4.3 The Fundamental Theorems of Calculus

- Many students think of the Fundamental Theorems of Calculus as being formulas, such as the following ones:

> 1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."
> 2. "If there exists F differentiable such that $F' = f$ on $[a, b]$, then
> $$\int_a^b f(t)\, dt = \int_a^b F'(t)\, dt = F(b) - F(a)."$$

- But these require the right hypotheses. We are given f as data. What hypotheses on f will make the first statement above true? What hypotheses on f make the second statement above true?
- That is what we study in this section.

Hypotheses of the first fundamental theorem of calculus

1. "Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\,dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$."

- This statement would at least make sense if f were merely in $\mathscr{R}[a, b]$.
- Actually, is that a true statement? It presupposes that if f is Riemann integrable on an interval $[a, b]$, then it is also Riemann integrable on any subinterval $[c, d]$ of $[a, b]$. Actually that is a true statement.

Exercise.

Let $f \in \mathscr{R}[a, b]$. Let $[c, d]$ be a subinterval of $[a, b]$. Prove that $f \in \mathscr{R}[c, d]$.

- Follows from the Partition Characterization of Integrability theorem.

Hypotheses of the first fundamental theorem of calculus

1. "Define $F : [a,b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\,dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a,b)$."

- Is it true that if f is merely Riemann integrable on $[a,b]$, then the above function F is differentiable and $F' = f$ on $[a,b]$?
- Not according to the following counterexample.

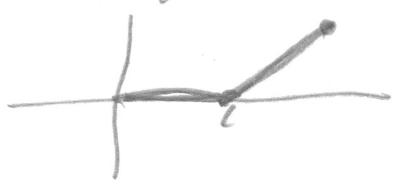
Exercise.

Let $f(t) := \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } 1 < t \le 2 \end{cases}$. Let $F(x) := \int_a^x f(t)\,dt$.

a) Calculate F.

b) Is F differentiable on $[0,2]$?

(a) $F(x) = \begin{cases} 0 & 0 \le t \le 1 \\ x-1 & 1 \le t \le 2. \end{cases}$



(b) No, F is not differentiable at $t=1$.

Theorem (First Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(t)\, dt$. Then F is differentiable and $F'(x) = f(x)$ for all $x \in (a, b)$.

Exercise.

Give the proof. Do it by proving directly from the definition of continuity of f that for each $x \in (a, b)$ we have

$$\lim_{h \to 0} \left| \frac{\int_x^{x+h} f(t)\, dt}{h} - f(x) \right| = 0.$$

- Note: We'll show on a homework assignment that the assumption of continuity on f is not necessary. That is, we can find f not continuous, yet for F as in the statement, F' exists for all x, and $F' = f$ for all x.

Theorem (First FTC) Let $f:[a,b]\to\mathbb{R}$ be continuous. Define $F:[a,b]\to\mathbb{R}$ by

$$F(x)=\int_a^x f(t)\,dt.$$

Then F is differentiable on (a,b) and $F'=f$.

Proof Since $f\in C[a,b]$, then $f\in\mathcal{R}[a,x]$ for any $x\in[a,b]$, so $F(x)$ is defined.

Fix $x_0\in(a,b)$. Since f is continuous at x_0, there exists $\delta>0$ such that for all t, if $|t-x_0|<\delta$, then $|f(t)-f(x_0)|<\varepsilon$.

Then for any h with $0<h<\delta$ we have

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right| = \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)\right|$$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h}\big(f(t)-f(x_0)\big)\,dt\right|$$

$$\le \frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|\,dt$$

$$< \frac{1}{h}\cdot\varepsilon\cdot h=\varepsilon.$$

This shows

$$\text{①}\quad \lim_{h\to 0^+}\frac{F(x_0+h)-F(x_0)}{h}=f(x_0).$$

If $-\delta<h<0$, then

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right| = \left|\frac{1}{h}\int_{x_0}^{x_0+h}\big(f(t)-f(x_0)\big)\,dt\right|$$

$$= \frac{1}{|h|} \left| - \int_{x_0+h}^{h} (f(t) - f(x_0))\,dt \right|$$

$$\leq \frac{1}{|h|} \int_{x_0+h}^{x_0} |f(t) - f(x_0)|\,dt$$

$$< \frac{1}{|h|} \cdot \varepsilon(-h) = \varepsilon.$$

This shows

$$② \lim_{h \to 0^-} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0).$$

By ① and ②, $F'(x_0) = f(x_0)$.

Hypotheses of the second fundamental theorem of calculus

- Let's now discuss the other of the fundamental theorems of calculus. It's something like:

> "If there exists F differentiable such that $F' = f$ on $[a, b]$, then
> $$\int_a^b f(t)\,dt = \int_a^b F'(t)\,dt = F(b) - F(a)"$$

- So what about the correct hypotheses on f?

Exercise.

In the above statement, let's say we merely assume that f is a function for which there exists F such that F is differentiable and $F' = f$. What else would we have to implicitly be asserting in order for the conclusion to have a chance to be true?

- We show next that this necessary hypothesis is also sufficient.

Theorem (Second Fundamental Theorem of Calculus)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose there exists $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$. Then $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$.

Exercise.

Write the proof of the second fundamental theorem of calculus.

Theorem (2nd FTC) Let $f:[a,b]\to\mathbb{R}$ be Riemann Integrable. Suppose there exists $F:[a,b]\to\mathbb{R}$ such that $F'(x)=f(x)$ for all $x\in[a,b]$. Then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Proof Let $\varepsilon>0$. By the Partition Characterization of Integrability Theorem, there exists a partition P of $[a,b]$ such that

$$\mathcal{U}(f,P)-L(f,P)<\varepsilon.$$

Of course

$$① \; L(f,P)\le \int_a^b f(t)\,dt \le \mathcal{U}(f,P).$$

Applying the Mean Value Theorem to F on each $[x_{i-1},x_i]$, there exists $c_i\in(x_{i-1},x_i)$ such that $F'(c_i)=\frac{F(x_i)-F(x_{i-1})}{\Delta_i}$, so $f(c_i)\Delta_i = F(x_i)-F(x_{i-1})$. Thus

$$② \; \sum_{i=0}^{n-1} f(c_i)\Delta_i = \sum_{i=0}^{n-1}\big(F(x_i)-F(x_{i-1})\big) = F(b)-F(a)$$

since the second sum in ② telescopes. But

$$L(f,P)\le \sum_{i=0}^{n-1} f(c_i)\Delta_i \le \mathcal{U}(f,P)$$

so using ②, we obtain

$$③ \; L(f,P)\le F(b)-F(a)\le \mathcal{U}(f,P).$$

By ① and ③ we get

$$\left| \int_a^b f(t)\,dt - (F(b)-F(a)) \right| \leq U(f,P) - L(f,P) < \varepsilon.$$

Since ε is arbitrary, it follows $\int_a^b f(t)\,dt = F(b)-F(a)$.

$\square$