- Recall the triangle inequality on $\mathbb{R}$:

$$|x + y| \leq |x| + |y| \qquad \text{for all } x, y \in \mathbb{R}.$$

- How would this generalize to $\mathbb{R}^2$?
- Let's view points of $\mathbb{R}^2$ as vectors: $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2)$ be vectors in $\mathbb{R}^2$. We define their "norms" as

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2}, \quad \|\vec{y}\| := \sqrt{y_1^2 + y_2^2}.$$

- The norm of the vector measures the length of the arrow representing the vector.
- Then the triangle inequality says:

$$\|(x_1, x_2) + (y_1, y_2)\| \leq \|(x_1, x_2)\| + \|(y_1, y_2)\|.$$

Exercise.

Explain by means of a sketch why you should believe the triangle inequality is true, and also explain where the name "triangle inequality" comes from.

- The triangle inequality says $\|(x_1 + y_1, x_2 + y_2)\| \leq \|(x_1, x_2)\| + \|(y_1, y_2)\|$.
- This says $\sqrt{(x_1 + y_1)^2 + (x_2 + y_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$.
- Squaring both sides, this is equivalent to

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 \leq (x_1^2 + x_2^2) + (y_1^2 + y_2^2) + 2\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

- Opening the left side and canceling off the common square terms gives $2(x_1 y_1 + x_2 y_2) \leq 2\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$. Using "dot product notation" this gives us the Cauchy-Schwarz inequality:

> Cauchy-Schwarz Inequality for vectors in $\mathbb{R}^2$
>
> $$\vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\|$$
>
> .

- **So working backwards, we see that we would have the triangle inequality for vectors in $\mathbb{R}^2$ if we could prove the above Cauchy-Schwarz Inequality**.

Cauchy Schwarz inequality and the triangle inequality in $\mathbb{R}^n$

- These things have obvious higher dimensional analogues. For vectors $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ their norm and dot products are defined by:

Norm and dot products in $\mathbb{R}^n$

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}, \qquad \|\vec{y}\| := \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2},$$
$$\vec{x} \cdot \vec{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

- Then we claim that the Cauchy-Schwarz Inequality holds and one can use it to deduce the triangle inequality in $\mathbb{R}^n$:

Cauchy-Schwarz inequality in $\mathbb{R}^n$: $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$

Triangle Inequality in $\mathbb{R}^n$: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

- We will show a proof that works in any $\mathbb{R}^n$. But we will do more. We'll will prove it for an "infinite dimensional" generalization of $\mathbb{R}^n$, the so-called $L^2 space$.

L^2-norm and inner products for $f, g : [a, b] \to \mathbb{R}$

- Let $f, g : [a, b] \to \mathbb{R}$. We think of a vector $\vec{x}$ as having components $x_1, x_2, \ldots, x_n$. We can think of f as also being a vector, except that it has infinitely many components:

  For each $x \in [a, b]$, think of $f(x)$ as representing the "x-th" component of f.

- Then we can define a new norm $\|f\|_2$ and a new dot product $< f, g >$ where the sums in the above box go over into integrals:

Norm and inner products for $f, g : [a, b] \to \mathbb{R}$

$$\|f\|_2 := \sqrt{\int_a^b (f(x))^2 \, dx}, \qquad \|g\|_2 := \sqrt{\int_a^b (g(x))^2 \, dx},$$

$$< f, g > := \int_a^b f(x) g(x) \, dx.$$

- Note that this new norm is determined by the inner product:

$$\|f\|_2^2 = < f, f > .$$

- Note also that the inner product is symmetric:

$$< f, g > = < g, f >$$

  and also linear in the first variable:

$$\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle , \qquad < cf, g > = c < f, g > .$$

Theorem

Let $f, g : [a, b] \to \mathbb{R}$. Suppose that $f, g \in \mathscr{R}[a, b]$. Then the following are true.

(i) We necessarily have that fg is also Riemann integrable on $[a, b]$.

(ii) (Cauchy-Schwarz Inequality) $| < f, g > | \leq \|f\|_2 \|g\|_2$.

(iii) (Triangle Inequality for $L^2[a, b]$) $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.

Exercise.

a) Let $a, b, c \in \mathbb{R}$ with $a > 0$. Say we know that for all $t \in \mathbb{R}$ we have

$$at^2 + bt + c \geq 0.$$

What can we say about a, b and c?

b) Write a proof of the theorem. You will find the first part of this exercise useful in doing the proof.

c) Use the ideas of this proof to write a proof of the triangle inequality in $\mathbb{R}^n$.

Definition

The set $L^2[a, b]$ is defined to be the set of functions $f : [a, b] \to \mathbb{R}$ such that $(f(x))^2$ is Riemann integrable on $[a, b]$, that is $\int_a^b (f(x))^2 \, dx$ exists and is finite.

- For functions $f \in L^2[a, b]$, the quantity $\|f\|_2 = \sqrt{\int_a^b (f(x))^2 \, dx}$ is called the L^2-norm of f.
- Note that it is almost a “norm” in the sense we have defined earlier in the course; do you see which property of a norm it fails to satisfy?
- The norm is defined by means of the “inner product” $< f, g >:= \int_a^b f(x)g(x) \, dx$, since $\|f\|_2^2 =< f, f >$.
- Thus we can view $L^2[a, b]$ as being a normed space (actually an “inner product space”).
- But it turns out that it is not a Banach space, i.e. not all Cauchy sequences converge.
- However, if we replace the Riemann integral by the Lebesgue integral, then the corresponding $L^2[a, b]$ becomes a complete space, so it is a Banach space. In fact it is then a “complete inner product space” and that is known as a Hilbert space.