- If $\{x_n\}_{n=1}^{\infty}$ is a sequence, say we choose a large $N \in \mathbb{N}$ and look at the members of the sequence x_n for any $n \geq N$. Let's informally call this "looking far out in the sequence".
- Then informally, the sequence is Cauchy provided given any $\varepsilon > 0$, if we look sufficiently far out in the sequence any pair of terms are within ε of each other.

Exercise.

To see if you understand what is a Cauchy sequence, consider the following sequence:

$$0, 1/2,\ 1,\ 2/3,\ 1/3,\ 0,\ 1/4,\ 2/4,\ 3/4,\ 1,\ 4/5, 3/5,\ 2/5,\ 1/5,\ 0,$$
$$1/6,\ 2/6,\ 3/6,\ 4/6,\ 5/6,\ 1,\ 6/7,\ 5/7,\ 4/7,\ 3/7,\ \ldots$$

a) Is it a Cauchy sequence? Why or why not?

b) What "closeness" property does this sequence satisfy?

Interval halving method for producing Cauchy sequences

- Let $I = [a, b]$ be a closed interval.
- We are going to use I to define a certain "nested decreasing sequence" of closed intervals.
- Let $I_1 = [a_1, b_1] := [a, b]$.
- Note that the midpoint of the interval is $\dfrac{a_1 + b_1}{2}$.
- I_1 has a left half $\left[a_1, \frac{a_1+b_1}{2}\right]$ and a right half $\left[\frac{a_1+b_1}{2}, b_1\right]$. Choose $I_2 = [a_2, b_2]$ to be either the left half or the right half of I_1 (I don't care which you choose).
- Note that $I_1 \supseteq I_2$ and I_2 is half as long as I_1.
- We would like to continue this process by next applying it to the new interval I_2 in order to produce I_3, etc.
- The formal way to do this is to use induction. We describe this on the next slide.

- We're using induction to continue this process of producing intervals.
- Let $n \geq 1$ and suppose that we have intervals $I_j = [a_j, b_j]$ for each j from 1 to n such that $I_1 \supseteq I_2 \cdots \supseteq I_n$ and each successive interval is half the length of the preceding one.
- Then I_n has a left half $\left[a_n, \frac{a_n+b_n}{2}\right]$ and a right half $\left[\frac{a_n+b_n}{2}, b_n\right]$.
- Choose $I_{n+1} = [a_{n+1}, b_{n+1}]$ to be either the left half or the right half of I_n (I don't care which you choose).
- Note that $I_n \supseteq I_{n+1}$ and I_{n+1} has half the length of I_n.
- By the Principle of Mathematical Induction, this process produces an infinite sequence of intervals $I_1, I_2, \ldots$ such that

$$I_1 \supseteq I_2 \supset I_3 \supseteq \ldots, \quad \text{and for each } j,\ I_{j+1} \text{ has half the length of } I_j.$$

- For each j, let x_j be any number in the interval I_j.
- Thus by this process we have produced an infinite sequence $\{x_j\}_{j=1}^{\infty}$.

We refer to the above method of producing an infinite sequence $\{x_j\}_{j=1}^{\infty}$ as the method of interval halving.

- We show in the next theorem that this sequence $\{x_j\}_{j=1}^{\infty}$ is necessarily a Cauchy sequence.

Theorem (Interval Halving Method Theorem)

Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed bounded intervals such that for each n, I_{n+1} is either the left half or the right half of I_n. For each n, let x_n be any point in I_n. Then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

- The Interval Halving Method Theorem gives us a nice way to produce Cauchy sequences.
- The idea of using a decreasing sequence of closed bounded intervals, each one either the left half or the right half of the previous one, is an idea we'll use several times.
- The next theorem gives another application of this method. See if you can prove it yourself by using this idea of producing such a sequence of intervals. You will have to figure out at each step whether you should pick the left or the right half of the previous interval.

Theorem

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with the following two properties:

- The sequence is decreasing: $(\forall n)[x_n \geq x_{n+1}]$
- The sequence is lower bounded: $(\exists M \in \mathbb{Z})(\forall n \in \mathbb{N})[x_n \geq M]$

Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

- The next theorem is another important result.
- It could be proved using a similar technique as the previous theorem, but a better approach is to try to deduce it from the theorem you just proved on the previous page,
- i.e. if you know that any decreasing lower bounded sequence is Cauchy, deduce from this that any increasing upper bounded sequence is Cauchy.

Theorem

Any increasing upper bounded sequence is a Cauchy sequence.

Must Cauchy sequences of rationals converge to a rational?

