- Let $a \in \mathbb{R}$, and let f be a real-valued function with domain an open interval containing a.
- This section concerns the approximation of $f(x)$ by means of polynomials.
- The aim is to study the behavior of $f(x)$ for x close to $x = a$, so it's natural to express any such polynomial in powers of $x - a$ rather than in x:

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n.$$

- Note that $p(a) = c_0$, so the lowest order coefficient is uniquely determined by the function value at a.

Problem

Can we recover all the coefficients of $p(x)$ using certain function values evaluated at a?

- $p'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + kc_k(x-a)^{k-1} + \cdots + n(x-a)^{n-1}$.
- Evaluating at a gives $\boxed{p'(a) = c_1.}$
- Differentiating again, we get

  $$p''(x) = 2c_2 + 3\cdot 2(x-a) + 4\cdot 3(x-a)^2 + \cdots + k\cdot(k-1)(x-a)^{k-2} + \cdots + n\cdot(n-1)(x-a)^{n-2},$$

  so evaluating at $x = a$ gives $\boxed{p''(a) = 2c_2}$.
- Let's differentiate once again in order to see the general pattern:

  $$p'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2(x-a) + \cdots + k\cdot(k-1)\cdot(k-2)(x-a)^{k-3} + \cdots + n\cdot(n-1)\cdot(n-2)(x-a)^{n-3},$$

  so putting $x = a$ gives $\boxed{p'''(a) = 3\cdot 2c_3}$.
- In general, for any k from 1 to n we get that $\boxed{p^{(k)}(a) = k!c_k}$, where the superscript denotes derivative of that order, so

  $$\boxed{c_k = \frac{p^{(k)}(a)}{k!}}$$

- So we've shown that for the polynomial $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$, the coefficients are determined by the formula

$$\boxed{c_k = \frac{p^{(k)}(a)}{k!}, k = 0, \dots, n.}$$

  Note that for $k = 0$ in the above formula, $p^{(0)}(a)$ is defined to be $p(a)$.
- Back now to the given function f. We would expect that we can say more about $f(x)$ if we know more of the derivatives of f at $x = a$.
- So if we wish to approximate f by a polynomial of degree n, it is natural to associate to f the polynomial whose value at $x = a$ and derivatives at $x = a$ up to order n agree with the corresponding values associated with f. This suggests the following definition.

Definition (Taylor polynomials)

Fix $n \in \mathbb{N}$. If f has derivatives up to and including order n, we associate to it the polynomial $P_n(x)$ defined by

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We call P_n the n-th order Taylor polynomial of f.

- Note that P_n is the unique polynomial of degree n whose value at a and derivatives at a up to order n agree with the corresponding values associated with f, i.e. $P_n(a) = f(a)$ and $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 1, \ldots, n$.

Definition: Taylor remainder

Fix $n \in \mathbb{N}$. If f has derivatives up to and including order n, we define the n-th Taylor remainder $R_n(x)$ by

$$R_n(x) := f(x) - P_n(x).$$

- The point is that we usually don't know $f(x)$ exactly, but we can calculate $P_n(x)$ exactly. If we then decide to approximate $f(x)$ by $P_n(x)$, $R_n(x)$ measures the error we make in doing this.
- So we view $P_n(x)$ as being a good estimate of $f(x)$ if $R_n(x)$ is close to 0.
- Typically we cannot know $R_n(x)$ exactly, so the next best thing is to get an upper bound on $|R_n(x)|$. In that way we know at worst how far $P(n(x)$ can be from $f(x)$.
- Taylor's Theorem below gives a formula for $R_n(x)$ which allows us to get an upper bound for $|R_n(x)|$ provided we know an upper bound on the absolute value of the $(n+1)-$th order derivative of f at $x = a$.
- So the more derivatives of f at "a" we can control, the better we understand the connection between $f(x)$ and the various Taylor polynomials at x.

Theorem (Taylor's Theorem)

Let $f : (a-r, a+r) \to \mathbb{R}$, where $a \in \mathbb{R}$ and $r > 0$. Fix $n \in \mathbb{N} \cup \{0\}$. Suppose for all $x \in (a-r, a+r)$ that f has derivatives at x of order up to and including $n+1$. Let $P_n(x) := \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!}$ denote the n-th order Taylor polynomial of f, and let $R_n(x) := f(x) - P_n(x)$ denote the n-th Taylor remainder. Then for each $x \in (a-r, a+r)$ there exists μ strictly between a and x such that $R_n(x) = f^{(n+1)}(\mu) \frac{(x-a)^{n+1}}{(n+1)!}$.

