In this section we look at some properties, some quite deep, shared by all continuous functions. They are known as the following:

1. Preservation of sign property
2. Intermediate Value Property
3. Uniform continuity property

1. Preservation of sign property:
    - Roughly says that if we know a function is continuous at a point and positive at that point, then it must be positive on some open interval containing that point.
    - This is a local property in that it doesn't require continuity anywhere other than at that point.
    - We will give the proof below in an exercise.
    - It is a very useful result, but not too deep in that it follows quite easily from the definition of continuity.

Before we comment on the Intermediate Value Property, do the following exercise.

Exercise.

The sets $(-\infty, 10)$, $(\infty, 10]$, $(1, 4)$, $[1, 7]$, $[1, \infty)$ are all intervals (and we could have included other kinds of intervals on this list). What property characterizes all of them? In other words, define the term "interval".

2. Intermediate Value Property:
   - This one applies to functions with domain an interval.
   - It says that if the function is continuous on some interval, then for any subinterval $I = (a, b)$ of the domain, the function takes on all values from $f(a)$ to $f(b)$.
   - Essentially it says that the continuous image of an interval is also an interval.
   - This is a deep property (which means it cannot be proven without making use of the completeness axiom of $\mathbb{R}$).
   - It is also one of the important theorems of the course.

3. Uniform continuity:

- Uniform continuity is a stronger statement about the function than it is merely continuous on its entire domain.
- Roughly it says that for any x_1, x_2 in the domain of f we can force $f(x_1)$ and $f(x_2)$ to be at most a certain prescribed distance ε apart by taking x_1 and x_2 in the domain close enough together as measured by a certain number δ, regardless of where we are in the domain. The point is that the same δ works uniformly on the domain.
- This is asking a lot of the function, so we have to pay the price by **insisting the domain is of a special sort**. It turns out what works is that the domain is a closed, bounded interval $[a, b]$. This is a deep theorem, and makes great use of the **compactness** of such intervals.

Theorem 1 (Preservation of sign)

Let f be a function and let $p \in D_f$. Suppose f is continuous at p. If $f(p) > 0$, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $x \in D_f \cap I_\delta(p)$, then $f(x) > 0$.

Exercise : Comments on the theorem and its proof

a) Intuitively, why do we expect it to be true?

b) If we weaken the continuity hypothesis, give a counterexample to show the resulting statement is false.

c) The continuity of f at p says what precisely? Use the $\varepsilon - \delta$ definition of continuity.

d) Are we making use of the continuity definition or are we proving that it holds here? Why?

e) If we could guarantee that $f(x) > f(p)/3$ for all x in some open interval centered at p, we would be done. So what ε should we work with in the $\varepsilon - \delta$ definition of continuity?

f) Write the proof of the theorem.

Theorem 2 (Intermediate Value Theorem)

Let I be an interval and f a function whose domain contains I. If f is continuous, then for all $a, b \in I$ with $a < b$ and all real numbers k, if k is strictly between $f(a)$ and $f(b)$, then there exists c such that $a < c < b$ and $f(c) = k$.

Exercise: Comments on the IVT and its proof

a) Draw some graphs of functions to illustrate what the theorem is saying.

b) Illustrate by sketch some counterexamples which show why we have the hypothesis that f is continuous.

c) Why is it sufficient to prove the result in case $f(a) < f(b)$?

d) Explain how to find a sequence of closed intervals $I_n = [a_n, b_n]$ such that
   - (i) $I_1 = [a, b]$
   - (ii) for all n, I_{n+1} is either the left half or right half of I_n
   - (iii) For all n, $f(a_n) \leq k \leq f(b_n)$

e) What does the Nested Intervals Theorem tell you about the sequence a_n and b_n?

f) What can you say about the sequences $f(a_n)$ and $f(b_n)$? Make the most of the continuity assumption of f.

g) Write the proof of the Intermediate Value Theorem.

Exercise: Existence of nth roots

Let k be a positive real number, and let $n \in \mathbb{N}$.

a) What is the formal definition of $k^{1/n}$, i.e. what does it mean to say a real number x satisfies

$$x = k^{1/n}?$$

b) Prove that the real number $k^{1/n}$ exists by making appropriate use of the Intermediate Value Theorem.

Hint: The IVT allows you to use two inequalities to deduce that a desirable equality can be achieved.

- Recall that f continuous means it is continuous at each $x_0 \in D_f$, i.e.

$$(\forall x_0 \in D_f)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[x \in D_f \text{ and } |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon].$$

- So the hard part is that once you have been given $\varepsilon > 0$, you have to figure out how to choose the δ so that once $x \in D_f$ is within δ of x_0, then $f(x)$ is within ϵ of $f(x_0)$.
- So we have to see how close we need to make x to x_0 (as measured by δ) in order to force $f(x)$ to be within ε of $f(x_0)$ (as prescribed by ε).
- It's possible that some x_0's would require a much smaller δ than others.
- The question is whether or not the same δ can be used for all $x_0 \in D_f$ simultaneously. If it can, we say f is uniformly continuous on D_f.

Definition

Let f be a real-valued function on a set D_f. We say that f is **uniformly continuous** if the following holds:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_1 \in D_f)\forall x_2 \in D_f)[|x_2 - x_1| < \delta \Longrightarrow |f(x_2) - f(x_1)| < \varepsilon].$$

- Note the difference between mere continuity and uniform continuity is just the order in which "$\forall x_1 \in D_f$" appears in the definition. But it makes a great difference.

Exercise

Consider the function $f(x) = 1/x$, $0 < x < 1$.

a) Illustrate by sketch why you believe f is not uniformly continuous.

b) Prove f is not uniformly continuous.

Fundamental Theorem on Uniform Continuity

Theorem Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

- This is an important result and often applied in real analysis books.
- The text has a nice proof using the Bolzano-Weierstrass theorem, so you should be sure to work through that proof.
- I'm going to show another proof which seems to me is more natural, and is very typical of so-called "compactness arguments".
- The idea is that given $\varepsilon > 0$, the statement of continuity at a single point $p \in D_f$ produces an open interval centered at p. As p varies over D_f, the set of all such open intervals gives an open cover of $[a, b]$.
- We can then pass to a finite subcover (why?), and this finite subcover allows us to produce a single δ that does what we'd like for the given ε (i.e. it allows us to confirm uniform continuity).

Exercise.

Write a proof of the theorem.