- We've seen that any convergent sequence is a Cauchy sequence. So if we specialize this to the rational numbers $\mathbb{Q}$, it says:

  > "Any sequence of rationals which converges to a rational number must be a Cauchy sequence."

- The entire motivation for defining the real numbers is the fact that the converse to this result is false. In fact

Theorem. There are Cauchy sequences of rational numbers which don't converge to a rational number.

- In the next few exercises we give a heuristic proof of this theorem. The details can be justified once we study infinite series in Chapter 5.

A Cauchy sequence of rationals doesn't necessarily converge to a rational

We make use of the following ideas concerning decimal expansions:

- A rational number is defined to be the ratio of two integers.
- Each rational number can be represented by a decimal expansion which is either terminating or repeating.
- Conversely, each decimal expansion which is either terminating or repeating represents a unique rational number.
- A decimal expansion which is neither terminating nor repeating does not represent a rational number.

Exercise.

a) Write down a few examples of terminating decimal expansions.

b) Write down a few examples of repeating, nonterminating decimal expansions.

c) If $a = 0.123572223796$ and $b = 0.12357721900457$, what is an upper bound for $|a - b|$? For this upper bound, look for the smallest reciprocal power of 10 you can get away with.

d) If $a = 0.342715$ and $b = 0.3422222\ldots$, what is an upper bound for $|a - b|$? Again, look for the smallest reciprocal power of 10 you can get away with.

e) Write down an example of a nonrepeating, nonterminating decimal expansion that uses only 0's and 1's.

f) For the example you just wrote down, write down a rational number that is within 0.001 of that number and which has a terminating decimal.

g) For the same example, write down a rational number that is within 0.000001 of that number and which has a nonterminating, repeating decimal.

A Cauchy sequence of rationals doesn't necessarily converge to a rational

Exercise.

Use the above ideas to construct a sequence of rational numbers which is a Cauchy sequence, yet which doesn't converge to a rational number.

- So the set of rational numbers has the property that convergent sequences are Cauchy, but there are Cauchy sequences of rationals which don't converge to a rational number.
- The previous exercise produced a Cauchy sequence in $\mathbb{Q}$ which could not converge in $\mathbb{Q}$ because the thing it was trying to converge to cannot be represented by a decimal expansion which is either terminating or repeating.
- So the rational number line has lots of holes in it, each hole corresponding to a Cauchy sequence in $\mathbb{Q}$ which is trying to converge but which cannot.
- The idea of creating $\mathbb{R}$ is to fill the holes in the rational line by allowing all Cauchy sequences to converge in $\mathbb{R}$.
- Note that the definition of "sequence", "convergence of a sequence", and "Cauchy sequence" work just as well in any Archimedean ordered field.

Definition of $\mathbb{R}$

We define $\mathbb{R}$ to be a set which is an Archimedean ordered field such that all Cauchy sequences converge.

So we've defined $\mathbb{R}$ to be any set which is an Archimedean ordered field such that all Cauchy sequences converge.

Comments

- Saying that all Cauchy sequences in $\mathbb{R}$ converge is usually abbreviated by saying that

$$\boxed{\text{“}\mathbb{R}\text{ is complete”.}}$$

  So using that terminology, we can define $\mathbb{R}$ to be a set which is a complete Archimedean ordered field.
- Existence: One can prove the existence of a complete Archimedean ordered field using set theory, but we will not do it in this course.
- Uniqueness: One can also prove that up to a renaming of the elements of $\mathbb{R}$, there is only one set which is a complete Archimedean ordered field. We will not deal with this in this course.

- The interval halving method is a useful way to produce Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, we know that (by definition) those Cauchy sequences converge to some real number. The specific problem we wish to solve will suggest how we should choose the intervals in the method and the specific point in each of those intervals.

Definition

- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. We call the sequence *increasing* (resp., *decreasing*) if for all n, $x_n \le x_{n+1}$ (resp., $x_n \ge x_{n+1}$). We write this symbolically as $x_n \nearrow$ (resp., $x_n \searrow$). In either case, we say that x_n is a *monotone sequence*.
- We call the sequence *strictly increasing* (resp., *strictly decreasing*) if for all n, $x_n < x_{n+1}$ (resp., $x_n > x_{n+1}$). We write this symbolically as $x_n \uparrow$ (resp., $x_n \downarrow$). In either case, we say that x_n is *strictly monotone*.

Definition

Let S be a subset of $\mathbb{R}$. We call S *upper bounded* (resp., *lower bounded*) if the following holds:

$$(\exists M \in \mathbb{R})(\forall x \in S)(x \le \text{ (resp., } \ge) \ M)$$

We call S *bounded* if it is both upper and lower bounded. Note this holds if and only if the following is true:

$$(\exists M \in \mathbb{R})(\forall x \in S)(|x| \le M).$$

- A sequence is bounded (resp., upper bounded, lower bounded) if the set of its values $S = \{x_n : n \in \mathbb{N}\}$ is a bounded (resp., upper bounded, lower bounded) set.