Exercise.

The notational convention is that $f^{(0)}(t) := f(t)$. What does Taylor's Theorem say in case $n = 0$? Why do you already know that the $n = 0$ case is true?

Theorem (Taylor's Theorem)

Let $f : (a-r, a+r) \to \mathbb{R}$, where $a \in \mathbb{R}$ and $r > 0$. Fix $n \in \mathbb{N} \cup \{0\}$. Suppose for all $x \in (a-r, a+r)$ that f has derivatives at x of order up to and including $n+1$. Let $P_n(x) := \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!}$ denote the n-th order Taylor polynomial of f, and let $R_n(x) := f(x) - P_n(x)$ denote the n-th Taylor remainder. Then for each $x \in (a-r, a+r)$ there exists μ strictly between a and x such that $R_n(x) = f^{(n+1)}(\mu) \frac{(x-a)^{n+1}}{(n+1)!}$.

Comments on the theorem and its proof:

- Recall that the proof of the Mean Value Theorem was obtained by subtracting off a certain straight line function from f (namely P_1) and applying Rolle's Theorem to the resulting function. The proof of Taylor's Theorem uses a similar trick, although that trick is done n times rather than once.
- Note a and x are fixed in the proof. It's natural to define the function

$$g(t) := f(t) - P_n(t) - K \, \frac{(t-a)^{n+1}}{(n+1)!},$$

  where K is the number for which $g(x) = 0$. Then our task is to prove there exists a number μ between a and x such that $K = f^{(n+1)}(\mu)$.
- Note that every one of $g(a), g'(a), g''(a), \ldots, g^n(a)$ is 0. Do you see why? Also $g(x) = 0$, by definition of K.
- This enables us to apply Rolle's theorem in turn to each of these functions $g^{(j)}$ each time producing a new μ_j which allows us to continue the process. After the n-th time, we get the number μ which we want and we are done.

Theorem (Taylor's Theorem)

Let $f:(a-r,a+r)\to\mathbb{R}$, where $a\in\mathbb{R}$ and $r>0$. Fix $n\in\mathbb{N}\cup\{0\}$. Suppose for all $x\in(a-r,a+r)$ that f has derivatives at x of order up to and including $n+1$. Let $P_n(x) := \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$ denote the n-th order Taylor polynomial of f, and let $R_n(x) := f(x)-P_n(x)$ denote the n-th Taylor remainder. Then for each $x\in(a-r,a+r)$ there exists μ strictly between a and x such that $R_n(x) = f^{(n+1)}(\mu)\frac{(x-a)^{n+1}}{(n+1)!}$.

Exercise.

Use the comments on the previous slide to write the proof. Note that this proof is similar, but more transparent than the proof given in our text. I got this proof from the famous book of Walter Rudin, "Principles of Mathematical Analysis".

- For a given specific function f, the Taylor polynomials P_n constitute a sequence of functions.
- If we can show that for each fixed x in interval $(a-r, a+r)$ we have $\lim_{n\to\infty} P_n(x) = f(x)$, then we have shown that the sequence of functions P_n converges pointwise to f.
- In order to show this pointwise convergence, we would typically use Taylor's Theorem. Here is an example.

Exercise.

a) Prove that for any $x > 0$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

b) Find the Taylor polynomials for the function $f(x) = e^x$ corresponding to $a = 0$. Use Taylor's Theorem to prove that the Taylor Polynomials converge pointwise to e^x for all $x \in \mathbb{R}$.

- Consider a sequence P_n of Taylor polynomials about $x = a$ corresponding to some function f.
- Taylor's Theorem can be used to decide whether or not for a given $x \in \mathbb{R}$ we have that the sequence $P_n(x)$ converges, so it tells us about pointwise convergence of the sequence of functions P_n.
- But it is desirable to have uniform rather than merely pointwise convergence.
- In this direction, the following are shown in Chapter 5:
  - There exists $r >$ (possibly $r = \infty$) such that for all x in the open interval $(a - r, a + r)$ the sequence $P_n(x)$ converges, i.e. P_n converges pointwise on this open interval. We call r the radius of convergence.
  - On any closed bounded subinterval of $(a - r, a + r)$ (i.e. on any subinterval of the form $[a - s, a + s]$ where $0 < s < r$) the convergence of the sequence P_n is uniform. We refer to this property as saying that the sequence P_n converges locally uniformly on $(a - r, a + r)$.
- With the local uniform convergence, we can then make use of the theorem in Chapters Three and Four concerning integration and differentiation of uniformly convergent sequences of functions, as long as we apply those theorems on the interval $[a - s, a + s]$.