- In this section we pursue a few of the ideas stated on the last slide of section 2.4.
- This means to take some of the ideas we've considered so far for real numbers and try to develop similar ideas in other settings, namely in **"function spaces"**.
- The first thing we developed for real numbers is the idea of a **sequence**, so we consider that first.

## Sequences of functions

- Let $D$ be any subset of $\mathbb{R}$.
- Suppose for each $n \in \mathbb{N}$ we have a real-valued function $f_n$ with $f_n : D \to \mathbb{R}$. *(Note: The term "function" will always mean real-valued function.)*
- We refer to $\{f_n\}_{n=1}^{\infty}$ as a sequence of functions on $D$.

**Exercise.**

Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$.

(i) Sketch the graph of a few terms of the sequence.

(ii) What is the apparent behavior of the sequence as you can see from the graph? Does it appear to go to some specific function?

## Various kinds of convergence of sequences of functions

- Let $\{f_n\}$ be a sequence of functions with domain $D$, and let $f$ be a function with domain $D$.
- We want to discuss convergence of the sequence $f_n$ to $f$. But there are many possible and different ways of such convergence, so writing $f_n \to f$ has no meaning by itself. We need to explain what kind of convergence it is.
- In this section we introduce and study two kinds of convergence, namely
  - ***pointwise convergence***
  - ***uniform convergence***.
- So rather than write "$f_n \to f$" (which has no meaning by itself), we might instead say "the sequence $f_n$ converges pointwise to $f$", or we might say "the sequence $f_n$ converges uniformly to $f$". These two types of convergence are not at all the same.

## Pointwise Convergence of a sequence of functions

With $D$, $f_n$ and $f$ as on previous slide, we say that the sequence $f_n$ converges pointwise to $f$ if for each $x \in D$, we have $f_n(x) \to f(x)$. In symbols:

$$(\forall x \in D)[f_n(x) \to f(x)].$$

In more detail, this means the following holds:

$$(\forall x \in D)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies |f_n(x) - f(x)| < \varepsilon].$$

We also refer to $f$ as the pointwise limit of the sequence $f_n$.

## Exercise.

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$.

(i) Does the sequence converge pointwise? If so, to what function $f(x)$?

(ii) What can you say about the continuity of the members of the sequence $f_n$ and the continuity of $f$?

- If each of the members of a sequence of functions have some desirable property and we know that the sequence $f_n$ converges in some way to $f$, it would be nice if we could then deduce that $f$ also has that property.
- For example, is it true in general that the pointwise limit of a sequence of continuous functions is continuous?
- Pointwise convergence is such a weak type of convergence that it often doesn't tell us much about the properties of the limit function.
- We'll see that uniform convergence of a sequence allows us to say more about the limit function.

**Definition**

Let $\{f_n\}, f$ all be functions with domain $D$. We say that $f_n$ converges uniformly to $f$ if the following is true:

$$\|f_n - f\|_\infty \to 0.$$

Equivalently, if the following is true:

$$(\forall \varepsilon > 0)(\exists \mathbb{N} \in \mathbb{N})(\forall x \in D)(\forall n \in \mathbb{N})[n \geq N \implies |f_n(x) - f(x)| < \varepsilon]$$

- So in the above definition, the same $N$ works simultaneously for all $x \in D$.
- Pointwise convergence merely says that for any $\varepsilon > 0$, if you first give me the $x \in D$, then there exists $N$, or maybe we should call it $N_x$ to emphasize that it depends on the specific $x$, such that for all $x \in D$, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N_x$.
- So if a sequence $f_n$ converges pointwise to $f$ but not uniformly, it means that for a given $\varepsilon$ and each $x$, we can find the $N_x$ with the appropriate property, but it is impossible to find a single $N$ that works for all $x$, so it must be that

$$\sup\{N_x : x \in D\} = \infty.$$

## Exercise

Draw graphs which illustrate the idea of the definition of uniform convergence of a sequence $f_n$ to a function $f$.

**Exercise**

Consider the sequence $f_n : [0, 1/2] \to \mathbb{R}$, $f_n(x) = x^n$.

(i) What is the pointwise limit of the sequence?

(ii) For each $n$, calculate $\|f_n\|_\infty$.

(iii) Explain how you know that the sequence converges uniformly.

(iv) If you change the domain from $[0, 1/2]$ to $[0, 1)$, prove that the convergence of the sequence is not uniform.

**Theorem** If a sequence $f_n$ converges uniformly on $D$ to a function $f$, then it also converges pointwise. However, the converse is false in general, that is, there exists a domain $D$ and a sequence of functions which is pointwise convergent but not uniformly convergent on $D$.

Exercise.

Prove the above theorem.

- Say $\{f_n\}$ is a sequence converging in some way to a function $f$.
- If we know that all of the $f_n$'s are continuous, pointwise converge doesn't tell us that $f$ is necessarily continuous. The next theorem tells us that if the convergence is uniform, then $f$ must be continuous.

**Theorem** Let $f_n$ a sequence of functions with domain $D$, and suppose that the sequence converges uniformly to a function $f$. If each $f_n$ is continuous, then $f$ is continuous.

- Simply put, it says that the uniform limit of continuous functions is continuous.
- The proof is a nice application of the triangle inequality.
- Start by giving yourself $\varepsilon > 0$. Use the uniform convergence convergence to produce $f_N$ which is uniformly within $\varepsilon/3$ of $f$.
- Use the continuity of $f_N$ and the triangle inequality to prove that $f$ is continuous.