Theorem (bounded monotone sequences)

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then x_n converges to a real number if any of the following is true:

(i) x_n is upper bounded and either increasing or strictly increasing;

(ii) x_n is lower bounded and either decreasing or strictly decreasing;

(iii) x_n is bounded and monotone or strictly monotone.

Exercise.

Write the proof of the above theorem by making appropriate use of some of the exercises we've already done above.

Exercise.

Prove that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ converges to a real number between 2 and 3. Do it by making use of the result of the previous exercise. Before doing it, recall the following two things:

- The binomial theorem
- The formula for an infinite geometric series

- We define $\sqrt{2}$ to be a positive real number x with the property that $x^2 = 2$.
- But defining it in this way, doesn't prove that there is such a real number $\sqrt{2}$.

Theorem There exists a positive real number x such that $x^2 = 2$.

Lemma 1. Suppose y_n is a sequence which converges to a number y. Then y_n^2 converges to y^2.

Lemma 2. Suppose y_n is a sequence which converges to a number y. Suppose that x is a number such that $y_n \leq x$ for all n. Then $y \leq x$.

Lemma 3. Suppose that x_n is a sequence that converges to a number x, and suppose y_n is a sequence such that the sequence $x_n - y_n$ converges to 0. Then y_n converges to x.

Exercise.

Prove the above theorem by first proving the three lemmas and then using the results of those lemmas. In your proof, make use of the interval halving method and clearly show how the completeness axiom of $\mathbb{R}$ is being used.

Definition: upper bound, lower bound, lub, glb

Let A be a subset of $\mathbb{R}$.

(i) A number M is called an ***upper bound*** of A if the following is true:

$$(\forall a \in \mathbb{R})[a \in A \Longrightarrow a \leq M].$$

(ii) A number m is called a ***lower bound*** of A if the following is true:

$$(\forall a \in \mathbb{R})[a \in A \Longrightarrow a \geq m].$$

(iii) A number M is called the ***least upper bound*** of A if the following two things are true:

  (a) M is an upper bound of A;
  (b) If M' is any upper bound of A, $M \leq M'$.

  If the least upper bound of A exists, it is denoted by $lub(A)$.

(iv) A number m is called the ***greatest lower bound*** of A if the following two things are true:

  (a) m is a lower bound of A;
  (b) If m' is any lower bound of A, then $m \geq m'$.

  If the greatest lower bound of A exists, it is denoted by $glb(A)$.

Theorem

Let A be a nonempty set with an upper bound. Then the least upper bound of A exists.

Exercise.

Use the method of interval halving and the completeness axiom to prove the above theorem.

Theorem

For any nonempty lower bounded set A, the greatest lower bound of A exists.

Exercise.

Prove the above theorem. But:

- Don't do it by going back to first principles, but rather by making use of the previous theorem.
- The idea is that if we define $-A$ to be $\{x \in \mathbb{R} : -x \in A\}$, then show that

$$glb(A) = -lub(-A).$$

- Let A be a nonempty set. We've seen that if A is upper bounded, then the completeness axiom implies that $lub(A)$ exists; If A is lower bounded, then the completeness axiom implies that $glb(A)$ exists.
- We wish to extend the definition of $lub(A)$ in case A is not upper bounded, and to extend the definition of $glb(A)$ in case A is not lower bounded.
- For this purpose we introduce $\sup(A)$ and $\inf(A)$.

Definition: sup and inf

Let A be a nonempty set. We define the supremum of A, denoted by $\sup(A)$ and the infimum of A, denoted by $\inf(A)$ by

$$\sup(A) = \begin{cases} lub(A) & \text{if } A \text{ is upper bounded} \\ \infty & \text{if } A \text{ is not upper bounded} \end{cases}$$

and

$$\inf(A) = \begin{cases} glb(A) & \text{if } A \text{ is lower bounded} \\ -\infty & \text{if } A \text{ is not lower bounded} \end{cases}$$

Exercise

(i) $\sup\{-1, -13, 5, 7, 9, 2\} =?$

(ii) $\inf\{-1, -13, 5, 7, 9, 2\} =?$

(iii) $\sup[6, 19.5]=?$

(iv) $\inf(\{x \in \mathbb{R} : x < 10\})=?$

(v) $\sup(\{1 - 1/n : n \in \mathbb{N}\})$

(vi) Let A be a nonempty subset of $\mathbb{R}$. Is it necessarily true that $\sup(A) \in A$? Give examples to illustrate.

