(S(v))$. So $\mu(S(v)) \leqslant C\sigma(S(v))$ for some constant C. Thus $\mathcal{M}_{(S_+,\sigma)} \subset \mathcal{M}_\sigma$. The last part will follow from Eq. 3.3 once we prove (3.3).

We now turn to Eq. 3.3. Since the inclusion $\mathcal{M}_\sigma \supset \mathcal{M}_{(F_+,\sigma)}$ holds for all $\sigma \in \mathcal{R}$ by parts (3.3) and (3.3), to prove (3.3) it suffices to prove the opposite inclusion when σ is optimal and, conversely, that such an inclusion implies the optimality of σ.

Suppose σ is optimal and μ is a σ-Carleson measure. Then for all $v \in T$,

$$\mu(v) \leqslant \mu(S(v)) \leqslant C\sigma(S(v)) \leqslant C\sigma_{|v|}.$$

Thus, if $f \in F_+$, then $\sum_{v\in T} f(v)\mu(v) \leqslant C \sum_{v \in T} f(v)\sigma_{|v|}$. This shows that μ is (F_+, σ)-Carleson (and hence also (S_+, σ)-Carleson).

Conversely, assume $\mathcal{M}_\sigma \subset \mathcal{M}_{(F_+,\sigma)}$ and let μ be the measure supported on a ray ω introduced in Lemma 6.1. We have proved there that μ is σ-Carleson, hence, by the hypothesis, it is (F_+, σ)-Carleson. Let $g_n = \delta_{v_n}$. Then

$$\frac{< g_n, \mu >}{< g_n, \sigma >} = \frac{\tau_n - \tau_{n+1}}{\sigma_n}$$

is bounded. On the other hand, splitting a sector as the union of its apex and the nearest neighbor subsectors, we see that $\tau_n = \sigma_n + q\tau_{n+1}$. Thus, $\tau_{n+1} = \frac{1}{q}(\tau_n - \sigma_n)$, so

$$\tau_n - \tau_{n+1} = \frac{q-1}{q}\tau_n + \frac{1}{q}\sigma_n .$$

It follows that τ_n/σ_n is bounded, proving that σ is optimal. This completes the proof of (c).

We now prove (3.3). Observe that, by definition of reference measure,

$$\sum_{n=0}^{\infty} q^n \sigma_n < \sigma_0 + \sum_{n=1}^{\infty} \frac{q+1}{q} q^n \sigma_n = \|\sigma\|.$$

Choose an increasing sequence $\{c_n\} \to \infty$ such that $\sum_{n=0}^{\infty} c_n \sigma_n q^n < \infty$. Let $\mu_n = c_n \sigma_n$ and let μ be the radial measure on T defined as $\mu(v) = \mu_{|v|}$. Notice that μ is finite. For each $h \in \mathcal{H}_+$, $h(o)$ is the average of the values of h on the neighbors of o. By iteration, $(q+1)q^{n-1}h(o) = \sum_{|v|=n} h(v)$. Then, for every radial measure ρ on T,

$$\langle h, \rho \rangle = \sum_{n \geqslant 0} \rho_n \sum_{|v|=n} h(v) = h(o) \|\rho\| .$$

Hence, as μ and σ are radial,

$$\langle h, \mu \rangle = h(o) \, \|\mu\| = Ch(o) \, \|\sigma\| = \langle h, \sigma \rangle$$

where $C = \|\mu\| / \|\sigma\|$. Therefore $\mu \in \mathcal{M}_{(H_+, \sigma)}$. But μ is not a σ-Carleson measure. Indeed, since $\{c_n\}$ is increasing,

$$\frac{\mu(S(v))}{\sigma(S(v))} = \frac{\sum_{k=0}^{\infty} \mu_{n+k} q^k}{\sum_{k=0}^{\infty} \sigma_{n+k} q^k} = \frac{\sum_{k=0}^{\infty} c_{n+k} \sigma_{n+k} q^k}{\sum_{k=0}^{\infty} \sigma_{n+k} q^k} \geqslant c_n \to \infty$$

as $n \to \infty$. We have thus shown that $\mathcal{M}_{(H_+, \sigma)} \not\subset \mathcal{M}_\sigma$.

