- Say we have a sequence $f_n(x)$ of functions defined on some interval, $[a, b]$. Let's say they converge in some sense to a function $f(x)$. We'd like to be able to deduce that f has certain properties provided we know that every one of f_n has such a property. For example, we've shown:

Theorem

(i) If f_n converges uniformly to f and each f_n is continuous, then f is continuous.

(ii) If f_n converges uniformly to f on $[a, b]$ and each $f_n \in \mathscr{R}[a, b]$, then $f \in \mathscr{R}[a, b]$ and furthermore

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

- We'd like a similar theorem for derivatives. We might conjecture a result such as the following:

  Let $I = (a, b)$ be an open interval. If f_n converges uniformly to f on I and each f_n is differentiable on I, then f is differentiable on I and f_n' converges uniformly to f'.

- Unfortunately this is not a theorem. The following exercise provides a counterexample.

Exercise.

Consider the sequence of functions $f_n(x) := \dfrac{\sin nx}{n}$ with domain $[0, 1]$.

a) Calculate $\|f_n\|_\infty$. Use it to deduce that $f_n \to 0$ uniformly.

b) Calculate $f_n{}'(x) = \cos nx$. How do you know this sequence does not converge uniformly to 0?

- Nevertheless there is a positive theorem of this sort, but we need to assume uniform convergence of f_n', not f_n.

Theorem

Let f_n be a sequence of functions with domain a bounded open interval I. Assume each of the following:

(i) Each f_n is differentiable on I and f_n' is continuous;

(ii) f_n' converges uniformly on I to some function g;

(iii) There exists $a \in I$ such that the sequence of numbers $f_n(a)$ converges to a real number.

Then f_n converges uniformly on I to some function f, and $f'(x) = g(x)$ for all $x \in (a, b)$.

- The theorem is a bit awkward, but it does give a way to say that if f_n converges in some way to f, then f_n' converges in some way to f'.

Theorem

Let f_n be a sequence of functions with domain a bounded open interval I. Assume each of the following:

(i) Each f_n is differentiable on I and f_n' is continuous;

(ii) f_n' converges uniformly on I to some function g;

(iii) There exists $a \in I$ such that the sequence of numbers $f_n(a)$ converges to a real number.

Then f_n converges uniformly on I to some function f, and $f'(x) = g(x)$ for all $x \in (a, b)$.

Comments

- Integration tends to make nice functions even nicer. So since differentiation can be viewed as the inverse procedure of integration (by the First Fundamental Theorem of Calculus), this suggests that differentiation turns function that are nice into functions that are not as nice. For example the function $f(x) = \sqrt{x}$ is uniformly continuous on $(0, 1)$ (even on $(0, \infty)$) but the derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$ is not even bounded on $(0, 1)$. The above theorem is another justification that differentiation doesn't necessarily preserve nice properties of functions.
- The theorem is still true if the hypothesis that f_n' be continuous be removed. It is this form of the result that is proved in Walter Rudin's classic book "Principles of Mathematical Analysis". However the assumption that f_n' be continuous makes for a simpler proof, and it allows us to make use of the first Fundamental Theorem of Calculus.

Theorem

Let f_n be a sequence of functions with domain a bounded open interval I. Assume each of the following:

(i) Each f_n is differentiable on I and f_n' is continuous;

(ii) f_n' converges uniformly on I to some function g;

(iii) There exists $a \in I$ such that the sequence of numbers $f_n(a)$ converges to a real number.

Then f_n converges uniformly on I to some function f, and $f'(x) = g(x)$ for all $x \in (a, b)$.

- Recall that the uniform limit of continuous functions is continuous, so the function g in the theorem is continuous. The trick is to now apply the first Fundamental Theorem of Calculus with integrand g and use that integral to define f. It's then a matter of showing that this choice of f works.

Exercise.

Write the proof.

Definition

Let $[a, b]$ be a closed bounded interval, and let $C^1[a, b]$ denote the set of functions $f : [a, b] \to \mathbb{R}$ such that f is differentiable and f' is continuous. For $f \in C^1[a, b]$, define $\|f\|$ by

$$\|f\| := \|f\|_\infty + \|f'\|_\infty.$$

Theorem

(i) $C^1[a, b]$ is a vector space of functions and the quantity $\|\cdot\|$ is a norm on $C^1[a, b]$.

(ii) $C^1[a, b]$ is complete relative to this norm. Consequently $C^1[a, b]$ with this norm is a Banach space.

Hints on the proof:

- For (i), after recalling the definition of norm it should be clear what to do.
- In the proof of (ii), we get to use the fact that $C[a, b]$ is a Banach space relative to the sup norm, i.e. if g_n is Cauchy relative to $\|\cdot\|_\infty$, then there exists $g \in C[a, b]$ such that g_n converges uniformly to g. This is something we proved in Chapter 2. Try to prove (ii) by applying this idea to both f_n and f_n' and then applying the previous theorem.

Exercise.

Write the proof.