- In this section we study the following questions:

  Let f_n be a sequence of functions all in $\mathscr{R}[a,b]$. Suppose we know that $f_n \to f$ in some specified type of convergence.

  (i) Is it necessarily true that the limit function f is in $\mathscr{R}[a,b]$?

  (ii) If so, is it true that $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as a sequence of numbers?

- If $f_n \to f$ pointwise:
  - We show by example that (i) above need not be true.
  - Another example will illustrate that even if (i) holds, (ii) need not hold.
- If $f_n \to f$ uniformly: We will prove that necessarily (i) and (ii) both hold.

Exercise 1.

We know that $\mathbb{Q}$ is a countable set. Since $\mathbb{Q} \cap [0,1]$ is an infinite subset of $\mathbb{Q}$, it is also a countable set. Let $\{r_m\}_{m=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{if } x \notin \{r_1, r_2, \ldots, r_n\} \end{cases}$$

a) To what function f does the sequence $\{f_n\}_{n=1}^{\infty}$ converge pointwise?

b) Is it true that f is Riemann integrable?

c) Is it true that each of f_n are Riemann integrable on $[0,1]$? If so, how much is $\int_0^1 f_n(x)\, dx$?

d) So what does this exercise show?

Exercise 2.

For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ n & \text{if } 0 < x \leq 1/n \\ 0 & \text{if } 1/n < x \leq 1 \end{cases}$$

a) To what function f does the sequence $\{f_n\}_{n=1}^{\infty}$ converge pointwise?

b) Is f Riemann integrable? If so, how much is $\int_0^1 f(x)\, dx$?

c) Is it true that each of f_n are Riemann integrable on $[0,1]$? If so, how much is $\int_0^1 f_n(x)\, dx$?

d) So what does this exercise show?

- After the above two negative examples, we give the following positive result.

Theorem

Let f_n be a sequence of functions in $\mathscr{R}[a,b]$. Suppose the sequence converges uniformly on $[a,b]$ to the function f. Then $f \in \mathscr{R}[a,b]$ and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

Idea of the proof:

- Note that there is no assumption of continuity on the f_n's, we only get to use the weaker hypothesis of Riemann integrability.
- Given an $\varepsilon > 0$ we can find f_N such that $\|f_N - f\|_\infty < \varepsilon$. We could hope to be able to use this to make similar statements about f which we could make about f_N.
- We should make use of the Partition Characterization of integrability. First write what this implies about f_N (namely that it implies the existence of a certain partition P of $[a,b]$ with nice properties) and then use the fact that $\|f_N - f\|_\infty < \varepsilon$ to see that what this says about $U(f,P)$ and $L(f,P)$.

Exercise.

Write a proof of the theorem.