- This section and the next give some of the results that are the best known to you from your calculus classes.
- If you make use of the results we have already proven and you use the hints provided, you should be able to come up with all of the proofs by yourself. It's a good exercise for you to do so.

## Definition

(i) Let $D \subset \mathbb{R}$, and let $x_0 \in D$. We say that $x_0$ is an interior point of $D$ provided there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq D$.

(ii) The interior of $D$, denoted by $D^0$, is the set of all interior points of $D$.

**Exercise.**

a) What is $D^0$ if $D = \mathbb{Z}$? How about if $D = [5, 17]$?

b) Explain why $D^0$ is an open set, and that it is the largest open set contained in $D$.

## Definition

Let $f : D \to \mathbb{R}$. Let $x_0$ be an interior point of $D$.

(i) We say that $f$ has a local maximum point at $x_0$ provided there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$ then $f(x) \leq f(x_0)$.

(ii) We say that $f$ has a local minimum point at $x_0$ provided there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$ then $f(x) \geq f(x_0)$.

(iii) We say that $f$ has a local extreme point at $x_0$ or that $x_0$ is a local extreme point of $f$ provided $f$ either has a local maximum point or a local minimum point at $x_0$.

**Theorem**

Let $f : D \to \mathbb{R}$, and let $x_0$ be an interior point of $D$. If $f$ is differentiable at $x_0$ and $f$ has a local extreme point at $x_0$, then $f'(x_0) = 0$.

**Some hints on the proof:**

- Say $x_0$ is a local maximum point of $f$. If $h$ is close enough to 0, what can you say about the sign of the quantity

$$f(x_0 + h) - f(x_0),$$

  whether or not $h$ is positive or negative?
- What can you say about the sign of $\dfrac{f(x_0 + h) - f(x_0)}{h}$?
- What does this tell you about $f'(x_0)$?

**Exercise.**

Write the proof of the theorem.

# Application of the basic result: Rolle's Theorem

**Theorem (Rolle's Theorem)**

Let $f \in C[a, b]$. Assume also that $f$ is differentiable on $(a, b)$. If $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

**Comments on Rolle's Theorem and its proof:**

- Make a sketch which illustrates what the theorem says.
- Don't forget what we have just finished proving on the previous slide.
- Why may we assume that $f \not\equiv 0$ on $[a, b]$?
- Why may we assume that $f$ has either a maximum or a minimum $x_0$ on $(a, b)$?
- What can we say about what happens at $x_0$? Why?

**Exercise.**

Write the proof of Rolle's Theorem.

**Theorem (The Mean Value Theorem)**

Let $f \in C[a, b]$. Assume also that $f$ is differentiable on $(a, b)$. Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Comments on the Mean Value Theorem and its proof:**

- Make a sketch which illustrates what the theorem says.
- The hypotheses are similar to the hypotheses of Rolle's Theorem. Do you see how to reduce the proof to an application of Rolle's Theorem?
- The idea is to subtract off something from $f$ so that the result would follow immediately from Rolle's Theorem. What to subtract off?
- Should be a straight line which agrees with $f$ at $a$ and $b$. What is the equation of that line?

**Exercise.**

Write the proof of the Mean Value Theorem.

**Theorem**

Let $f$ be differentiable on an open interval $I = (a, b)$.

(i) If $f' \equiv 0$ on $I$, then $f$ is constant.

(ii) If $f' \equiv g'$ on $I$, then $f$ and $g$ differ by a constant function.

(iii) If $f'(x) \geq 0$ for all $x \in I$, then $f$ is increasing on $I$.

(iv) If $f'(x) > 0$ for all $x \in I$, then $f$ is strictly increasing on $I$.

Exercise.

a) Write the proof of the above theorem. Each of them follow easily if the right results are used.

b) Prove that the converse of (iii) is true. Note that you don't need any powerful tools to prove this.

c) Show that the converse of (iv) is not true in general. Do so by supplying an appropriate counterexample.

- So if we have $f \in C[a,b]$ such that $f$ is differentiable on $(a,b)$ and we calculate that $f'(x) < 0$ for all $x \in (a,b)$, then $f$ must be strictly decreasing, and that implies that for all such $x$,

$$f(a) > f(x) > f(b).$$

**Exercises**

Use the derivative to prove each of the following.

a) $\sin x < x$ for all $x > 0$
*Hint: Let $f(x) = \sin x - x$, and calculate $f'(x)$.*

b) $\cos x > 1 - \frac{x^2}{2}$ for all real $x$.
*(Hint: Don't forget the results you've just finished proving. Let $f(x) = \cos x - 1 + x^2/2$ for $x > 0$; calculate $f'(x)$ and deduce the result for $x > 0$. Now explain how to extend to $\mathbb{R}$. )*

# Some Applications of these ideas

**Exercise.**

Let $f : U \to \mathbb{R}$, where $U$ is an open subset of $\mathbb{R}$. Suppose that $f$ is differentiable and for some $M \in \mathbb{R}$ we have that $|f'(x)| \leq M$ for all $x \in U$. Prove that $f$ is uniformly continuous.
*(Hint: Let $a, b$ be in the domain of $f$ with $a < b$. Apply the Mean Value Theorem to $f$ on this interval; using the hypothesis what are you able to deduce? And why does this now allow you to deduce $f$ is uniformly continuous?)*