4.4 Uniform Convergence of Sequences of Functions and the Derivative

- Say we have a sequence $f_n(x)$ of functions defined on some interval, $[a, b]$. Let's say they converge in some sense to a function $f(x)$. We'd like to be able to deduce that f has certain properties provided we know that every one of f_n has such a property. For example, we've shown:

Theorem

(i) If f_n converges uniformly to f and each f_n is continuous, then f is continuous.

(ii) If f_n converges uniformly to f on $[a, b]$ and each $f_n \in \mathscr{R}[a, b]$, then $f \in \mathscr{R}[a, b]$ and furthermore

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

- We'd like a similar theorem for derivatives. We might conjecture a result such as the following:

  Let $I = (a, b)$ be an open interval. If f_n converges uniformly to f on I and each f_n is differentiable on I, then f is differentiable on I and f_n' converges uniformly to f'.

- Unfortunately this is not a theorem. The following exercise provides a counterexample.

Exercise.

Consider the sequence of functions $f_n(x) := \dfrac{\sin nx}{n}$ with domain $[0, 1]$.

a) Calculate $\|f_n\|_\infty$. Use it to deduce that $f_n \to 0$ uniformly.

b) Calculate $f_n{}'(x) = \cos nx$. How do you know this sequence does not converge uniformly to 0?

(a) $\|f_n\|_\infty = \frac{1}{n}$ since for each x, $\left|\frac{\sin nx}{n}\right| \leq \frac{1}{n}$, but taking $x = \frac{\pi}{2n}$, $f_n(x) = \frac{\sin n(\frac{\pi}{2n})}{n} = \frac{1}{n}$.

(b) $f_n'(x) = \cos nx$ doesn't even converge pointwise.
so $\{f_n\}$ converges uniformly to 0 (since $\|f_n\|_\infty = \frac{1}{n} \to 0$ as $n \to \infty$)

f_n' is differentiable, yet f_n' doesn't even converge pointwise, let alone uniformly.

Theorem

Let f_n be a sequence of functions with domain a bounded open interval I. Assume each of the following:

(i) Each f_n is differentiable on I and f_n' is continuous;

(ii) f_n' converges uniformly on I to some function g;

(iii) There exists $a \in I$ such that the sequence of numbers $f_n(a)$ converges to a real number.

Then f_n converges uniformly on I to some function f, and $f'(x) = g(x)$ for all $x \in (a, b)$.

- Recall that the uniform limit of continuous functions is continuous, so the function g in the theorem is continuous. The trick is to now apply the first Fundamental Theorem of Calculus with integrand g and use that integral to define f. It's then a matter of showing that this choice of f works.

Exercise.

Write the proof.

• the reason we need (iii) is avoid something like $f_n(x) \equiv n$: Then f_n satisfies (i) and (ii) (since $f_n' \equiv 0$), but $\{f_n\}$ does not converge even pointwise, since $f_n(x) \to \infty$ all x.

main steps • Define (a) $\boxed{f(x) = f(a) + \int_a^x g(t)\,dt}$ Then use

first FTC to observe $f' = g$.

• Use 2nd FTC to observe

(b) $\boxed{f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt}$

• Subtract (a) and (b). Then use the uniform convergence of $f_n' \to g$ to show $f_n \to f$ uniformly.

Proof of the main theorem of 4.4

Theorem (Differentiability and Sequences of Functions)
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions with domain a bounded open interval I. Assume each of the following hold:

(i) Each f_n is differentiable and f_n' is continuous;

(ii) $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on I to some function g;

(iii) There exists $a \in I$ and $L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(a) = L.$$

Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on I to some function f and $f' = g$.

Proof. Since the uniform limit of continuous functions is continuous, g must be continuous. Define $f : I \to \mathbb{R}$ by

$$① \quad f(x) := L + \int_a^x g(t)\,dt.$$

Since g is continuous, it follows from the First FTC that f is differentiable and

$$② \quad f'(x) = g(x) \text{ for all } x \in I.$$

Thus it only remains for us to prove $f_n \to f$ uniformly on I.

Since f_n' is assumed continuous, it is then integrable on the interval containing a and x for any $x \in I$, so by the Second FTC,

(3) $f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt$ for each $x \in I$.

Fix $x \in I$ with $a < x$. Subtracting (1) and (3) we get

$$\begin{aligned} |f_n(x) - f(x)| &= \left| f_n(a) - L + \int_a^x (f_n'(t) - g(t))\,dt \right| \\ &\leq |f_n(a) - L| + \int_a^x |f_n'(t) - g(t)|\,dt \\ &\leq |f_n(a) - L| + \|f_n' - g\|_\infty (x-a) \\ &\leq |f_n(a) - L| + \|f_n' - g\|_\infty \cdot |I|, \end{aligned}$$

where $|I|$ denotes the length of I. If $x < a$, a similar proof also shows

$$|f_n(x) - f(x)| \leq |f_n(a) - L| + \|f_n' - g\|_\infty \cdot |I|.$$

Since the right side does not depend on x, it follows

(4) $$\|f_n - f\|_\infty \leq |f_n(a) - L| + \|f_n' - g\|_\infty |I|.$$

Since $f_n' \to g$ uniformly, it follows from (4) that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, i.e. $f_n \to f$ uniformly on I.

$\square$