Exercise.

The definitions of upper and lower bound of a set A still make sense if $A = \emptyset$. If you follow through on those definitions, what is the natural choice for defining $\sup(\emptyset)$ and $\inf(\emptyset)$?

Exercise.

Let A be a nonempty set. Answer the following questions using nothing more than the definitions of supremum and infimum.

a) Suppose λ is a number such that $\lambda < \sup(A)$. What can you deduce about the set A?

b) Suppose λ is a number such that $\lambda > \sup(A)$. What can you deduce about the set A?

c) Suppose λ is a number such that $\lambda < \inf(A)$. What can you deduce about the set A?

d) Suppose λ is a number such that $\lambda > \inf(A)$. What can you deduce about the set A?

Theorem

Let x_n be a sequence. Let $A = \{x_n : n \in \mathbb{N}\}$.

(i) If x_n is an increasing upper bounded sequence, then $\lim_{n\to\infty} x_n = lub(A)$.

(ii) If x_n is a decreasing lower bounded sequence, then $\lim_{n\to\infty} x_n = glb(A)$.

Exercise.

a) Use the above exercise to write a proof of the theorem.

b) How must you modify the statement in (i) if we leave out "upper bounded"? How must you modify the statement in (ii) if we leave out "lower bounded"?

- If x_n is a sequence, it's quite possible that x_n does not have a limit, so if we don't know whether or not the sequence converges, we don't have the right to use the notation $\lim_{n\to\infty} x_n$.
- But we study here associated limiting values of the sequence each of which always exist (either as a real number, ∞, or $-\infty$) and they are denoted by $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

$\limsup_{n\to\infty} x_n$

- For each $N \in \mathbb{N}$, let $T_N := \{x_n : n \geq N\} = \{x_N, x_{N+1}, x_{N+2}, \dots\}$. We call T_N the *Nth* tail of the sequence.
- If the sequence x_n is not upper bounded, then $\sup(T_N) = \infty$ for every N; if the sequence x_n is upper bounded, then $\sup(T_N)$ is a real number.
- Note that $T_{N+1} \subseteq T_N$ for each $N \in \mathbb{N}$.
- Thus by the exercise on the previous slide, $\sup(T_{N+1}) \leq \sup(T_N)$ and so if x_n is upper bounded, $\sup(T_N)$ is a decreasing sequence of real numbers. Thus it converges either to a real number or to $-\infty$.

Definition

If x_n is any sequence, we define

$$\limsup_{n\to\infty} x_n := \begin{cases} \infty & \text{if the sequence } x_n \text{ is not upper bounded} \\ \lim_{N\to\infty} \sup(\{x_N, x_{N+1}, x_{N+2}, \dots\}) & \text{if the sequence } x_n \text{ is upper bounded} \end{cases}$$

- Exercise. Calculate $\limsup_{n\to\infty}(-1)^n$

- We similarly observe that with $T_N := \{x_N, x_{N+1}, x_{N+2}, \dots\}$ then since $T_N \subseteq T_{N+1}$, it follows that $\inf(T_{N+1}) \geq \inf(T_N)$, so the sequence $\inf(T_N)$ is an increasing sequence.
- If the sequence x_n is not lower bounded, then $\inf(T_N) = -\infty$ for every N. But if x_n is lower bounded, the sequence $\inf(T_N)$ is an increasing sequence and so converges either to a real number or ∞. Thus we define $\liminf_{n\to\infty} x_n$ as follows:

Definition

If x_n is any sequence, we define

$$\liminf_{n\to\infty} x_n := \begin{cases} -\infty & \text{if the sequence } x_n \text{ is not lower bounded} \\ \lim_{N\to\infty} \inf(\{x_N, x_{N+1}, x_{N+2}, \dots\}) & \text{if the sequence } x_n \text{ is lower bounded} \end{cases}$$

- Exercise. Calculate $\liminf_{n\to\infty}(-1)^n$
- Exercise. Write down an example of a sequence x_n such that $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 1$.

- So one of the benefits is that for any given sequence x_n, even if $\lim_{n\to\infty} x_n$ does not exist (in which case we have no right to use the notation $\lim_{n\to\infty} x_n$), both $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ do exist.
- The other benefit is the following theorem.

Theorem

Let x_n be any sequence.

(i) It is always the case that $-\infty \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \infty$.

(ii) The sequence x_n converges if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ in which case the common value is $\lim_{n\to\infty} x_n$.

Proof.

Proof of (i).

Proof of (ii)

$\square$