Theorem (Heine-Borel) Closed bounded intervals are compact.

Proof. We argue by contradiction. Suppose $[a,b]$ is a closed and bounded interval which is not compact. Then there exists a family of sets $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that

1. $[a,b] \subseteq \bigcup_{\alpha \in A} U_\alpha$;
2. for all $\alpha \in A$, U_α is an open set;
3. for any finite subset B of A, $[a,b]$ is **not** a subset of $\bigcup_{\alpha \in B} U_\alpha$.

We now inductively construct a sequence of intervals $\{I_n\}_n = \{[a_n, b_n]\}_n$ with the following properties:

1. $I_1 = [a,b]$;
2. For each n, I_{n+1} is either the left or the right half of I_n;
3. For each n, I_n is not a subset of $\bigcup_{\alpha \in B} U_\alpha$ for any finite subset B of A.

Basis step. Choose $I_1 = [a,b]$. Then by assumption, property 3 holds for $n=1$.

Inductive Step Let $n \in \mathbb{N}$ and suppose we have $I_1, \dots, I_n$ satisfying ①, ②, and ③. Look at the left and right halves of I_n. If both of these were covered by finitely many of the sets in $\mathcal{U}$, then so would I_n be covered by finitely many of the sets of $\mathcal{U}$. But by the inductive hypothesis, this does not happen. So we define

$$I_{n+1} = \begin{cases} \text{Left half of } I_n & \text{if the left half cannot be covered by finitely many elements of } \mathcal{U} \\ \text{right half of } I_n & \text{otherwise} \end{cases}$$

This completes the proof of the induction.

By the Nested Intervals Theorem, there exists a unique $x \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

In particular, $x \in [a,b]$, so since $\mathcal{U}$ covers $[a,b]$, there exists $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$.

Since U_{α_0} is open, there exists $r>0$ such that

$$I_r(x) \subseteq U_{\alpha_0}.$$

Since $b_n - a_n \to 0$, we can choose n_0 such that $b_{n_0} - a_{n_0} < r$. I claim that

$$I_{n_0} \subseteq I_r(x).$$

To see why, let $w \in I_{n_0}$. Since also $x \in I_{n_0}$, then

$$|w-x| < b_{n_0} - a_{n_0} < r,$$

so $w \in I_r(x)$. This proves the claim.

Thus $I_{n_0} \subseteq I_r(x) \subseteq U_{\alpha_0}$, and so a single set of $\mathcal{U}$ contains I_{n_0}. This contradicts property 3 of I_{n_0}, and so we are done.

$\square$