- Given that we have convergence of some sequences, we study here what we can prove about the convergence of various algebraic combinations of those sequences.

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(i) $c\, x_n \to c\, L$, where c is any constant;

(ii) $x_n + y_n \to L + M$;

(iii) $x_n\, y_n \to L\, M$;

(iv) If $L \neq 0$, then $x_n \neq 0$ for sufficiently large n, and $\dfrac{1}{x_n} \to \dfrac{1}{L}$;

(v) If $M \neq 0$, then $\dfrac{x_n}{y_n} \to \dfrac{L}{M}$.

- We will prove all of these as exercises.

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(i) $c\, x_n \to c\, L$, where c is any constant.

Comments on proof of (i)

- Given that we can make $|x_n - L|$ small for large n, we must show that we can make $|c\, x_n - c\, L|$ small.
- So for a given $\varepsilon > 0$, how small must we make $|x_n - L|$ in order that $|c\, x_n - c\, L| < \varepsilon$?

Exercise.

Write the proof of (i).

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(ii) $x_n + y_n \to L + M$;

Comments on proof of (ii)

- We must show that if we can make $|x_n - L|$ small for all sufficiently large n, and we can make $|y_n - M|$ small for sufficiently large n, then we can make $|(x_n + y_n) - (L + M)|$ suitably small for all sufficiently large n.
- Given that we can make $|x_n - L|$ smaller than a given positive real number for sufficiently large n,and we can make $|y_n - M|$ smaller than a given positive real number for sufficiently large n, how big should n be so that we are sure that both $|x_n - L|$ and $|y_n - M|$ are smaller than a given positive number ε?
- If we can force $|x_n - L|$ and $|y_n - M|$ both to be small for sufficiently large n, how can we be sure that $|(x_n + y_n) - (L + M)|$ is also small? What tool should we use?

Exercise.

Write the proof of (ii).

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then the following hold:

(iii) $x_n\, y_n \to L\, M$;

Comments on proof of (iii)

- We must show that if we can make $|x_n - L|$ small for all sufficiently large n, and we can make $|y_n - M|$ small for sufficiently large n, then we can make $|x_n\, y_n - L\, M|$ suitably small for all sufficiently large n.
- The trick is to add and subtract the right thing so that after simplifying we get a sum of terms each of which we can force to be suitably small for all large enough n.
- Try adding and subtracting $y_n L$ under the absolute value bars. This gives $|x_n\, y_n - L\, M| = |x_n y_n - y_n L + y_n L - LM| = |y_n(x_n - L) + L(y_n - M)|$.
- Now apply the triangle inequality to get $|x_n\, y_n - L\, M| \leq |y_n|\, |x_n - L| + |L|\, |y_n - M|$.
- How do you know that you can force the term $|y_n|\, |x_n - L|$ to be suitably small? What property of the convergent sequence y_n should you make use of to do it?

Exercise.

Write the proof of (iii).

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(iv) If $L \neq 0$, then $x_n \neq 0$ for sufficiently large n, and $\dfrac{1}{x_n} \to \dfrac{1}{L}$;

Comments on proof of (iv)

- We first need to ensure that knowing $L \neq 0$ is enough to deduce $x_n \neq 0$ for all n after a while.
- So this means proving that $|x_n|$ is bounded away from 0 for all n after a while.
- Since $L \neq 0$ and the terms of x_n get close to L for all n sufficiently large, it must be possible to prove $x_n \neq 0$ for n large. But how to prove it?
- Try to do it using the reverse triangle inequality.
- Next must show that if we can make $|x_n - L|$ small for all sufficiently large n, then we can make $|1/x_n - 1/L|$ suitably small for all sufficiently large n.
- Rewriting we get $|1/x_n - 1/L| = \left|\dfrac{x_n - L}{x_n\, L}\right|$.
- Use the facts that $|x_n|$ is bounded away from 0 and that we can make $|x_n - L|$ as small as we wish to show that for all sufficiently large n we can make $\left|\dfrac{x_n - L}{x_n\, L}\right|$ suitably small for all n after a while.

Exercise.

Write the proof of (iv).

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(v) If $M \neq 0$, then $\dfrac{x_n}{y_n} \to \dfrac{L}{M}$.

Comments on proof of (v)

- This part of the theorem is the most complex one of the theorem.
- However, do you see that now that we have proved the other parts of the theorem we can prove part (v) very easily?

Exercise.

Write the proof of (v).

Exercise

Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $x_n \to 0$, but we specify nothing more about the sequence y_n.

a) It is not necessarily true that $x_n\, y_n \to 0$. Intuitively why don't you believe that in general $x_n\, y_n$ has to converge to 0?

b) Give a few specific counterexamples to illustrate what can go wrong.

c) What additional property could you assign to the sequence y_n so that one can prove that $x_n\, y_n \to 0$? Try to make your property as weak a condition on y_n as you can.

Exercise

Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $x_n \to 0$ and y_n is bounded.

a) Write a direct proof that $x_n\, y_n \to 0$.

b) Now write a simpler proof that $x_n\, y_n \to 0$ that makes use of the squeeze theorem.

- A useful result is

  Theorem. If $x_n \to L$ and $y_n \to M$ where L and M are real numbers, and if in addition $x_n < y_n$ for all n, then $L \leq M$.

- A special case of this is when $y_n = 0$ for all n:

  if $x_n \to L$ for some real number L and $x_n < 0$ for all n, then $L \leq 0$.

- We proved this last result earlier in the course.

Exercise

Write a proof of the theorem which makes use of the above result. Make your proof as simple as possible, but show all relevant details.