- Intuitively a function is continuous provided you can draw its graph without lifting your pencil.
- But what to do if we have a function whose graph cannot be drawn? How to decide if it is continuous?

For example, how to decide continuity of each the following functions?

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.
2. Each positive rational number can be written uniquely in the form m/n, where m and n have no factors in common. For the following function, we agree to write each positive rational in this way.

   Define $g : (0,1) \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \dfrac{1}{n} & \text{if } x \text{ rational of the form } \dfrac{m}{n} \end{cases}$
3. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(12^n x)}{2^n}$

- These functions all exist, yet there is no way we can draw any of their graphs.
- So how can we decide whether or not they are continuous?

Or how to answer questions like the following?

1. Does there exist $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational number?

2. Does there exist $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational number and discontinuous at every irrational number?

- These examples make the case that we need a precise definition of continuity.

- The right way to define continuity is to make use of limits, so we begin by defining what we mean by

$$\lim_{x \to a} f(x) = L.$$

- In section 2.2 we will discuss the connection of limits and continuity.
- You might expect the definition of "$\lim_{x \to a} f(x) = L$" to be

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon].$$

- But the definition must have something to do with the domain of f, and even though we may not consider here domains other than intervals, we'd like to have the flexibility to define limits of functions with more complicated domains than just intervals.
- In addition we'd like to be able to take limits at points which can be approached by points of the domain of f.
- For this reason we introduce the idea of a **cluster point** of a set.

Definition

Let D be a nonempty subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. We say that "a" is a **cluster point** of D provided the following is true:

$$(\forall \delta > 0)(\exists x \in D \setminus \{a\})[0 < |x - a| < \delta].$$

Equivalently, "a" is a cluster point of D provided there exists a sequence x_n in $D \setminus \{a\}$ such that $x_n \to a$.

- Before we look at some examples, do the following exercise.

Exercise

Say x_n is a sequence of real numbers converging to the real number L. Suppose that a, b are numbers such that $a \leq x_n \leq b$ for all n. Prove (using only the definition of convergence) that $a \leq L \leq b$.

Definition

Let D be a nonempty subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. We say that "a" is a **cluster point** of D provided the following is true:

$$(\forall \delta > 0)(\exists x \in D \setminus \{a\})[0 < |x - a| < \delta].$$

Equivalently, "a" is a cluster point of D provided there exists a sequence x_n in $D \setminus \{a\}$ such that $x_n \to a$.

Exercise

Find all the cluster points of the following sets. Prove you are correct.

1. $A = \{0, 1, 2, 3, 4, 5\}$
2. $B = (0, 10)$
3. $C = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$.
4. Can you generalize the result in the previous part?
5. $\mathbb{Z}$
6. $\mathbb{Q}$.

Definition

Let f be a real-valued function with domain denoted by D_f. Let $a \in \mathbb{R}$ be a cluster point of D_f. Let L be a real number. Then we define $\lim_{x \to a} f(x) = L$ by

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D_f)[0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon].$$

- Since we know quite a bit about sequences, it would be convenient to characterize the definition of limit in terms of convergence of certain sequences.

Theorem. Let f be a real-valued function on some domain $D_f \subseteq \mathbb{R}$, let $a \in \mathbb{R}$ be a cluster point of D_f, and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim_{x \to a} f(x) = L$;

(ii) For every sequence x_k in $D_f \setminus \{a\}$, if $x_k \to a$, then $f(x_k) \to L$.

Comments on the proof

$\Longrightarrow$: This is done with a direct proof using the working definitions of $\lim_{x \to a} f(x) = L$ and convergence of sequences. But be sure you are clear on what you are assuming to be true and what you are trying to prove.

$\Longleftarrow$: Our textbook proves this direction by the method of contradiction. But I think it's just as easy to do it using contraposition, i.e. assuming that (i) is false, see if you can prove that (ii) is false.

Theorem. Let f be a real-valued function on some domain $D_f \subseteq \mathbb{R}$, let $a \in \mathbb{R}$ be a cluster point of D_f, and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim_{x \to a} f(x) = L$;

(ii) For every sequence x_k in $D_f \setminus \{a\}$, if $x_k \to a$, then $f(x_k) \to L$.

Exercise.

Write the proof of the above theorem.

The sequential characterization of limits makes it easy to prove the following result.

Theorem. Let f, g be two real-valued functions with domains D_f, D_g. Let $a \in \mathbb{R}$ be a cluster point of $D_f \cap D_g$. Suppose that L and M are numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then

(i) $\lim_{x \to a} f(x) + g(x) = L + M$,

(ii) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

(iii) $\lim_{x \to a} f(x)/g(x) = L/M$ provided $M \neq 0$ and a is a cluster point of $D_{f/g}$.

Exercise.

Write the proof of the above theorem.

Definition

Let f be a real-valued function with domain D_f, and let L be a real number.

(i) (Left-sided limit) Suppose that $a \in \mathbb{R}$ is a cluster point of $D_f \cap (-\infty, a)$. We write that $\lim_{x \to a^-} f(x) = L$ provided for every sequence x_n in $D_f \cap (-\infty, a)$, if $x_n \to a$ then $f(x_n) \to L$.

(ii) We have a similar definition for right-sided limits.

(iii) (Finite limit at ∞) Suppose that D_f is not bounded above. We write that $\lim_{x \to \infty} f(x) = L$ provided for every sequence x_n in D_f, if $x_n \to \infty$ then $f(x_n) \to L$.

Theorem (Monotone Sequence Characterization of Limits)

Let f be a real-valued function with domain D_f, and let L be a real number.

(i) Let $a \in \mathbb{R}$ be a cluster point of $D_f \cap (-\infty, a)$. Suppose that for every **increasing** sequence x_n in $D_f \cap (-\infty, a)$, if $x_n \to a^-$ then $f(x_n) \to L$. Then $\lim_{x \to a^-} f(x) = L$.

(ii) Let $a \in \mathbb{R}$ be a cluster point of $D_f \cap (a, \infty)$.Suppose that for every **decreasing** sequence x_n in D_f, if $x_n \to a^+$ then $f(x_n) \to L$. Then $\lim_{x \to a^+} f(x) = L$.

- In order to prove this we will make use of the following theorem, which perhaps seems obvious to you.

Theorem (Monotone Subsequence Theorem)

Every sequence of numbers has a monotone subsequence.

Exercise.

Write the proof of the Monotone Subsequence Theorem and then use it to write the proof of the Monotone Sequence Characterization of Limits theorem.