1.6 The Nested Intervals Theorem

- We've seen several applications of the interval halving method.
- Starting with a closed interval $I_1 = [a_1, b_1]$, the method produces a sequence of closed intervals $I_n = [a_n, b_n]$, such that for each $n \geq 1$, I_n is either the left half or the right half of I_{n-1}. If we randomly pick a point $x_n \in I_n$ for each n, then the resulting sequence is necessarily a Cauchy sequence.
- Thus by the completeness axiom of $\mathbb{R}$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some real number x.

Exercise.

Prove that $x \in \displaystyle\bigcap_{n=1}^{\infty} I_n$.

- If you look at the proof of the interval halving method, you will see that for the proof to work, it isn't important that each interval is chosen as the right or left half of the preceding interval.
- All that matters is that
  i. They are all closed intervals;
  ii. The intervals are nested, i.e. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$
  iii. The length of the intervals converges to 0, i.e. $b_n - a_n \to 0$.
- This is essentially what the Nested Intervals Theorem says.

Theorem (Nested Intervals Theorem)

Let $I_n = [a_n, b_n]$ be a sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots,$

(ii) $b_n - a_n \to 0$ as $n \to \infty$.

Then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one real number x. Moreover both sequences a_n and b_n converge to x.

- So the theorem says that if the intervals are closed and satisfy (i) and (ii), then the intersection of all of the intervals cannot be empty, and in fact there is exactly one real number x which lies in all of them.

Theorem (Nested Intervals Theorem)

Let $I_n = [a_n, b_n]$ be a sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$,

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Then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one real number x. Moreover both sequences a_n and b_n converge to x.

Comments on the proof of the Nested Intervals Theorem

- There are four things to prove:
  (i) The sequence a_n converges to some number x.
  (ii) The sequence b_n also converges and its limit is x.
  (iii) The number x lies in I_n for every n.
  (iv) There cannot be more than one real number that lies in every one of the sets I_n.
- (i) Do you see why a_n is a Cauchy sequence?
- (ii) Do you see how to use results from section 1.4 to deduce this one?
- (iii) This is very much like the exercise we did on the first slide.
- (iv) Say (in order to obtain a contradiction) that x and y lie in I_n for every n, where $x \neq y$. Using the assumption $x \neq y$, do you see how to produce a useful positive real number from x and y? Do you see how to use that positive real number to obtain a contradiction?

Theorem (Nested Intervals Theorem)

Let $I_n = [a_n, b_n]$ be a sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots,$

(ii) $b_n - a_n \to 0$ as $n \to \infty$.

Then $\displaystyle\bigcap_{n=1}^{\infty} I_n$ consists of exactly one real number x. Moreover both sequences a_n and b_n converge to x.

Exercise.

Use the comments to write the proof of the Nested Intervals Theorem.

Exercise

a) The theorem is false if we replace the condition that the intervals I_n be closed by the condition that they be open. Show a counterexample which shows that the intersection can be chosen to be empty.

b) The theorem is false if we omit the condition that $b_n - a_n \to 0$. Give a counterexample.

Definition

Let S be a subset of $\mathbb{R}$. The set S is said to be **dense** in $\mathbb{R}$ if for every real number x there exists a sequence s_k consisting entirely of elements of S such that $s_k \to x$.

Theorem. $\mathbb{Q}$ is dense in $\mathbb{R}$.

- We will give two proofs of this. The first proof uses the interval halving method, and the second proof is more direct.

Comments on our first proof of the exercise

Let x be a real number.

- We must prove that there is a sequence in $\mathbb{Q}$ which converges to x.
- If $x \in \mathbb{Q}$, why does the result easily follow?
- So assume that $x \notin \mathbb{Q}$.
- Explain why there is an interval $I_1 = [a_1, b_1]$ such that $x \in I_1$ and both a_1 and b_1 are rational.
- If we apply the interval halving method beginning with the interval I_1, why is it true that all the subsequent intervals $I_n = [a_n, b_n]$ have endpoints that are rational?
- Relative to the given x, how should the intervals in the interval halving method be selected?

Theorem. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise.

Write a proof of the theorem using the comments above.

Before we give a second proof of the exercise, we first prove a theorem which establishes an equivalent formulation that a set S is dense in $\mathbb{R}$.

Theorem

Let S be a subset of $\mathbb{R}$. Then S is dense in $\mathbb{R}$ if and only if every open interval contains a point of S.

Comments on the proof of the theorem

- You have to prove two things here:
  - (i) Assuming that S is dense in $\mathbb{R}$, you have to prove that any open interval contains at least one point of S.
  - (ii) Assuming that S has the property that every open interval contains at least one point of S, you have to prove that S is dense in $\mathbb{R}$.
- (i) Give yourself an open interval I. Let x be any point of I. Assuming that $x \notin S$, how can you use the density of S to deduce there are lots of points of S in I?
- (ii) Let x be any point of $\mathbb{R}$. Assuming that $x \notin S$, how can you use the fact that every open interval contains a point of S to deduce that there is a sequence of points of S converging to S?

Exercise.

Use the above comments to write a proof of this theorem.

Now let's return to the exercise and give a second proof using this alternate formulation of density.

Theorem. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Comments on the second proof of the exercise.

- For this proof we have to show that every open interval (a, b) contains a rational number m/n.
- The m and the n have a different role to play. Think of n as determining a "grid size" $1/n$, so that the various multiples of m/n allow us to partition $\mathbb{R}$ into closely spaced rationals (the spacing determined by how large is n).
- The closer is a and b, the more challenging it is to find a rational in between, and so the smaller the required grid size.
- A measure of the closeness of a and b is the quantity $\varepsilon := b - a$.
- This suggests that the grid size should be less than ε, so we'd like to choose $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$. How do we know we can do that?
- Notice that since $0 < 1/n < b - a$ then $nb - na > 1$.
- Since the spacing between nb and na is more than 1, what does that allow us to do?

Exercise.

Use the above comments to write a second proof that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Comments on the construction of $\mathbb{R}$ from $\mathbb{Q}$

- So every real number can be obtained as a limit of a sequence of rational numbers.
- This suggests a way to prove the existence of a set having all of the desired properties of the real numbers.
- The advantage of working with Cauchy sequences is that it gives a condition of convergence of a sequence without specifying what the sequence converges to.
- And one can formulate what is a Cauchy sequence of rationals without any mention of real numbers.
- Consider the set consisting of Cauchy sequences of rationals.
- We define a relation on the set as follows: we say that the Cauchy sequence x_n is related to the Cauchy sequence y_n provided $|x_n - y_n| \to 0$. It's not hard to prove that this is in fact an equivalence relation.
- By the Equivalence Class Theorem, this partitions the set of Cauchy sequences of rationals into disjoint equivalence classes.
- One then defines $\mathbb{R}$ to be the set of distinct equivalence classes, i.e. a real number is defined to be an equivalence class. One can place a structure on this set and prove it has all the properties that $\mathbb{R}$ is supposed to have.
- This process of producing $\mathbb{R}$ from equivalence classes of Cauchy sequences in $\mathbb{Q}$ is known as "the completion of $\mathbb{Q}$".
- It can be done in greater generality. Given any set X on which there is enough structure to talk about "Cauchy sequences", but which has the property that not all Cauchy sequences converge, one can "complete X" to obtain a bigger set $\overline{X}$ such that all Cauchy sequence in $\overline{X}$ do converge. This set $\overline{X}$ is obtained as equivalence classes of Cauchy sequences in X.