- This section gives another application of the interval halving method, this time to a particularly famous theorem of analysis, the *Heine – Borel Covering Theorem*.
- It also introduces two very important kinds of sets, namely *open sets* and *compact sets*.
- The Heine-Borel theorem says that closed bounded intervals $[a, b]$ are examples of compact sets.
- The concept of open set is what is needed in order to define convergence and to formulate the idea of continuity.
- One can formulate the definition of open set in other settings where various notions of convergence are needed:
  - For example, it is formulated in $\mathbb{R}^n$ in order to study multivariable calculus.
  - In a branch of mathematics known as "functional analysis" where we study sets of functions, we're interested in convergence of sequences of functions, so one requires a notion of open set in that setting.
- The compact sets are typically infinite, but they have a property in common with finite sets with very far-reaching applications.

Notation

Let $x \in \mathbb{R}$ and let $r > 0$. The notation $I_r(x)$ refers to the open interval centered at x of radius r, that is

$$I_r(x) := \{y \in \mathbb{R} : |y - x| < r\} = \{y \in \mathbb{R} : -r < y - x < r\} = (x - r, x + r).$$

Exercise.

Consider the set $U = \{x \in \mathbb{R} : 7 < x < 16\}$.

a) Sketch this set on a number line.

b) Identify this set using the "I" notation of the above definition.

c) Now identify $I_7(1.3)$ using set-builder notation and then using interval notation. Sketch it on a number line.

Definition

Let O be a subset of $\mathbb{R}$. We call O an **open set** if for each x in O there exists an open interval centered at x which is contained in O. Thus O is open provided

$$(\forall x \in O)(\exists r > 0)[I_r(x) \subseteq O].$$

- Note that the r in the above definition will usually depend on the given x.

Exercise.

Let $a, b \in \mathbb{R}$ such that $a < b$.

a) Prove that $\{4\}$ is not an open set.

b) Prove that the closed interval $[a, b]$ is not open.

c) Prove that the open interval (a, b) is an open set.

- A similar argument as the one used in the exercise shows that half open intervals like $(-\infty, b)$ and (a, ∞) are also open sets.

Definition: Indexed families of sets and their union

- Let A be any nonempty set (the "indexing set") and for $\alpha \in A$, say we have a set U_α which is a subset of $\mathbb{R}$. We call
$$\mathscr{U} = \{U_\alpha : \alpha \in A\}$$
a family of sets indexed by A.
- The union of all of the sets in the indexed family $\mathscr{U}$ is defined by
$$\bigcup_{\alpha \in A} U_\alpha := \{x \in \mathbb{R} : (\exists \alpha \in A)[x \in U_\alpha]\}.$$

- Note that a typical element of $\mathscr{U}$ is an entire subset of $\mathbb{R}$,
- whereas a typical element of the union of the elements of $\mathscr{U}$ is a real number (since the union is a subset of $\mathbb{R}$),
- so $\mathscr{U}$ and the union of the elements of $\mathscr{U}$ are very different kinds of objects.

Exercise.

Define the following collection of sets:

$$U_1 = \{1, 3\},\ U_2 = \{1/2, 7, 0\},\ U_3 = \{9, 10, 11\},\ U_4 = \{1, 7, 10, 15\}.$$

Let's view this as an indexed collection of sets $\mathscr{U}$. In the following, make sure to use correct set notation (in this case, listing notation).

a) Identify the indexing set A.

b) Identify the family of sets which we've called $\mathscr{U}$.

c) Identify $\bigcup_{i \in A} U_i$.

d) Identify an element in $\mathscr{U}$, any element will do.

e) Identify any element in $\bigcup_{i \in A} U_i$.

Definition of cover

Let S be a subset of $\mathbb{R}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an indexed family of sets. We say that **the family of sets is a cover of S (or that the family covers S)** provided

$$S \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

Exercise.

Look again at the family $U_1 = \{1, 3\}$, $U_2 = \{1/2, 7, 0\}$, $U_3 = \{9, 10, 11\}$, $U_4 = \{1, 7, 10, 15\}$ of the previous exercise.

a) Describe what are all the sets S which are covered by this family $\{U_1, U_2, U_3, U_4\}$.

b) Write down a few different sets S which are covered by this family of sets.

c) Write down a subset of $\mathbb{R}$ which is not covered by this family of sets.

d) Write down a subset S of $\mathbb{R}$ which is covered by $\{U_1, U_2, U_3, U_4\}$, but which is also covered by $\{U_1, U_2, U_3\}$.

- In part (d) of the previous exercise, we refer to $\{U_1, U_2, U_3\}$ as being a **subcover** of the cover $\{U_1, U_2, U_3, U_4\}$ because it is a subset of the original cover $\{U_1, U_2, U_3, U_4\}$ and it is also a cover of the set S written down in that exercise.

Definition

Let S be a subset of $\mathbb{R}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an indexed family of sets which is a cover of S. Let $B \subseteq A$, i.e. B is a subset of the indexing set A. If it is the case that $\mathscr{U} = \{U_\alpha : \alpha \in B\}$ is also a cover of S, then we say that $\mathscr{U} = \{U_\alpha : \alpha \in B\}$ is a **subcover** of the cover $\mathscr{U} = \{U_\alpha : \alpha \in A\}$.

Exercise.

a) Write down a specific cover of $S = \mathbb{R}$ consisting of finitely many open intervals.

b) Write down a specific cover of $\mathbb{R}$ consisting of countably many open intervals such that this cover does not have a subcover consisting of finitely many intervals.

Definition

Let S be a subset of $\mathbb{R}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an indexed family of sets which is a cover of S. We call it an **open cover** provided all of the sets U_α which make up the cover are open sets.

