4.3 The Fundamental Theorems of Calculus

- Many students think of the Fundamental Theorems of Calculus as being formulas, such as the following ones:

1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."
2. "If there exists F differentiable such that $F' = f$ on $[a, b]$, then

$$\int_a^b f(t)\, dt = \int_a^b F'(t)\, dt = F(b) - F(a)."$$

- But these require the right hypotheses. We are given f as data. What hypotheses on f will make the first statement above true? What hypotheses on f make the second statement above true?
- That is what we study in this section.

- Recall that of the conditions $f \in \mathscr{R}[a,b]$ and $f \in C[a,b]$, we have

$$f \in C[a,b] \Rightarrow f \in \mathscr{R}[a,b]$$
$$f \in \mathscr{R}[a,b] \not\Rightarrow f \in C[a,b]$$

Do you recall how to prove the first of these, and what is a counterexample for the second?

Hypotheses of the first fundamental theorem of calculus

1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."

- This statement would at least make sense if f were merely in $\mathscr{R}[a, b]$.
- Actually, is that a true statement? It presupposes that if f is Riemann integrable on an interval $[a, b]$, then it is also Riemann integrable on any subinterval $[c, d]$ of $[a, b]$. Actually that is a true statement.

Exercise.

Let $f \in \mathscr{R}[a, b]$. Let $[c, d]$ be a subinterval of $[a, b]$. Prove that $f \in \mathscr{R}[c, d]$.

1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."

- Is it true that if f is merely Riemann integrable on $[a, b]$, then the above function F is differentiable and $F' = f$ on $[a, b]$?
- Not according to the following counterexample.

Exercise.

Let $f(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } 1 < t \leq 2 \end{cases}$. Let $F(x) := \int_a^x f(t)\, dt$.

a) Calculate F.

b) Is F differentiable on $[0, 2]$?

1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."

- So the above exercise shows that in the boxed statement above, it is not sufficient to merely put the hypothesis that $f \in \mathscr{R}[a, b]$.
- So let's view this as the following problem:

Problem:

Say we have this unknown function $f(x)$, and what we know about it are the values of all of the integrals $\int_a^x f(t)\ \ dt$ for all $x \in [a, b]$. Can we determine each $f(x)$ from this information? What condition on f is sufficient to achieve this?

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- Well the idea is that if we know $\int_a^x f(t)\,dt$ for each x, then we also know $\int_x^{x+h} f(t)\,dt$ for each $x \in (a, b)$ and h sufficiently close to 0. Do you see why?
- If f is merely assumed to be continuous, then for h close to 0, we expect the integral $\int_x^{x+h} f(t)\,dt$ to be close to $f(x) \cdot h$, and so we would expect to be able to prove that $\lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$.
- This solves the above boxed problem, and then to finish the proof of the theorem on the previous slide (it becomes a theorem if we assume in addition that f is continuous), it is enough to recognize $\frac{\int_x^{x+h} f(t)\,dt}{h}$ as a difference quotient of F.
- In fact it is quite easy to use the continuity of f to make the above rigorous. We do so on the next slide.
- I wanted to mention that our textbook and just about every calculus book proves it by making use of the so-called “mean value theorem for integrals”. In order to prove that theorem you have to make use of the Extreme Value Theorem and the Intermediate Value Theorem. This is pretty “heavy artillery”, but the proof we give doesn't require any of this, and is really quite simple.

Theorem (First Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\ dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$.

Exercise.

Give the proof. Do it by proving directly from the definition of continuity of f that for each $x \in (a, b)$ we have

$$\lim_{h \to 0} \left| \frac{\int_x^{x+h} f(t)\ dt}{h} - f(x) \right| = 0.$$

Hypotheses of the second fundamental theorem of calculus

- Let's now discuss the other of the fundamental theorems of calculus. It's something like:

> "If there exists F differentiable such that $F' = f$ on $[a, b]$, then
> $$\int_a^b f(t)\, dt = \int_a^b F'(t)\, dt = F(b) - F(a)''$$

- So what about the correct hypotheses on f?

Exercise.

In the above statement, let's say we merely assume that f is a function for which there exists F such that F is differentiable and $F' = f$. What else would we have to implicitly be asserting in order for the conclusion to have a chance to be true?

- We show next that this necessary hypothesis is also sufficient.

Theorem (Second Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose there exists $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Some comments and steps of the proof:

- Notice that this theorem is stronger than the one that appears in most calculus books where the hypothesis is the more stringent one that f should be continuous.
- Because we have developed the theory of integration a bit differently from the text, we have to modify the text's proof a bit, but it still works out nicely.
- Give yourself $\varepsilon > 0$; we'll try to prove that $\left| \int_a^b f(x)\, dx - (F(b) - F(a)) \right| < \varepsilon$. Since ε is arbitrary, that would imply the conclusion of the theorem.
- Begin by writing down what the "Partition Characterization of Riemann Integrability Theorem" says about f relative to the given ε. It produces a special partition P.
- Note that any number that lies between $U(f, P)$ and $L(f, P)$ must be within ε of $\int_a^b f(x)\, dx$.
- Apply the Mean Value Theorem to F on each of the intervals $[x_{i-1}, x_i]$ of P. This produces special points c_i in each subinterval $[x_{i-1}, x_i]$ of the partition. Using these c_i's, write down the associated "Riemann sum", and note that it must lie between $U(f, P)$ and $L(f, P)$. Then make use of the fact that $f(c_i) = F'(c_i)$. This causes the Riemann sum to be a telescoping sum, and the result follows.

Theorem (Second Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose there exists $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Exercise.

Write the proof of the second fundamental theorem of calculus.