Suppose now that σ is not good. Then $\sum_n q^n \tau_n = \infty$, where $\tau_n = \sigma(S(v))$ with $|v| = n$. Fix a ray $\omega \in \Omega$ and define μ to be 0 off ω and $\mu(\omega_n) = \tau_n - \tau_{n+1}$. We have proved in Lemma 6.1 that μ is a σ-Carleson measure. By the way μ is defined,

$$\int K_\omega \, d\mu = \sum_{n=0}^{\infty} q^n \mu(\omega_n) = \sum_{n=0}^{\infty} q^n (\tau_n - \tau_{n+1}) . \tag{6.1}$$

Fix $m \in \mathbb{N}$. Then,

$$\begin{aligned} \sum_{n=0}^{m} q^n (\tau_n - \tau_{n+1}) &= \sum_{n=0}^{m} q^n \tau_n - \sum_{n=1}^{m+1} q^{n-1} \tau_n \\ &= \tau_0 + \sum_{n=1}^{m} (q^n - q^{n-1}) \tau_n - q^m \tau_{m+1} \\ &= \tau_0 - q^m \tau_{m+1} + \left(1 - \frac{1}{q}\right) \sum_{n=1}^{m} q^n \tau_n . \end{aligned} \tag{6.2}$$

Denote by B_m the ball of radius m in T centered at o and observe that

$$q^m \tau_{m+1} = \frac{1}{q+1} \, \sigma(T \backslash B_m).$$

Since σ is a finite measure, letting $m \to \infty$, we see that $q^m \tau_{m+1} \to 0$. But the third term on the right side in Eq. 6.2 tends to ∞, because σ is not good, hence the same happens to the left hand sides of Eqs. 6.2 and 6.1. It follows that μ is not an (H_+, σ)-Carleson measure.

Conversely, suppose that σ is good and set $A = \sum_n q^n \tau_n$. Let μ be a σ-Carleson measure. Fix ω on Ω, and for each $n \in \mathbb{N}$, let $W_n = S(\omega_n) \backslash S(\omega_{n+1})$. Then

$$\begin{aligned} \mu(K_\omega) &= \int K_\omega(v) \, d\mu(v) = \sum_n \int_{W_n} K_\omega(v) \, d\mu(v) \\ &\leqslant \sum_n \int_{W_n} q^n \, d\mu(v) \leqslant \sum_n \int_{S_{v_n}} q^n \, d\mu(v) \leqslant \sum_n q^n \tau_n = A . \end{aligned}$$

Since $\mu(K_\omega)$ is bounded and $\sigma(K_\omega)$ is independent of ω, it follows that $\mu(K_\omega)/\sigma(K_\omega)$ is bounded as well. This implies that μ is an (H_+, σ)-Carleson measure and proves the desired inclusion. $\square$

We now turn to the proofs of Theorems 34 and 3.5 which provide some support for our conjecture. We begin with the following lemma.

Lemma 6.2 *Let f be a nonnegative subharmonic function on T. Define the circular sums $F_n := \sum_{|v|=n} f(v)$, $n \geqslant 0$. Let $A_0 := f(0)$ and for $n \geqslant 1$, let $A_n := F_n/((q+1)q^{n-1})$ denote the circular averages of f. Then both F_n and A_n are increasing.*

Proof Let $n = |v|$, denote by v^+ any of the offspring of v, that is its forward neighbors $v^+ > v$, $|v^+| = n+1$. Let us take the sum over all vertices v in the circle S_n of radius $n > 0$, that is at distance n from o, of the subharmonicity condition

$$f(v) \leqslant \frac{f(v^-) + \sum_{v^+} f(v^+)}{q+1} .$$

Since each forward neighbor v^+ appears only once in this sum over S_n, but the same vertex v^- is repeated q times, we obtain

$$\sum_{|v|=n} f(v) \leqslant \frac{q \sum_{|u|=n-1} f(u) + \sum_{|w|=n+1} f(w)}{q+1} .$$

That is, the circular sums $F_n := \sum_{|v|=n} f(v)$ satisfy the inequality

$$F_n \leqslant \frac{F_{n+1} + qF_{n-1}}{q+1} \tag{6.3}$$

for $n > 0$, and $F_0 = f(o) \leqslant F_1/(q+1)$. Note that the last inequality implies $F_1 - F_0 \geqslant qF_0$. Let us rewrite (6.3) as

$$F_{n+1} \geqslant (q+1)F_n - qF_{n-1} , \tag{6.4}$$

that is,

$$F_{n+1} - F_n \geqslant q(F_n - F_{n-1}) . \tag{6.5}$$

Since $F_1 - F_0 \geqslant qf(o) \geqslant 0$, it follows from Eq. 6.5 that $F_{n+1} - F_n \geqslant 0$ for all n. Therefore $\{F_n\}$ is increasing. By Eq. 6.4 we see that $A_{n+1} \geqslant ((q+1)/q)A_n - (1/q)A_{n-1}$, that is, $A_{n+1} - A_n \geqslant q^{-1}(A_n - A_{n-1})$. Since $A_1 - A_0 = \frac{1}{q+1}\left(\sum_{|v|=1} f(v)\right) - f(o) \geqslant 0$, the same argument shows that the averages A_n are increasing. $\square$

