Theorem (Cauchy's Generalized Mean Value Theorem)

Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Assume that $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t)}{g'(t)}.$$

Exercise.

a) In the statement of the theorem, how do we know we are not dividing by 0 when we write $g(b) - g(a)$?

b) Explain why this is a generalization of the Mean Value Theorem.

c) For the proof, if we consider the function

$$h(x) := f(a) + \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)],$$

why do we have the right to apply Rolle's Theorem to the function $f(x) - h(x)$? What do you get if you do apply Rolle's Theorem to $f - h$?

d) Complete the proof of the Generalized Mean Value Theorem.

Versions 1 and 2 of L'Hôpital's rule

- The Generalized Mean Value Theorem is the key to proving the various versions of L'Hôpital's Rule.

Theorem (L'Hôpital's rule)

(i) (Version 1) Let f and g be continuous on $[a, b]$, differentiable on (a, b), with $g'(x) \neq 0$ for any $x \in (a, b)$. Let $L \in \mathbb{R}$. If $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and equals L.

(ii) (Version 2) Let f and g be differentiable on (b, ∞), with $g'(x) \neq 0$ for any $x \in (b, \infty)$. Let $L \in \mathbb{R}$. If $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals L.

Hints on the proof:

- To prove (i): Do the hypotheses tell us anything about $f(a)$ and $g(a)$? Write down what the conclusion of the Generalized Mean Value Theorem gives you, and see if you can complete the proof.
- To prove (ii): Consider the functions $F(u) := f(1/u)$ and $G(u) := g(1/u)$ for u near 0 on the right. Apply version 1 of L'Hôpital's Rule to F/G and see if that does what is needed.

Exercise.

Write the proofs.

Theorem (L'Hôpital's rule, Version 3)

Let f and g be differentiable on (b, ∞), with $g'(x) \neq 0$ for any $x \in (b, \infty)$. Let $L \in \mathbb{R}$. If $\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x)$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals L.

Exercise.

Write the proof. This one is more technically demanding to prove than the other two versions of L'Hôpital's rule. But if you try it yourself you'll have a greater appreciation of the details when we go over it in class.

Theorem (L'Hôpital's rule, Version 4)

Let f and g be differentiable on $(0, a)$, with $g'(x) \neq 0$ for any $x \in (0, a)$. Let $L \in \mathbb{R}$. If $\lim_{x \to 0^+} f(x) = \infty = \lim_{x \to 0^+} g(x)$ and $\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to 0^+} \frac{f(x)}{g(x)}$ exists and equals L.

Exercise.

The proof of this one follows from Version 3 using a similar trick as we used to get Version 2 from Version 1. Write the proof.

Exercise.

Of course you saw lots of examples of the use of L'Hôpital's rule in your calculus classes. But do this exercise, being sure to clearly explain everything needed to justify all of the steps. Feel free to use the properties of the natural logarithm and exponential functions, in particular the differentiation formulas for each of them, and the resulting fact that they are all continuous.

1) Say we know that $g : [0, \infty) \to \mathbb{R}$ is continuous, and $\lim_{x \to 0^+} g(x) = L$. What theorems allow us to deduce that $\lim_{x \to 0^+} e^{g(x)} = e^L$?
2) Prove that $\lim_{x \to 0} x^x = 1$.