- Shortly we will give the formal definitions concerning continuity of a functions. In the definitions, we distinguish between
  - (i) what it means that a given function is continuous at a point of its domain
  - (ii) and what it means that a given function is continuous on its domain.
- Points "a" of the domain are either cluster points or not cluster points.
- Cluster points "a" have the property that there are sequences in $D_f \setminus \{a\}$ converging to a;
- Points of D_f which are not cluster points cannot be approached in $D_f \setminus \{a\}$, so they are also referred to as **isolated points** of D_f, i.e. they have the property that

$$(\exists \delta > 0)[D_f \cap (a-\delta, a+\delta) = \{a\}].$$

Exercise.

If $D_f = ((0,4] \setminus \{2\}) \cup \{10\}$, then what are the cluster points and what are the isolated points of D_f?

Definition

Let f be a real-valued function with domain D_f. Let $a \in D_f$. We say that f **is continuous at** a **provided**

(i) a **is an isolated point of** D_f **or**

(ii) a **is a cluster point of** D_f **and** $\lim_{x \to a} f(x) = f(a)$.

Exercise.

Recall we have a formulation of the statement $\lim_{x \to a} f(x) = f(a)$ in terms of sequences provided a is a cluster point of D_f. We claim that f is continuous at $a \in D_f$ if and only if for every sequence x_n in D_f, if $x_n \to a$ then $f(x_n) \to f(a)$.
Prove that this sequential characterization of continuity works even if a is an isolated point of D_f.

Definition

Let f be a real-valued function with domain D_f. Let $a \in D_f$. We say that **f is continuous at a provided**

(i) **a is an isolated point of D_f or**

(ii) **a is a cluster point of D_f and $\lim_{x \to a} f(x) = f(a)$.**

Exercise.

Writing down the ε, δ definition of limit gives a third way to say that f is continuous at $a \in D_f$, namely f is continuous at $a \in D_f$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}[x \in D_f \cap (a - \delta, a + \delta) \Longrightarrow |f(x) - f(a)| < \varepsilon].$$

Explain why this definition automatically predicts continuity at any isolated point a of D_f.

So to recap, we have

Three Equivalent formulations of continuity at a point $a \in D_f$

1. (ε, δ formulation of continuity) f is continuous at $a \in D_f$ provided

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[x \in D_f \cap (a - \delta, a + \delta) \Longrightarrow |f(x) - f(a)| < \varepsilon].$$

2. (sequential criterion of continuity) f is continuous at $a \in D_f$ provided for every sequence x_n in D_f, if $x_n \to a$ then $f(x_n) \to f(a)$.

3. (limit formulation of continuity) f is continuous at $a \in D_f$ provided
   (i) a is an isolated point of D_f or
   (ii) a is a cluster point of D_f and $\lim\limits_{x \to a} f(x) = f(a)$.

- We are free to use any one of the above three characterizations in making use of the definition, any one which is most convenient for us.

- In the following definition, we distinguish between a function being continuous at a particular point a in D_f and f just simply being continuous.

Definition of continuity

Let $A \subseteq D_f$. We say that f is continuous on A if the following holds:

$$(\forall a \in A)[f \text{ is continuous at } a].$$

If f is continuous on D_f, we simply say "f is continuous".

- We often consider the set of continuous functions on an interval I. We let $C(I)$ denote the set of continuous functions on I, and if we write I for example as $[a, b]$, we simply write $C[a, b]$ (leaving out the additional parentheses).

- We'll find the sequential characterization of continuity particularly useful since we've already seen some useful theorems concerning sequences.

Exercise.

a) The simplest example of a continuous function is $f(x) = x$. Prove the continuity of it in the most elementary proof you can think of.

b) What set of functions can we generate from $f(x) = x$ using the operations of addition, multiplication, and division? If you were to try to prove this rigorously, what theorem would you need to use?

c) In order to prove continuity of the functions in (b), what theorem should we be motivated to try to prove?

The sequential characterization of continuity and theorems proved in the previous section make the following theorem immediate:

Theorem (Arithmetic properties of continuity)

Let f and g be continuous at $a \in D_f \cap D_g$. Then we have each of the following:

1. $f \pm g$ is continuous at a.
2. $f \cdot g$ is continuous at a.
3. f/g is continuous at a provided $g(a) \neq 0$.

The following are some examples you should work through:

Examples.

1. Our text has several examples of continuity on page 48.
2. On page 49 in the exercises are several examples of continuity to work through (2.20, 2.21, 2.23, 2.24, 2.25).
3. Problem 2.26 shows why monotone increasing functions are continuous except possibly on countable sets. It's an important result, and one you should be aware of.

Exercises.

Define $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$

a) Draw the graph of f (if it's not too much trouble).

b) For which x values do you believe f is continuous?

c) Prove your claim in (b) above.

Exercise

Each positive rational number can be written uniquely in the form m/n, where m and n have no factors in common. For the following function, we agree to write each positive rational in this way. Define

$$f : (0,1) \to \mathbb{R} \text{ by } f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \dfrac{1}{n} & \text{if } x \text{ rational of the form } \dfrac{m}{n} \end{cases}$$

a) Prove that f is discontinuous at every rational in $(0,1)$.

b) Prove that f is continuous at every irrational in $(0,1)$.
   Hints for question 2:
   (i) Start with $x_0 \in (0,1) \setminus \mathbb{Q}$ and $\varepsilon > 0$. You must produce $\delta > 0$ such that for all x, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = f(x) < \varepsilon$.
   (ii) Let $E_\varepsilon := \left\{ \dfrac{m}{n} \in (0,1) : \dfrac{1}{n} \geq \varepsilon \right\}$. How big a set is E_ε? Can you count how many elements there are in it?
   (iii) Is there a number $\min \left\{ \left| \frac{m}{n} - x_0 \right| : \frac{m}{n} \in E_\varepsilon \right\}$? How do you know?
   (iv) If you took δ to be $1/2$ of the minimum on the previous line, does it do what we'd like it to do?