- One aim is to give a deeper understanding of calculus and why the theorems of the subject are true.
- But the techniques introduced here extend far beyond this aim.
- They are the techniques of a branch of mathematics called “analysis”, which is the study of infinite processes and limits.
- This includes subjects involving limiting operations, continuity, infinite sums, differentiability, etc.
- Includes ode’s, pde’s, complex analysis, harmonic analysis, measure theory, probability theory, stochastic processes, stochastic differential equations, potential theory, point set topology, numerical analysis, approximation theory, calculus of variations, functional analysis, and others.
- To read advanced books or papers in these subjects, you need to be able to fill in lots of missing steps, and that involves adapting the techniques such as we learn in this course.
- So the proofs of the results we learn here are important, as they are often adapted to proving results in these other subjects.

- To get a deeper understanding of calculus, we need a precise understanding of what is the set of real numbers.
- We define the real numbers to be a set satisfying a specific set of axioms.
- First we define what is an "Archimedean Ordered Field".
- There is more than one number system that is an Archimedean ordered field, but after adding one more axiom, the Completeness Axiom, we will have a set that in some sense uniquely satisfies all of the axioms, **and we will call that set the real numbers**.

Archimedean Ordered Field

An Archimedean Ordered Field is any set $\mathbb{F}$ with two binary operations $+$ and $\cdot$ and an order relation $<$ satisfying the following ten properties:

(1) *Closure*:

- (i) $(\forall a, b \in \mathbb{F})[a + b \in \mathbb{F}]$
- (ii) $(\forall a, b \in \mathbb{F})[a \cdot b \in \mathbb{F}]$

(2) *Commutativity*:

- (i) $(\forall a, b \in \mathbb{F})[a + b = b + a]$
- (ii) $(\forall a, b \in \mathbb{F})[a \cdot b = b \cdot a]$

(3) *Associativity*:

- (i) $(\forall a, b, c \in \mathbb{F})[a + (b + c) = (a + b) + c]$
- (ii) $(\forall a, b, c \in \mathbb{F})[a \cdot (b \cdot c) = (a \cdot b) \cdot c]$

(4) *Distributivity*: $(\forall a, b, c \in \mathbb{F})[a \cdot (b + c) = (a \cdot b) + (a \cdot c)]$

(5) *Identity*: $(\exists 0, 1 \in \mathbb{F})[(0 \neq 1) \wedge ((\forall a \in \mathbb{F})(a + 0 = a \text{ and } a \cdot 1 = a))]$

(6) *Inverses*:

- (i) (Additive inverses) $(\forall a \in \mathbb{F})[(\exists - a \in \mathbb{F})[a + (-a) = 0]$
- (ii) (Multiplicative inverse) $(\forall a \in \mathbb{F})[a \neq 0 \Rightarrow (\exists a^{-1} \in \mathbb{F})[a \cdot a^{-1} = 1])]$

(7) *Transitivity*: $(\forall a, b, c \in \mathbb{F})[(a < b \text{ and } b < c) \Rightarrow a < c]$

(8) *Preservation of Order*:

- (i) $(\forall a, b, c \in \mathbb{F})[a < b \Rightarrow a + c < b + c]$
- (ii) $(\forall a, b, c \in \mathbb{F})[(a < b \text{ and } c > 0) \Rightarrow a \cdot c < b \cdot c]$

(9) *Trichotomy*: $(\forall a, b \in \mathbb{F})[$ exactly one of the following occurs: $a < b, a = b, a > b]$

(10) *Archimedean Property*: Define $\mathbb{N}$ to be the smallest subset of $\mathbb{F}$ such that $1 \in \mathbb{F}$ and is closed under $+$. Then $(\forall \varepsilon > 0)(\forall M > 0)(\exists n \in \mathbb{N})[n\varepsilon > M]$

Exercise

Which of the above axioms are satisfied by each of the following sets (with the usual operations of addition and multiplication)?

- the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$
- the set of integers, $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$
- the set of rational numbers, $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\}$
- the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- the set of complex numbers, $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$
- the set of real numbers, $\mathbb{R}$

Exercise

Is there more than one example of an Archimedian Ordered Field?

Exercise

There are lots of familiar properties of arithmetic which don't appear in Axioms 1-10 of an Archimedean Ordered Field. Prove that each of the following properties hold in any field.

(f-i) Uniqueness of the identity elements 0 and 1.

(f-ii) Uniqueness of additive and multiplicative inverses.