**Exercise.**

Write a proof of the theorem.

**Exercise**

Give a simple proof that the sequence $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$ does not converge uniformly.

## Exercise

Consider the sequence $f_n : (0,\infty) \to \mathbb{R}$, $f_n(x) = x^{\frac{1}{n}}$.

a) Prove that this sequence converges pointwise to 1.

b) Prove that the convergence is not uniform.

Here are a few hints:

- Do you see why we could easily finish the proof of (a) if we knew it for $x > 1$? If you let $y_n := x^{1/n} - 1$, you need to show that $y_n \to 0$. See if you can do it by applying the binomial theorem in the right way and making an estimate.
- For (b), show that it is already not uniformly convergent for $x \in (0,1]$. Do it by calculating the supremum of $1 - f(x)$ for $x \in (0,1]$.

Exercise

Consider the sequence $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^{1+\frac{1}{n}}$. It follows from the previous slide that $f_n$ converges pointwise to $x$. Prove that the convergence of this sequence is uniform.

- We'll do it by actually calculating the supremum $\|x - f_n(x)\|_\infty$ and just observing that it goes to 0 as $n \to \infty$.
- For this we'll make use of some calculus results you learned in Calc I, even though we haven't yet proved them.
- The result is the "second derivative test" for finding absolute max of a function. It says essentially that for $f$ a function with domain an open interval such that
  - $f$ has exactly one critical point $x$ (i.e. a unique $x$ such that $f'(x) = 0$ and
  - $f''(x) < 0$

  then $x$ is a point of absolute maximum of $f$.

- What properties of $\mathbb{R}$ are needed in order to discuss convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ to a number $x$?
- We need to be able to calculate $|x_n - x|$ in order to measure how far apart the term $x_n$ is from the limiting value $x$.
- To calculate $x_n - x$ we need the "vector space '' structure of $\mathbb{R}$
- and to calculate $|x_n - x|$ we need the "norm" structure of $\mathbb{R}$.
- So we should be able to formulate the idea of convergence in any normed space.

Recall the definition of a **normed space** which we introduced in section 2.4:

## Definition

Let $V$ be a real vector space. Let $\|\cdot\|$ denote a real-valued function on $V$ whose value at $v \in V$ is written as $\|v\|$. We say that $\|\cdot\|$ is a **norm** if the following three properties hold:

(i) (Positive Definite) $(\forall v \in V)(\|v\| \geq 0$ and $(\forall v \in V)[\|v\| = 0 \Leftrightarrow v = 0]$

(ii) $(\forall \alpha \in \mathbb{R})(\forall v \in V)[\|\alpha v\| = |\alpha| \; \|v\|]$

(iii) (Triangle Inequality) $(\forall v, w \in V)[\|v + w\| \leq \|v\| + \|w\|]$

If there is a norm $\|\cdot\|$ on $V$, we refer to $(V, \|\cdot\|)$ as a **real normed space**.

We next formulate convergence and Cauchy sequences in any normed space:

## Definition

Let $(V, \|\cdot\|)$ be a real normed space. Let $v_n$ be a sequence in $V$, and let $v \in V$.

(i) We say that the sequence $v_n$ converges in norm to $v$ provided $\|v_n - v\| \to 0$, i.e.

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies \|v_n - v\| < \varepsilon].$$

(ii) We say that the sequence $v_n$ is a Cauchy sequence provided the following is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \implies \|v_m - v_n\| < \varepsilon].$$

**Theorem** In any normed space, if $v_n$ converges to $v$ in norm, then $v_n$ is a Cauchy sequence.

**Exercise.**

Write the proof of the above theorem.

- Once again we look at the converse of the above result, which would say that every Cauchy sequence converges. This is false in general; there are lots of normed spaces for which it fails.
- We give a special name to the normed spaces for which it is true.

**Definition**

We say that a normed space is ***complete*** provided every Cauchy sequence is convergent in norm. A complete normed space is referred to as a ***Banach space***.

Exercise.

a) What is an example of a Banach space which we've so far studied in this course?
b) What is an example of a normed space which is not a Banach space?

In section 2.4 we saw that $C[a, b]$ with the sup norm is an example of a normed space. We now show that it has the stronger property of being a Banach space.

**Theorem** Let $[a, b]$ be a closed bounded interval. Let $C[a, b]$ be the normed space of continuous real-valued function on $[a, b]$, equipped with the sup norm. Then $C[a, b]$ is a Banach space.

Some hints on the proof:

- We must give ourselves a sequence $f_n$ which is Cauchy relative to the sup norm.
- Do you see why it's true that for each $x \in [a, b]$, we have $f_n(x)$ is a Cauchy sequence of real numbers?
- Why does this allow us to associate a new real number which we will call $f(x)$?
- Now try to prove that the sequence $f_n$ converges in norm to $f$.
- How do you know that $f \in C[a, b]$?

**Exercise.**

Write the proof of the above theorem.

# Something to think about

- Earlier in this section we wrote down an example of a finite dimensional Banach space and a finite dimensional normed space which is not a Banach space.
- We just proved that $C[a, b]$ with the sup norm is a Banach space, and it is an infinite dimensional Banach space.
- Can you think of an example of an infinite dimensional normed space which is not a Banach space?