Proof of Theorem 3.4 Let σ be a reference measure, and let μ be a radial σ-Carleson measure. We must show that $\mu \in \mathcal{M}_{(S_+,\sigma)}$. We start by recalling the Abel partial summation formula (see, e.g. [14], [1, Theorem 8.27]): if $a_n, b_n \in \mathbb{R}$ ($n \in \mathbb{N}$) and $B_k = \sum_{n=0}^k b_n$, then

$$\sum_{n=1}^k a_n b_n = a_k B_k - a_0 B_0 - \sum_{n=1}^k (a_n - a_{n-1}) B_{n-1} .$$

For $k > n$, $B_k - B_{n-1} = \sum_{i=n}^{k} b_i$, and the formula becomes

$$\sum_{n=1}^{k} a_n b_n = \Big(a_k - \sum_{n=1}^{k}(a_n - a_{n-1})\Big)B_k - a_0 B_0 + \sum_{n=1}^{k}(a_n - a_{n-1})\sum_{i=n}^{k} b_i$$

$$= a_0(B_k - B_0) + \sum_{n=1}^{k}(a_n - a_{n-1})\sum_{i=n}^{k} b_i\,.$$

Let $B = \lim_k B_k = \sum_{n=0}^{\infty} b_n$ and assume $B < \infty$. Adding $a_0 B_0$ and letting $k \to \infty$ gives

$$\sum_{n=0}^{\infty} a_n b_n = a_0 B + \sum_{n=1}^{\infty}(a_n - a_{n-1})\sum_{i=n}^{\infty} b_i \tag{6.6}$$

$$= \sum_{n=0}^{\infty}(a_n - a_{n-1})\sum_{i=n}^{\infty} b_i\,, \tag{6.7}$$

where we define $a_{-1} = 0$.

Let $v \in T$, $|v| = n \geqslant 1$. Let μ_n and σ_n denote the values of μ and σ on vertices of length n. Since μ is radial, we have $\mu(S(v)) = \sum\limits_{r=0}^{\infty} q^r \mu_{n+r} = \sum\limits_{j=n}^{\infty} q^{j-n}\mu_j$, so $q^n\mu(S(v)) = \sum\limits_{j=n}^{\infty} q^j \mu_j$. Similarly, $q^n\sigma(S(v)) = \sum\limits_{j=n}^{\infty} q^j\sigma_j$. Thus our assumption that μ is σ-Carleson implies that the tails satisfy

$$\sum_{j=n}^{\infty} \mu_j q^j = q^n\mu(S(v)) \leqslant Cq^n\sigma(S(v)) = C\sum_{j=n}^{\infty}\sigma_j q^j\,.$$

Now let $f \geqslant 0$ be a subharmonic function on T, and A_n be its circular averages. By Lemma 6.2, A_n is increasing. Therefore

$$\sum_{n=0}^{\infty}(A_n - A_{n-1})\sum_{j=n}^{\infty}\mu_j q^j \leqslant C\sum_{n=0}^{\infty}(A_n - A_{n-1})\sum_{j=n}^{\infty}\sigma_j q^j\,.$$

But then, the formula of summation by parts (6.6) with $a_n = A_n$ and $b_n = \mu_n q^n$ followed by $b_n = \sigma_n q^n$ implies that

$$\sum_{n=0}^{\infty} A_n\mu_n q^n \leqslant C\sum_{n=0}^{\infty} A_n\sigma_n q^n,$$

which yields $\langle f, \mu\rangle \leqslant C\langle f, \sigma\rangle$. $\square$

Proof of Theorem 3.5 Let $f_k := f(\omega_k)$, and set $f_{-1} = f_{-2} = 0$. Because of subharmonicity, the sequence f_k is increasing (there cannot be local maxima), $f_1 \geqslant (q+1)f_0$, and $(q+1)f_{k-1} \leqslant f_{k-2} + f_k$, that is $f_k - qf_{k-1} \geqslant f_{k-1} - f_{k-2}$ for all $k \geqslant 0$. Therefore

$$\frac{f_n}{q^n} = \sum_{j=0}^{n}\left(\frac{f_j}{q^j} - \frac{f_{j-1}}{q^{j-1}}\right) = \sum_{j=0}^{n}\frac{f_j - qf_{j-1}}{q^j} \geqslant \sum_{j=0}^{n}\frac{f_{j-1} - f_{j-2}}{q^j}.$$