- So the covers in the previous exercise were open covers (since they consisted of open intervals, and as we showed earlier, open intervals are open sets).

- In this section, we are particularly interested in
  - open covers of sets
  - the number of elements in the indexing set of that open cover

Definition

For an open cover $\{U_\alpha : \alpha \in A\}$ of a set $S \subseteq \mathbb{R}$, let's call it

- **a finite open cover** of S if A is a finite set;
- **a countable open cover** of S if A is a countable set;
- **an uncountable open cover** of S if A is an uncountable set.

Exercise.

Write down specific open covers of $\mathbb{R}$ of the following types:

a) A finite open cover

b) A countable open cover that does not have a finite subcover

c) A countable open cover that does have a finite subcover

d) An uncountable open cover

Theorem. Let $\{U_\alpha : \alpha \in A\}$ be a nonempty family of open sets (where the indexing set A can have any cardinality whatsoever) . Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set.

Exercise.

Prove the above theorem.

Theorem. A nonempty subset U of $\mathbb{R}$ is open if and only if it is a union of a family of nonempty open intervals.

Exercise.

Prove the above theorem.

Let's review the definition of open cover of a set and finite subcover of an open cover of a set:

Open cover of a set

Let S be any subset of $\mathbb{R}$. An **open cover** of S is a family of sets U_α indexed by some set A such that the following hold:

(i) U_α is open for each $\alpha \in A$;

(ii) $S \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Finite subcover of an open cover of a set

Let S be any subset of $\mathbb{R}$ and let $\{U_\alpha : \alpha \in A\}$ be an open cover of S. We say that this open cover **has a finite subcover** if there exists a set B such that the following two things hold:

- B is a finite subset of A;
- $\{U_\alpha : \alpha \in B\}$ is a cover of S.

Exercise.

From an "economics" point of view, explain in words what is the benefit of a given open cover $\{U_\alpha : \alpha \in A\}$ of a set S having a finite subcover?

How do open covers of sets typically arise?

- Let S be any set.
- For each $x \in S$, let U_x be any open set which contains x.
- Then $\{U_x : x \in S\}$ is an open cover of S which is indexed by the points of S.

Why might it be desirable for an open cover of a set to have a finite subcover?

- In the above method of producing open covers, each U_x might arise in an attempt to describe some phenomenon associated with that x value, for example in describing the behavior of some function near x.
- The family of sets in the open cover is making lots of local statements about the behavior of that function.
- But say we would like to make a single global statement about the behavior of the function.
- Then it would be desirable that the above open cover has a finite subcover,
- i.e. that there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ of A such that S is covered by the associated sets $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_N}$, which means that

$$S \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_N}.$$

- We might be able to use this finite family of sets to make a global statement about the given function.

Exercise.

Let S be a nonempty set with the property that every cover of S has a finite subcover.

a) Does the set $\{1, 2, 3, \ldots, 1000\}$ have this property?

b) Does the set $[0, 1]$ have this property?

c) What kind of set must S be if it has the above property?

In the above exercise, if we modify the condition so that we allow only open covers, we arrive at the definition of a compact set:

Definition

A subset S of $\mathbb{R}$ is called **compact** provided every open cover of S has a finite subcover. This means that for any open cover $\{U_\alpha : \alpha \in A\}$ of S, there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ of A such that $S \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_N}$.

- For any set S, we can always get a cover of S by simply taking $\{\{x\} : x \in S\}$.
- But this is not an open cover of S, because singleton sets are not open subsets of $\mathbb{R}$.
- In the definition of compactness, we're "fattening" up the sets in this particular cover of S by insisting that they be open sets.

Exercise.

a) Prove that $\{1, 2, 3\}$ is compact.

b) Prove that $\mathbb{R}$ is not compact.

c) Prove that $(0, 1)$ is not compact.

- A similar argument as in (a) above shows that any finite set is compact.
- After doing the above exercise, you might wonder if there exist any infinite subsets of $\mathbb{R}$ which are compact.
- The point of the Heine-Borel theorem is that it shows that there are lots of infinite sets which are compact.

Theorem (Heine-Borel). Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed bounded interval $[a, b]$ is a compact set.

- So the theorem says that any open cover of $[a, b]$ must have a finite subcover.
- Even though the statement is very elegant and simply stated, the proof of it is nontrivial (i.e. does not follow easily from the definitions of compact and closed interval).
- It is another nice application of the interval halving method.
- If I describe the main ideas of the proof, you might be able to come up with it yourself.

Comments on the proof

- We have to give ourselves an open cover $\{U_\alpha : \alpha \in A\}$ of $[a, b]$, and we must deduce that there is a finite subcover, i.e. finitely many of the sets in the open cover already cover $[a, b]$.
- The proof is by contradiction. We assume no finite subcover exists, and we deduce a contradiction.
- We use the method of interval halving. If $I_1 := [a, b]$, consider the left half and the right half of I_1. At least one of those two intervals also doesn't have a finite subcover of the original cover. Let I_2 be either the left or right half, choosing it so that it doesn't have a finite subcover taken from the original open cover.
- Continuing in this way (using induction), we get an infinite decreasing sequence of intervals I_n none of which has a finite subcover of the original open cover.
- Apply the Nested Intervals Theorem to obtain an x in the intersection of all of these intervals I_n.
- That x must be in at least one of the sets U_α, since they cover $[a, b]$.
- Use that fact that U_α is open to show that one of the intervals I_n is entirely contained in U_α (if n is chosen large enough). Do you see why this is the contradiction we were looking for?

Theorem (Heine-Borel). Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed bounded interval $[a, b]$ is a compact set.

Exercise.

Write the proof of the Heine-Borel theorem.