$\boxed{\text{Homework 16}}$

$$T: R[0,2\pi] \to \mathbb{R},$$

$$T(f) = \int_0^{2\pi} f(x)\cos x \, dx.$$

(c) $f_n \to f$ in L_2 means

$$\|f_n - f\|_2 \to 0 \text{ as } n \to \infty.$$

<u>convergence in $\mathbb{R}$</u>

The sequence in $\mathbb{R}$ is $\{T(f_n)\}_n$. To show convergence to $T(f)$ we calculate $|T(f_n) - T(f)|$ and show $\to 0$ as $n \to \infty$.

So $\|T(f_n) - T(f)\|_2$ has ~~no~~ meaning.

* on previous homework you showed

$$|T(f)| \le \sqrt{\pi}\|f\|_2.$$

How to use that to show $T(f_n) \to T(f)$?

$$|T(f_n) - T(f)| = |T(f_n - f)| \le \sqrt{\pi}\, \|f_n - f\|_2$$

$$\to 0 \text{ as } n \to \infty$$

Sequence characterization of Continuity

Say $g: R \to R$.

Then g continuous if and only if for every convergent sequence, say $x_n \to x$, it follows $g(x_n) \to g(x)$.

g continuous $\Longleftrightarrow$ $\forall \{x_n\}$ $\forall x$, if $x_n \to x$ then $g(x_n) \to g(x)$.

on Homework 6:

$T: R[0,2\pi] \to R$,

for all $\{f_n\}$, f in $R[0,2\pi]$

$f_n \to f$ in the L_2-norm,

then $T(f_n) \to T(f)$.

we call T <u>continuous</u>.

First FTC

$f:[a,b]\to\mathbb{R}$, f continuous,

Define $F(x)=\int_a^x f(t)\,dt$.

Then F is differentiable and $F'=f$.

Is "converse" true.

i.e. Say f is integrable. Define $F(x)=\int_a^x f(t)\,dt$.

Say $F'=f$ all x. Does this imply f is continuous?

No

ex. Let $F(x)=\begin{cases} 0 & x=0 \\ x^2\sin\frac{1}{x} & 0<x\le 1 \end{cases}$

On H/6: · F is continuous on $[0,1]$

· F is differentiable, but

· F' not continuous at 0.

· F' integrable

· We show $F(x)=\int_0^x F'(t)\,dt$.

we have:

let $\varepsilon > 0$.

Then
$$\int_0^x F'(t)\,dt = \int_0^\varepsilon F'(t)\,dt + \int_\varepsilon^x F'(t)\,dt$$

$$= \int_0^\varepsilon F'(t)\,dt + F(x) - F(\varepsilon)$$

(2nd FTC).

$$\therefore \left| \int_0^x F'(t)\,dt - F(x) \right| = \left| \int_0^\varepsilon F'(t)\,dt - F(\varepsilon) \right|$$

$$\leq \left| \int_0^\varepsilon F'(t)\,dt \right| + |F(\varepsilon)|$$

$$\leq \int_0^\varepsilon |F'(t)|\,dt + |F(\varepsilon)|$$

$$\leq \|F'\|_\infty \cdot \varepsilon + |F(\varepsilon)|$$

$\to 0$ as $\varepsilon \to 0$ since F is <u>continuous</u> at 0 and $F(0) = 0$

But $\left|\int_0^x F'(t)\,dt - F(x)\right|$ is independent of ε. So it must be 0, i.e.

$$\boxed{F(x) = \int_0^x F'(t)\,dt}$$

So the counterexample is obtained by taking

$$f(x) = F'(x).$$