But then

$$\begin{aligned}\langle f,\sigma\rangle &= \sum_{n\geqslant 0} f_n\sigma_n \geqslant \sum_{n=0}^{\infty}\sum_{j=0}^{n}(f_{j-1}-f_{j-2})\sigma_n q^{n-j}\\ &= \sum_{j=1}^{\infty}\sum_{n=j}^{\infty}(f_{j-1}-f_{j-2})\sigma_n q^{n-j} = \sum_{j=1}^{\infty}(f_{j-1}-f_{j-2})\tau_j\\ &\geqslant B\sum_{j=1}^{\infty}(f_{j-1}-f_{j-2})\tau_{j-1} \geqslant \frac{B}{C}\sum_{j=1}^{\infty}\sum_{m=j-1}^{\infty}(f_{j-1}-f_{j-2})\mu_m\\ &= \frac{B}{C}\sum_{m=0}^{\infty}\mu_m\sum_{j=1}^{m+1}(f_{j-1}-f_{j-2}) = \frac{B}{C}\sum_{m=0}^{\infty} f_m\mu_m = \frac{B}{C}\langle f,\mu\rangle\,.\end{aligned}$$

□

We next give a proof of our theorem of Carleson type for the Bergman space.

Proof of Theorem 3.6 To prove (i), let $v \in T$ and assume that σ satisfies (3.5). By Lemma 5.1, $q/(q+1) \leqslant f_v$ on $S(v)$ and $0 \leqslant f_v \leqslant 1$ on T. So $f_v \in A^p(\sigma)$, hence, for all $\mu \in \mathcal{M}_{(A^p(\sigma),\sigma)}$,

$$\begin{aligned}\frac{q}{q+1}\mu(S(v)) &\leqslant \sum_{w\in S(v)}|f_v(w)|^p\mu(w) \leqslant \sum_{w\in T}|f_v(w)|^p\mu(w)\\ &\leqslant C\sum_{w\in T}|f_v(w)|^p\sigma(w) \leqslant C\;(1+C_{\sigma,p})\,\sigma(S(v)).\end{aligned}$$

To prove (ii), we will let $C_{\varepsilon,p}$ denotes any real number depending at most on ε and p but possibly varying from line to line. Fix $|v| = n > 0$. If $w \in T\setminus S(v)$, let $k = |w\wedge v|$ and $j = d(w, w\wedge v)$. Then $|w| = d(o, w\wedge v) + d(w\wedge v, w) = k + j$ and $d(v,w) = d(v, w\wedge v) + d(w\wedge v, w) = n - k + j$. Therefore, by Lemma 5.1,

$$\begin{aligned}\sum_{w\in T\setminus S(v)}|f_v(w)|^p\sigma(w) &\leqslant \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} q^{(k-n-j)p}\;\varepsilon^{k+j}\;q^j\\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} q^{-np}\;(q^p\varepsilon)^k\;(q^{-(p-1)}\varepsilon)^j\\ &= C_{\varepsilon,p}\sum_{k=0}^{n-1} q^{-np}(q^p\varepsilon)^k \qquad (\text{since } q^{-(p-1)}\varepsilon < q^{-p} < 1)\\ &= C_{\varepsilon,p}q^{-np}(q^p\varepsilon)^n \qquad (\text{since } q^p\varepsilon > 1)\\ &= C_{\varepsilon,p}\varepsilon^n = C_{\varepsilon,p}\sigma(v) < C_{\varepsilon,p}\sigma(S(v)).\end{aligned}$$

□

Finally, we give the proof of Theorem 3.2 in which we show how to construct non-optimal reference measures.

Proof of Theorem 3.2 Define recursively an increasing sequence of positive integers by $x_0 = 0$ and for $k \geqslant 0$,

$$x_{k+1} = x_k + (1 + x_k)\, b_{x_k}. \tag{6.8}$$

Moreover, define the sequences $f_k : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega_k > 0$ as follows. The function f_0 is supported in the interval $[x_0\,,\, x_1]$ and its graph is the descending segment that connects the points $(x_0\,,\, 1)$ and $(x_1\,,\, 0)$. Choose $x_0 < \omega_0 < x_1$ and define the piecewise linear function f_1 on $[x_0\,,\, x_2]$ that coincides with f_0 on $(x_0\,,\, \omega_0)$ and whose graph in the interval $[\omega_0\,,\, x_2]$ is the line segment connecting the points $(\omega_0\,,\, f_0(\omega_0))$ and $(x_2\,,\, 0)$: clearly, this second line segment decreases at a lower rate than the first, and so f_1 is convex. We choose ω_0 sufficiently close to x_1 so that

$$x_1 - \omega_0 < 1\,,\;\; f_1(\omega_0) = f_0(\omega_0) \leqslant 2 f_1(x_1), \quad \text{and} \int_{\omega_0}^{x_2} f_1 < 1\,.$$

