- We develop things differently from our text, combining the material from sections 3.1 and 3.2.

Definition: Partition of $[a, b]$ and associated terminology

(i) By a partition P of the interval $[a, b]$ we mean a finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

(ii) The above partition divides $[a, b]$ into n intervals $I_i := [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Let's refer to each of these intervals as "the intervals of the partition". We denote the length of the *ith* such subinterval by Δ_i, that is $\Delta_i := x_i - x_{i-1}$.

(iii) (Norm of a partition) For the above partition, we define the norm $\|P\|$ of it to be the length of the longest of the partitioning subintervals, that is

$$\|P\| := \max\{\Delta_i : i = 1, \ldots, n\}.$$

(iv) (Refinement of a partition) Let P, P' be any two partitions of $[a, b]$. We say that P' is a refinement of P if $P \subseteq P'$.

(v) (Common refinement of two partitions) Let P', P'' be two partitions of $[a, b]$. The common refinement of P' and P'' is the partition $P' \cup P''$.

Definition: Upper and Lower Sums

Let $f : [a, b] \to \mathbb{R}$, and assume that f is bounded. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

- For each i, let $M_i := \sup \{f(x) : x_{i-1} \le x \le x_i\}$ and $m_i := \inf\{f(x) : x_{i-1} \le x \le x_i\}$.
- Corresponding to this partition, define the upper sum $U(f, P)$ and the lower sum $L(f, P)$ by

$$U(f, P) := \sum_{i=1}^{n} M_i \, \Delta_i, \qquad L(f, P) := \sum_{i=1}^{n} m_i \, \Delta_i.$$

Theorem (Refinement Theorem)

Let P be a partition of $[a, b]$ and let P' be any refinement of P. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Consequently,

$$|U(f, P') - L(f, P')| \leq |U(f, P) - L(f, P)|.$$

Hints on the proof:

- Do you see that $L(f, P') \leq U(f, P')$ is obvious?
- Do you see that $L(-f, P) = -U(f, P)$ and $U(-f, P) = -L(f, P)$?
- Do you see how the previous bullet allows us to reduce the proof to showing that $U(f, P') \leq U(f, P)$?

Exercise.

Write the proof of the theorem.

Definition: Upper and Lower Integrals

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the upper integral $\overline{\int_a^b} f(x)\,dx$ and the lower integral $\underline{\int_a^b} f(x)\,dx$ by

$$\overline{\int_a^b} f(x)\,dx := \inf\{U(f, P) : P \text{ a partition of } [a, b]\}$$
$$\underline{\int_a^b} f(x)\,dx := \sup\{L(f, P) : P \text{ a partition of } [a, b]\}$$

Exercise.

a) Let $f(x) = \begin{cases} 5 & \text{if } x < 4 \\ 7 & \text{if } x \geq 4 \end{cases}$ Calculate $\overline{\int_4^{12}} f(x)\,dx$ and $\underline{\int_4^{12}} f(x)\,dx$.

b) Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ Calculate $\overline{\int_4^{12}} f(x)\,dx$ and $\underline{\int_4^{12}} f(x)\,dx$.

Theorem (Basic properties of upper and lower integrals)

(i) Both the upper and lower integrals exist as real numbers.

(ii) We always have $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

(iii) For any partition P of $[a, b]$, we have $L(f, P) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f, P)$ and so consequently $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(f, P) - L(f, P)$.

Exericse.

Write the proof of the theorem.

Definition

- Let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is Riemann integrable on $[a, b]$ if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

- If f is Riemann integrable, then $\int_a^b f(x)\,dx$ denotes the common value of $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$.
- The set of all Riemann integrable functions on $[a, b]$ is denoted by $\mathscr{R}[a, b]$.

- A useful criterion that $f \in \mathscr{R}[a, b]$ is given by the following theorem:

Theorem (Partition Characterization of Integrability)

Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Idea of the proof:

- $\Rightarrow$ follows from the Refinement Theorem and the definitions of sup and inf.
- $\Leftarrow$ follows from part (iii) of the theorem on the previous slide.

Exercise.

Write the proof of the theorem.

Theorem (Linearity of the integral)

Let $f, g \in \mathscr{R}[a,b]$. Then we have each of the following.

(i) $f+g \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(ii) $-f \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.

(iii) For any real α, we have $\alpha f \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

Hints:

(i) Show first that for any partition P of $[a,b]$ we have $U(f+g,P) \le U(f,P) + U(g,P)$ and $L(f+g,P) \ge L(f,P) + L(g,P)$.

