Some Applications of these ideas

- So if we have $f \in C[a, b]$ such that f is differentiable on (a, b) and we calculate that $f'(x) < 0$ for all $x \in (a, b)$, then f must be strictly decreasing, and that implies that for all such x,

$$f(a) > f(x) > f(b).$$

Exercises

Use the derivative to prove each of the following.

a) $\sin x < x$ for all $x > 0$
Hint: Let $f(x) = \sin x - x$, and calculate $f'(x)$.

b) $\cos x > 1 - \frac{x^2}{2}$ for all real x.
(Hint: Don't forget the results you've just finished proving. Let $f(x) = \cos x - 1 + x^2/2$ for $x > 0$; calculate $f'(x)$ and deduce the result for $x > 0$. Now explain how to extend to $\mathbb{R}$.)

(a) Let's apply the Mean Value Thm to $f(x) = \sin x$ on $[0, x]$. Then there exists $c \in (0, x)$ such that

$$\frac{\sin x - \sin 0}{x - 0} = \cos c, \quad \text{so} \quad \sin x = (\cos c) \cdot x \leq x$$

If $0 < x < \frac{\pi}{2}$ then $\cos c < 1$, so $\sin x < x$. If $x \geq \frac{\pi}{2} > 1$, then $\sin x \leq 1 < x$. So in any case $\sin x < x$.

(b) Let apply MVT to $\cos x + \frac{x^2}{2}$ on $[0, x]$, then there exists $c \in (0, x)$ s.t. $\frac{\cos x + \frac{x^2}{2} - 1}{x - 0} = -\sin c + c$

By (a) $c - \sin c > 0$, so $\frac{\cos x + \frac{x^2}{2} - 1}{x} > 0$. Since $x > 0$, this gives $\cos x + \frac{x^2}{2} - 1 > 0$, i.e. $\cos x > 1 - \frac{x^2}{2}$

Note $\cos(-x) = \cos x$ and $(-x)^2 = x^2$; thus $\cos x \geq 1 - \frac{x^2}{2}$ for all x.

Some Applications of these ideas

Exercise.

Let $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}$. Suppose that f is differentiable and for some $M \in \mathbb{R}$ we have that $|f'(x)| \leq M$ for all $x \in U$. Prove that f is uniformly continuous.
(Hint: Let a, b be in the domain of f with $a < b$. Apply the Mean Value Theorem to f on this interval; using the hypothesis what are you able to deduce? And why does this now allow you to deduce f is uniformly continuous?)

Let $\varepsilon > 0$. Let $x_1, x_2 \in U$ with $|x_1 - x_2| < \frac{\varepsilon}{M}$.
By MVT there exists c between x_1 and x_2 such that $f(x_1) - f(x_2) = (x_1 - x_2) f'(c)$. Then

$|f(x_1) - f(x_2)| = |f'(c)|\,|x_1 - x_2| \leq M\delta = \varepsilon.$

Thus f is uniformly continuous.