- The Extreme Value Theorem deals with the question of when we can be sure that for a given function f,
  (1) the values $f(x)$ don't get too big or too small,
  (2) and f takes on both its absolute maximum value and absolute minimum value.
- We'll see that it gives another important application of the idea of compactness.

Definition

A real-valued function f is called **bounded** if the following holds:

$$(\exists m, M \in \mathbb{R})(\forall x \in D_f)[m \leq f(x) \leq M].$$

If in the above definition we only require the existence of M then we say f is **upper bounded**; and if we only require the existence of m then say that f is **lower bounded**.

Exercise

Phrase boundedness using the terms supremum and infimum, that is, try to complete the sentences

"f is upper bounded if and only if ......"

"f is lower bounded if and only if ......"

"f is bounded if and only if ......"

using the words supremum and infimum somehow.

Some examples

Exercise

Give some examples (in pictures) of functions which illustrates various things:

a) A function can be continuous but not bounded.

b) A function can be continuous, but might not take on its supremum value, or not take on its infimum value.

c) A function can be continuous, and does take on both its supremum value and its infimum value.

d) A function can be discontinuous, but bounded.

e) A function can be discontinuous on a closed bounded interval, and not take on its supremum or its infimum value.

- You might notice that the above negative examples involving continuous functions all have domains which are not closed bounded intervals. This suggests that compactness has something to do with it.

- So a function can be continuous and bounded, yet if $M = \sup\{f(x) : x \in D_f\}$ and $m = \inf\{f(x) : x \in D_f\}$, there may be no x value in $[a, b]$ such that $f(x) = M$ or $f(x) = m$.

Definition

Let f be a real-valued function with some domain D_f. Let $M = \sup\{f(x) : x \in D_f\}$ and $m = \inf\{f(x) : x \in D_f\}$. If there exists $x \in D_f$ such that $f(x) = M$, we say that f has a maximum value, and if there exists $x \in D_f$ such that $f(x) = m$, we say that f has a minimum value.

- The Extreme Value Theorem gives two conditions (i.e. hypotheses in the theorem) which together guarantee that a given function has both a maximum and minimum value. The conditions involve
  (i) a continuity assumption on f
  (ii) a compactness assumption on D_f.
- The examples on the previous slide illustrate that both of these conditions are needed in order to get a theorem.

Theorem (Extreme Value Theorem)

Let f be a real-valued continuous function with domain a closed bounded interval $[a, b]$. Then f is bounded, and f has both a maximum and minimum value on $[a, b]$.

- This theorem is one of the most important of the subject.
- The proof will make use of the Heine-Borel theorem, the Bolzano-Weierstrass theorem, and the sequential characterization of continuity.

- First observe that if we were able to prove the partial result that any such f must be upper bounded and it has a maximum value, then we could deduce from this that any such f must be lower bounded and have a minimum value.
- The proof of this partial result splits separately into two parts:
  1. Proving f is upper bounded;
  2. Proving f has a maximum value.

1. How to prove f is upper bounded?

- This uses the fact that f isn't merely continuous, but even uniformly continuous.
- We write down what the uniform continuity of f says when you take $\varepsilon = 1$ in the definition of uniform continuity. The resulting δ which we get from this is an important tool which we make use of in the next step.
- We put an open interval of radius δ about each point of $[a, b]$. This family of open intervals gives an open cover of $[a, b]$ from which we can extract a finite subcover.
- We use the centers of the intervals in the finite subcover to reduce the proof of upper boundedness of f to checking the biggest value of $f(x)$ as x varies over those finitely many centers.

2. How to prove f has a maximum value?

- From the first part of the proof, we know that $\sup\{f(x) : a \leq x \leq b\} < \infty$.
- By definition of *supremum*, we can find a sequence $\{x_n\}_{n=1}^{\infty}$ in $[a, b]$ such that $f(x_n)$ converges to $\sup\{f(x) : a \leq x \leq b\}$.
- If we knew that $\{x_n\}_{n=1}^{\infty}$ converged to a point x in $[a, b]$, we could deduce from the continuity of f that $f(x_n)$ converges to $f(x)$, and so we would be done.
- However, $\{x_n\}_{n=1}^{\infty}$ need not converge, but we can deduce that a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of it converges to a point x of $[a, b]$, and that is just as good.

Theorem (Extreme Value Theorem)

Let f be a real-valued continuous function with domain a closed bounded interval $[a, b]$. Then f is bounded, and f has both a maximum and minimum value on $[a, b]$.

Exercise.

Write the proof of the Extreme Value Theorem.

Exercise

Prove that $f(x) = x^2 - x + 1 + \cos x$ has a minimum value on $\mathbb{R}$.