(ii) Recall $U(-f,P) = -L(f,P)$ and $L(-f,P) = -U(f,P)$.

(iii) Easy in case $\alpha > 0$. The case of $\alpha < 0$ follows from this and (ii).

Exericise.

Prove the theorem.

Theorem (Gluing theorem)

Let $a < b < c$. Let $f : [a, c] \to \mathbb{R}$. Suppose that $f \in \mathscr{R}[a, b]$ and $f \in \mathscr{R}[b, c]$. Then $f \in \mathscr{R}[a, c]$ and $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$.

Exercise.

Write the proof of the gluing theorem.

- We've only defined $\int_a^b f(x)\, dx$ in case $a < b$.
- But we want to define $\int_a^b f(x)\, dx$ even if $a > b$, and furthermore we want the above Gluing Theorem to be true, regardless of the order of the numbers a, b and c. This motivates us to make the following definition:

Definition

Let $a > b$. Then we define $\int_a^b f(x)\, dx$ as follows: $\int_a^b f(x)\, dx := -\int_b^a f(x)\, dx.$

Exercise.

Assuming we want the gluing theorem to hold regardless of the order of a, b, c, why were we forced to make the above definition?

Theorem

For any three different real numbers a, b, c and any f which is Riemann integrable on any of the three closed intervals we can form from these three numbers, we have

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Exercise.

Make use of the definition on the previous slide (and whatever else is needed) to prove this theorem.

- It's of interest for us to know lots of examples of bounded functions which are Riemann integrable.
- The following theorem guarantees that the set of Riemann integrable functions includes at least the set of continuous functions.

Theorem (Integrability of continuous functions)

Let f be continuous on $[a, b]$. Then $f \in \mathscr{R}[a, b]$.

Hints on the proof:

- The proof follows if we make the right use of
  - the uniform continuity of f (how do you know f is uniformly continuous?)
  - and the Partition Characterization of Integrability.

Exercise.

Write the proof.

- Another general class of Riemann integrable functions is given by the next theorem.

Theorem (Integrability of monotone functions)

Let f be a monotone function on $[a, b]$. Then $f \in \mathscr{R}[a, b]$.

Hints on the proof:

- Monotone functions need not be continuous, so cannot use ideas of continuity.
- Do you see why it's sufficient to just prove it for f monotone increasing?
- For any partition of $[a, b]$ and any associated interval $[x_{i-1}, x_i]$ of that partition, how much is the supremum of $f(x)$ as x varies over that interval? How much is the infimum of $f(x)$ as x varies over that interval?
- For any such partition, how much is $U(f, P) - L(f, P)$? (Do you know what is a telescoping sum?)
- Recall the Partition Characterization of Integrability.

Exercise.

Write the proof.

Exercise

a) Write down an example of a function on some interval $[a, b]$ (or just the sketch of a function) which is Riemann integrable but not continuous. How do you know your example is Riemann integrable?

b) Let $f : [a, b] \to \mathbb{R}$ be given by $f = g - h$, where $g, h : [a, b] \to \mathbb{R}$ are both monotone increasing. Is f Riemann integrable? Why or why not?

Exercise.

(a) Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Suppose that f is continuous everywhere on $[a, b]$ except at the point c. Prove that f is Riemann integrable on $[a, b]$.

(b) What generalization of this result can be proved with a simple modification to the proof in (a).

Exercise.

Define $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$.

a) Where is f discontinuous?

b) Explain why $f \notin \mathscr{R}[0,1]$.

Exercise

Consider the following function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases}$.

a) Where are the discontinuities of f? So how many discontinuities does f have?

b) Prove that $f \in \mathscr{R}[0,1]$.

- The above example shows that a function can have lots of discontinuities and still be Riemann integrable.
- An interesting question is to characterize the set of points on which a Riemann integrable function can be discontinuous.

Theorem

Let $f, g : [a, b] \to \mathbb{R}$.

a) Suppose that $f \in \mathscr{R}[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

b) Suppose that f and g are in $\mathscr{R}[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

c) If $f \in \mathscr{R}[a, b]$, then $|f| \in \mathscr{R}[a, b]$ and furthermore $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. However, the converse is false; there exists f such that $|f| \in \mathscr{R}[a, b]$, but $f \notin \mathscr{R}[a, b]$.

Exercise.

Write the proof.