(f-iii) $(\forall a \in \mathbb{F})[a \cdot 0 = 0]$

(f-iv) $(\forall a, b, c \in \mathbb{F})[a \neq 0 \text{ and } b \neq 0 \Rightarrow a \cdot b \neq 0]$

(f-v) $(\forall a \in \mathbb{F})[-1 \cdot a = -a]$

(f-vi) $(\forall a \in \mathbb{F})[-(-a) = a]$

(f-vii) $(\forall a, b \in \mathbb{F})[(-a) \cdot (-b) = a \cdot b]$ *(Hint: Begin by proving that $(-1) \cdot (-1) = 1$.)*

(f-viii) $(\forall a, b \in \mathbb{F})[-(a \cdot b) = (-a) \cdot b = a \cdot (-b)]$

Exercise

Prove that the following properties hold in any ordered field.

(o-i) $(\forall a, b, c, d \in \mathbb{F})[a < b \text{ and } c < d \Longrightarrow a + c < b + d]$

(o-ii) $(\forall a, b \in \mathbb{F})[a < b \Longleftrightarrow -b < -a]$

(o-iii) $(\forall a \in \mathbb{F})[a > 0 \Longleftrightarrow -a < 0]$

(o-iv) $(\forall a, b, c \in \mathbb{F})[a < b \text{ and } c < 0 \Longrightarrow a \cdot c > b \cdot c]$

(o-v) $(\forall a \in \mathbb{F} \smallsetminus \{0\} : a^2 > 0]$

(o-vi) $1 > 0$

(o-vii) $(\forall a, b \in \mathbb{F} \smallsetminus \{0\})[a < 0 \text{ and } b > 0 \Longrightarrow ab < 0]$

(o-viii) $(\forall a, b \in \mathbb{F} \smallsetminus \{0\})[a < 0 \text{ and } b < 0 \Longrightarrow ab > 0$

Why do we ask that $0 \neq 1$ in our definition of an Archimedean ordered field?

1. Since we are trying to define a model for $\mathbb{R}$, we need the associated set $\mathbb{N}$ to be infinite. But if it were true that $0 = 1$, then $\mathbb{N}$ would be $\{0\}$. So this is not a possibility we want to allow.
2. Does the property of $0 \neq 1$ follow automatically from the other axioms?

   No, there does exist a set $\mathbb{F}$ which satisfies all axioms of an Archimedean ordered field except for $0 \neq 1$.

   Example. Take $\mathbb{F} = \{0\}$; define $0 + 0 = 0$ and $0 \cdot 0 = 0$.

   - Then $\mathbb{F}$ satisfies all of the field axioms, except for the one $0 \neq 1$ (recall that a conditional statement is true if its hypothesis is false).
   - Clearly $\mathbb{F}$ satisfies trichotomy, and since all other order axioms are conditional statements with a false hypothesis, they are also true.
   - The Archimedean property is also a conditional statement with a false hypothesis, since it can be rewritten as

     $$(\forall \varepsilon \in \mathbb{F})(\forall M \in \mathbb{F})[(\varepsilon > 0 \text{ and } M > 0) \Longrightarrow ((\exists n \in \mathbb{N})(n\varepsilon > M))]$$

     so $\mathbb{F}$ satisfies this as well.

Exercise

Let $\varepsilon > 0$ be a given positive number in an Archimedean ordered field $\mathbb{F}$. Prove that there exists $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$.

Definition (Absolute value)

Let a be an element of an Archimedean ordered field $\mathbb{F}$. Then $|a|$, the absolute value of a, is defined by

$$|a| = \begin{cases} a & \text{if a } \geq 0 \\ -a & \text{if a } < 0 \end{cases}$$

Exercise

Prove that for all $a \in \mathbb{F}$ each of the following hold:

a) $|a| = |-a|$.

b) $a \leq |a|$

Exercise

Let $x \in \mathbb{F}$ such that the following statement is true:

$$(\forall \varepsilon > 0)[|x| < \varepsilon].$$

Prove that $x = 0$.

Theorem (Triangle Inequality) For any two elements a, b of an Archimedean Ordered Field $\mathbb{F}$, we have

$$|a + b| \leq |a| + |b|.$$

Exercise

Prove the triangle inequality.

Exercise

Let a, b be any elements of an Archimedean Ordered Field $\mathbb{F}$. Deduce each of the following from the triangle inequality.

a) $|a - b| \leq |a| + |b|$

b) (reverse triangle inequality) $|a - b| \geq |a| - |b|$ and $|a + b| \geq |a| - |b|$

- In section 1.3 we will introduce a property that an Archimedean ordered field may or may not have, called the ***Completeness Axiom***. If it satisfies that axiom, we will call it a complete Archimedean ordered field.
- Part of the motivation will be that the set of rational numbers does not satisfy the Completeness Axiom.

Definition: Set of real numbers

The set of real numbers is defined to be a set which is a complete Archimedean ordered field.