Suppose for $k \geqslant 1$ we have chosen $f_0, f_1, \ldots, f_k$ and $\omega_j \in (x_j, x_{j+1})$ for $0 \leqslant j \leqslant k-1$. The inductive step is to choose $\omega_k \in (x_k, x_{k+1})$ (we explain below additional conditions on ω_k) and define f_{k+1} to be the piecewise linear function on $[x_0\,,\, x_{k+2}]$ that coincides with f_k on $[x_0\,,\, \omega_k]$ and whose graph on $[\omega_k\,,\, x_{k+2}]$ is the linear segment from $(\omega_k\,,\, f_k(\omega_k))$ to $(x_{k+2}\,,\, 0)$. The rate of descent of each segment in the graph of f_{k+1} is less than the previous one, so f_{k+1} is convex. Choose ω_k sufficiently close to x_{k+1} so that

$$x_{k+1} - \omega_k < 1 \text{ and} \int_{\omega_k}^{x_{k+2}} f_{k+1} < \frac{1}{(k+1)^2}\,.$$

From elementary geometric considerations we obtain that

$$f_{k+1}(\omega_k) \leqslant 2 f_{k+1}(x_{k+1}).$$

Indeed, since $x_{k+1} - \omega_k < 1 < x_{k+2} - x_{k+1}$, we see that $\frac{x_{k+2}-\omega_k}{x_{k+2}-x_{k+1}} \leqslant 2$, and so

$$f_{k+1}(\omega_k) = \left(\frac{x_{k+2} - \omega_k}{x_{k+2} - x_{k+1}}\right) f_{k+1}(x_{k+1}) \leqslant 2 f_{k+1}(x_{k+1}).$$

Now let $f(x) = \lim_k f_k(x)$. The function f coincides with f_k on $[x_0\,,\, \omega_k]$, is strictly decreasing and convex, and

$$\int_0^\infty f < \int_0^{\omega_1} f + \sum_k \frac{1}{(k+1)^2} < \infty\,. \tag{6.9}$$

Moreover, the graph of f on $[\omega_k\,,\, x_{k+2}]$ is a segment of negative slope $D_+ f(\omega_k)$ that intersects the x-axis at x_{k+2}. Thus, using Eq. 6.8, we have

$$-D_+ f(\omega_k) = \frac{f(\omega_k)}{x_{k+2} - \omega_k} < \frac{f(\omega_k)}{x_{k+2} - x_{k+1}} = \frac{f(\omega_k)}{(1 + x_{k+1})\, b_{x_{k+1}}}.$$

Hence, for $k \geqslant 0$,

$$\frac{f(\omega_k)}{|D_+ f(\omega_k)|\, b_{x_{k+1}}} > x_{k+1}\,. \tag{6.10}$$

Next, define $a_n = f(n)$. By Eq. 6.9, $\sum_n a_n < \infty$. It is easy to verify that $\{a_n\}$ is $P-$subharmonic on $\mathbb{N}$, since f is convex and decreasing on $\mathbb{R}_+$. Finally, let $k \in \mathbb{N}$ and $n = x_k$. Observe that $x_{k-1} < \omega_{k-1} < x_k$, the piecewise linear function f is linear in the interval $[\omega_{k-1}\,,\, \omega_k]$, and x_k belongs to this interval. Thus $Df(n) = Df(x_k) = D_+ f(\omega_k)$. We may also assume that also $x_k + 1 \in [x_k\,,\, \omega_k]$, because $x_{k+1} - x_k$ diverges and $x_{k+1} - \omega_k < 1$, so also $\omega_{k+1} - x_k$ diverges. But then f is linear on the interval $[x_k\,,\ x_k+1] = [a_n\,,\ a_n+1]$, and so $a_n - a_{n+1} = f(n) - f(n+1) = Df(n) = Df(x_k) = D_+ f(\omega_{k-1})$. Therefore, by Eq. 6.10,

$$\frac{a_n}{(a_n - a_{n+1})\, b_n} = \frac{f(x_k)}{|D_+ f(\omega_{k-1})|\, b_{x_k}} \geqslant \frac{1}{2}\, \frac{f(\omega_{k-1})}{|D_+ f(\omega_{k-1})|\, b_{x_k}} > \frac{x_k}{2} \to \infty\,.$$

□

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