Hints:

(i) *Begin by completing the square.*

(ii) *You can't immediately make use of the EVT because $\mathbb{R}$ isn't a closed bounded interval. However, you can reduce the problem to a problem on a closed bounded interval using the expression for $f(x)$ in (i).*

Corollary of the Extreme Value Theorem

Let f be a real-valued continuous function with domain a closed bounded interval $[a, b]$. Then the range of f is a closed bounded interval.

Exercise.

Write the proof of this corollary.

Definition

Let $a < b$ be real numbers. We denote by $C[a, b]$ the set of real-valued continuous functions with domain $[a, b]$.

- We are going to recognize some additional structures on $C[a, b]$ that it has in common with $\mathbb{R}$.
- We see that it has a "vector space" structure (which means we can do arithmetic with it), but we will see on the next few slides that it also has a structure which allows us to talk about convergence in the way we did so with $\mathbb{R}$.
- The idea of using these two different kinds of structures (vector space and topological) to study an object is what one does in the subject of functional analysis.

Definition

Let f be any real-valued function on some domain D. We define the "sup-norm" of f by

$$\|f\|_\infty = \sup\{|f(x)| : x \in D\}.$$

Note: Our textbook refers to the sup-norm of f instead as $\|f\|_{sup}$. One sees both notations used, but $\|f\|_\infty$ is more common (and easier to type!).

- Note that $\|f\|_\infty < \infty$ if and only if f is bounded.

Theorem Let f, g be bounded functions on $[a, b]$, and let α be a real number. Then

1. $\|f\|_\infty = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.
2. $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$.
3. (triangle inequality) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Theorem Let f, g be bounded functions on $[a, b]$, and let α be a real number. Then

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3. (triangle inequality) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

- (1) and (2) follow quite easily from the definition of supremum.
- For (3), try to show that the right side is an upper bound of the set of values $\{|f(x) + g(x)| : x \in [a, b]\}$.

Exercise.

Write the proof of the theorem.

- The set of **bounded real-valued functions** on a fixed interval $[a, b]$ forms what is called a **vector space**. This means that if f, g are any two such functions, then $f + g$ is such a function, αf is such a function for any real number α, and furthermore basic rules of arithmetic hold (these rules are described on page 59 of the text, we won't list them here).
- Similarly if we look instead at the **continuous real-valued functions** on $[a, b]$, these also form a vector space. Do you see why? What property of continuity is needed here?
- The three properties described on the previous slide are worth abstracting and giving a name. We do that next.

Definition

Let V be a real vector space. Let $\|\cdot\|$ denote a real-valued function on V whose value at $v \in V$ is written as $\|v\|$. We say that $\|\cdot\|$ is a **norm** if the following three properties hold:

(i) (Positive Definite) $(\forall v \in V)(\|v\| \geq 0$ and $(\forall v \in V)[\|v\| = 0 \Leftrightarrow v = 0]$

(ii) $(\forall \alpha \in \mathbb{R})(\forall v \in V)[\|\alpha v\| = |\alpha| \, \|v\|]$

(iii) (Triangle Inequality) $(\forall v, w \in V)[\|v + w\| \leq \|v\| + \|w\|]$

If there is a norm $\|\cdot\|$ on V, we refer to $(V, \|\cdot\|)$ as a **real normed space**.

Exercise

Write down three different real normed spaces which we've studied.

- So $\mathbb{R}$ and $C[a, b]$ share the property that they both form real normed spaces.
- There are many other examples of real normed spaces you will see in future.
- One of the themes of the subject of analysis is to formulate and prove things about general normed spaces which one has already studied for the real numbers.
- One can hope to be able to do this because in several of the proofs we have already studied with $\mathbb{R}$, it isn't always the special properties of $\mathbb{R}$ that we use, but instead just the three norm properties (think of our use of the triangle inequality) and these also hold in any normed space.
- For example in $C[a, b]$,
  - It is a normed space, so we know how to measure the distance between elements of $C[a, b]$.
  - We can talk about sequences $\{f_n\}_{n=1}^{\infty}$.
  - We can formulate convergence of sequences.
  - We can formulate the idea of Cauchy sequences.
  - We can formulate what it would mean to say $C[a, b]$ is complete, and ask if it is true.
  - We can formulate what is an open subset of $C[a, b]$.
  - We can formulate what would be a dense subset of $C[a, b]$ and if so, are there some natural dense subsets of $C[a, b]$ (analogous to $\mathbb{Q}$ being dense in $\mathbb{R}$)?
  - Even better, are there any countable dense subsets of $C[a, b]$?
  - We can formulate the idea of compactness in $C[a, b]$.