Definition: Sequence

A ***sequence*** is defined to be a function from $\mathbb{N}$ to $\mathbb{R}$. If a sequence is named x, we will refer to $x(n)$ as x_n. We will usually denote the entire sequence by $\{x_n\}_{n=1}^{\infty}$ or more simply by $\{x_n\}_n$, but sometimes by abuse of notation we may denote the entire sequence by x_n.

- We also think of a sequence as being an endless list of real numbers.
- So the sequence $x_n = 1/n$ gives rise to the list

$$1, 1/2, 1/3, 1/4, 1/5, 1/6, \dots.$$

- We're particularly interested in the behavior of the terms of the sequence far out in the list, i.e. x_n for very large n.

Definition: Convergence and divergence of a sequence

(i) Let $\{x_n\}_n$ be a sequence and L a real number. We say that x_n converges to L provided

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies |x_n - L| < \varepsilon].$$

(ii) If $\{x_n\}_n$ converges to L, we write $x_n \to L$, or $\lim_{n\to\infty} x_n = L$.

(iii) We say the sequence $\{x_n\}_n$ converges provided there exists a real number L such that $x_n \to L$. In symbols this says,

$$(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies |x_n - L| < \varepsilon].$$

(iv) If $\{x_n\}_n$ is a sequence and and it does not converge, then we say that $\{x_n\}_n$ diverges.

Exercise 1.2.1.

a) Informally, what does it mean to say that the sequence x_n converges to the number L?

b) Suppose the following statement is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies |x_n - L| < 100\varepsilon]$$

Is it true that $x_n \to L$?

c) Write in symbols and in words what it means to say that a sequence does not converge to a number L.

d) Informally, what does it mean to say that a sequence does not converge to a given number L?

Exercise 1.2.2.

Consider the statement $1/n \to 0$.

(i) Intuitively why do you believe it is true?
(ii) Write a proof that it is true.

Exercise 1.2.3.

Consider the statement $1/\sqrt{n} \to 0$.

(i) Intuitively why do you believe it is true?

(ii) Write a proof that it is true.

- In working with limits of sequences (and limits in general), we will often make use of the **triangle inequality** (and some of its variations) and the **reverse triangle inequality**. These say that for all real numbers x and y,

$$\text{Triangle inequality:} \quad |x-y| \leq |x|+|y|, \quad |x+y| \leq |x|+|y|$$

$$\text{Reverse triangle inequlaity:} \quad |x-y| \geq |x|-|y|, \quad |y|-|x| \geq |x-y|$$

You might find it interesting to take note of how many times you make use of these inequalities this semester.

Exercise.

Consider the statement that a convergent sequence has a unique number to which it converges.

a) Intuitively why do you believe this is true?

b) Write a proof that it is true.

Exercise.

Fix r such that $0 < r < 1$. Consider the statement that $r^n \to 0$.

a) Intuitively why do you believe the statement is true?

b) Write a proof that it is true. Make use of properties of the natural logarithm and exponential functions (even though you haven't yet been given rigorous definitions).

c) Since we haven't rigorously developed the definition and properties of the log and exponential functions, we should try to write a proof that doesn't make use of them. So write a proof that $r^n \to 0$ which does not make use of logs or exponentials.

Exercise.

Consider the statement that $(-1)^n$ diverges.

a) Intuitively why do you believe it is true?

b) Write a proof that it is true.

Definition: Cauchy sequence

A sequence x_n is called a ***Cauchy sequence*** if the following is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \implies |x_m - x_n| < \varepsilon]$$

Exercise.

a) Intuitively what does it mean to say that a sequence is a Cauchy sequence?
b) Write in symbols and in words what it means to say that a sequence is not Cauchy.
c) Intuitively what does it mean to say that a sequence is not Cauchy?

Theorem.

If a sequence is convergent, then it is a Cauchy sequence.

Proof.

"Theorem: If a sequence is convergent, then it is a Cauchy sequence."

Exercise.

a) What does the contrapositive of the above theorem say? Is it true?

b) Earlier in this section we proved that the sequence $(-1)^n$ is divergent. Give a simpler alternate proof.

Definition: Bounded sequence

A sequence x_n is called ***bounded*** if the following is true:

$$(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})[|x_n| < M]$$

If a sequence is not bounded, we say it is ***unbounded***. The sequence is called ***upper bounded*** if

$$(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})[x_n < M]$$

and ***lower bounded*** if

$$(\exists M \in \mathbb{N})(\forall n \in \mathbb{N})[x_n > -M].$$

Note that as a consequence, a sequence is bounded if and only if it is both upper bounded and also lower bounded. But note that a sequence can be neither upper nor lower bounded, so one is not the negation of the other.

Exercise.

a) Informally what does boundedness of a sequence say about the sequence?

b) Give an example of a bounded sequence which is not convergent.

c) Write down in symbols and in words what it means to say that a sequence is unbounded.

d) Give an example of an unbounded sequence.

Theorem.

If a sequence is a Cauchy sequence, then it is bounded.

Proof.

Definition: Divergence to ∞, Divergence to $-\infty$

A sequence is said to diverge to ∞ if the following is true:

$$(\forall M > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies x_n > M]$$

The notation $x_n \to \infty$ indicates that the sequence x_n diverges to ∞. Similarly, we say that the sequence diverges to $-\infty$, and write $x_n \to -\infty$ if the following is true:

$$(\forall M > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \implies x_n < -M]$$

Exercise.

a) Informally, what does it mean to say that $x_n \to \infty$?

b) True or false: If $x_n \to \infty$, then x_n is unbounded.

c) Can you give an example of an unbounded sequence x_n such that x_n doesn't diverge to ∞ or $-\infty$? If not explain why not, and if true give such an example.

d) Is it true or false that every bounded sequence is a Cauchy sequence? If it is true prove it, and if false then give a counterexample.

e) Can you give an example of a Cauchy sequence which is not bounded?