Theorem (Cauchy's Generalized Mean Value Theorem)

Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Assume that $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t)}{g'(t)}.$$

Exercise.

a) In the statement of the theorem, how do we know we are not dividing by 0 when we write $g(b) - g(a)$?

b) Explain why this is a generalization of the Mean Value Theorem.

c) For the proof, if we consider the function

$$h(x) := f(a) + \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)],$$

why do we have the right to apply Rolle's Theorem to the function $f(x) - h(x)$? What do you get if you do apply Rolle's Theorem to $f - h$?

d) Complete the proof of the Generalized Mean Value Theorem.

ⓐ If $g(a) = g(b)$, then by Rolle's Theorem there would exist x between a and b such that $f'(x) = 0$.

But we've assumed $g'(x) \neq 0$ for any x.

ⓑ If we take in particular $g(x) = x$ it becomes the Mean Value Theorem.

Theorem (Generalized Mean Value Theorem)

Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Assume that $g'(x) \neq 0$ for any $x \in (a,b)$. Then there exists $t \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(t)}{g'(t)}.$$

Proof. If it were true that $g(a) = g(b)$, then by Rolle's Theorem there would exist $x \in (a,b)$ such that $g'(x) = 0$, contradicting our assumption. Thus $g(a) \neq g(b)$.

Let $h(x) = f(x) - \left[f(a) + \left(\frac{f(b)-f(a)}{g(b)-g(a)} \right)(g(x) - g(a)) \right]$. Then h is continuous on $[a,b]$, differentiable on (a,b), $h(a) = 0 = h(b)$. Thus by Rolle's Theorem, there exists $t \in (a,b)$ such that $h'(t) = 0$. This means

$$f'(t) - \left(\frac{f(b)-f(a)}{g(b)-g(a)} \right) g'(t) = 0,$$

and this says

$$\frac{f'(t)}{g'(t)} = \frac{f(b)-f(a)}{g(b)-g(a)}. \quad \square$$

Versions 1 and 2 of L'Hôpital's rule

- The Generalized Mean Value Theorem is the key to proving the various versions of L'Hôpital's Rule.

Theorem (L'Hôpital's rule)

(i) (Version 1) Let f and g be continuous on $[a, b]$, differentiable on (a, b), with $g'(x) \neq 0$ for any $x \in (a, b)$. Let $L \in \mathbb{R}$. If $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and equals L.

(ii) (Version 2) Let f and g be differentiable on (b, ∞), with $g'(x) \neq 0$ for any $x \in (b, \infty)$. Let $L \in \mathbb{R}$. If $\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals L.

Hints on the proof:

- To prove (i): Do the hypotheses tell us anything about $f(a)$ and $g(a)$? Write down what the conclusion of the Generalized Mean Value Theorem gives you, and see if you can complete the proof.
- To prove (ii): Consider the functions $F(u) := f(1/u)$ and $G(u) := g(1/u)$ for u near 0 on the right. Apply version 1 of L'Hôpital's Rule to F/G and see if that does what is needed.

Exercise.

Write the proofs.

we only show the proof of version 1.

Theorem (L'Hôpital's rule, 1st Version)

Let f and g be continuous on $[a,b]$, differentiable on (a,b), with $g'(x) \neq 0$ for any $x \in (a,b)$. Let $L \in \mathbb{R}$. If $\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} g(x) = 0$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.

Proof Since f and g are continuous at a, then $f(a) = \lim_{x\to a^+} f(x) = 0$ and $g(a) = \lim_{x\to a^+} g(x) = 0$.

By the Generalized Mean Value Theorem, for each $x \in (a,b)$ there exists $t_x \in (a,x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(t_x)}{g'(t_x)}.$$

As $x \to a$, then $t_x \to a$ also, so

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(t_x)}{g'(t_x)} = L. \quad \square$$