- This section and the next give some of the results that are the best known to you from your calculus classes.
- If you make use of the results we have already proven and you use the hints provided, you should be able to come up with all of the proofs by yourself. It's a good exercise for you to do so.

Definition

(i) Let $D \subset \mathbb{R}$, and let $x_0 \in D$. We say that x_0 is an interior point of D provided there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq D$.

(ii) The interior of D, denoted by D^0, is the set of all interior points of D.

Exercise.

a) What is D^0 if $D = \mathbb{Z}$? How about if $D = [5, 17]$?

b) Explain why D^0 is an open set, and that it is the largest open set contained in D.

(a) $\mathring{\mathbb{Z}} = \emptyset$, $\mathring{[5,17]} = (5,17)$

(b) • Let $x_0 \in \mathring{D}$. Then $\exists \delta > 0$ s.t. $I_{x_0,\delta} \subseteq D$. Since $I_{x_0,\delta}$ is open, it follows each point of $I_{x_0,\delta}$ is in $\mathring{D}$. Thus D_0 is open.

• If G is any open subset of D, then by definition of "open", each point of G lies in $\mathring{D}$. Thus $G \subseteq \mathring{D}$.

A basic result of this section

Theorem

Let $f : D \to \mathbb{R}$, and let x_0 be an interior point of D. If f is differentiable at x_0 and f has a local extreme point at x_0, then $f'(x_0) = 0$.

Some hints on the proof:

- Say x_0 is a local maximum point of f. If h is close enough to 0, what can you say about the sign of the quantity

$$f(x_0 + h) - f(x_0),$$

  whether or not h is positive or negative?
- What can you say about the sign of $\dfrac{f(x_0 + h) - f(x_0)}{h}$?
- What does this tell you about $f'(x_0)$?

Exercise.

Write the proof of the theorem.

- we're given that $f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$
- Then it's a matter of checking $\frac{f(x_0+h) - f(x_0)}{h} \leq 0$

for $h > 0$ and $\frac{f(x_0+h) - f(x_0)}{h} \geq 0$ for $h < 0$.
provided h is small enough (if we assume x_0 a point of local max).

Theorem (Local Extremum) Let $f: D \to \mathbb{R}$ and let x_0 be an interior point of. If f is differentiable at x_0 and f has a local extremum at x_0, then $f'(x_0) = 0$.

Proof We write the proof in case f has a local max. at x_0. The proof in case f has a local min. is similar.

Since f has a local max. at x_0, there exists $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $f(x) \leq f(x_0)$.

Let $h \in \mathbb{R}$ with $|h| < \delta$. If $h > 0$, then $f(x_0+h) - f(x_0) \leq 0$, so $\dfrac{f(x_0+h) - f(x_0)}{h} \leq 0$. Thus

$$① \; f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} \leq 0.$$

If $h < 0$, then we still have $f(x_0+h) - f(x_0) \leq 0$, but now $\dfrac{f(x_0+h) - f(x_0)}{h} \geq 0$. Thus

$$② \; f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} \geq 0.$$

From ① and ② we conclude $f'(x_0) = 0$, so we are done. $\square$

Theorem (Rolle's Theorem)

Let $f \in C[a,b]$. Assume also that f is differentiable on (a,b). If $f(a) = f(b) = 0$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Comments on Rolle's Theorem and its proof:

- Make a sketch which illustrates what the theorem says.
- Don't forget what we have just finished proving on the previous slide.
- Why may we assume that $f \not\equiv 0$ on $[a,b]$?
- Why may we assume that f has either a maximum or a minimum x_0 on (a,b)?
- What can we say about what happens at x_0? Why?

Exercise.

Write the proof of Rolle's Theorem.

- In previous theorem we showed that at a local extreme point, x_0, if f is differentiable at x_0 then $f'(x_0) = 0$.
- So you need to argue that either f is constant or f has a local extreme point.

Theorem (Rolle's Theorem) Let $f \in C[a,b]$. Assume also that f is differentiable on (a,b). If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof Since f is continuous on $[a,b]$, it follows from the Extreme Value Theorem that there exist $x_1, x_2 \in [a,b]$ such that f has a maximum value at x_1 and a minimum value at x_2. If both x_1 and x_2 occur at an endpoint of $[a,b]$, then since $f(a) = f(b)$ we deduce f is constant and so $f' \equiv 0$ on (a,b). In this case we can take c to be any point of (a,b).

Thus we may assume that either x_1 or x_2 lies in (a,b). It follows that f has a local extremum at a point of (a,b), and so by the Local Extremum Theorem f' is 0 at that point. Thus in this case we can take c to be the point of that local extremum. $\square$

Application of Rolle's Theorem: The Mean Value Theorem

Theorem (The Mean Value Theorem)

Let $f \in C[a, b]$. Assume also that f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Comments on the Mean Value Theorem and its proof:

- Make a sketch which illustrates what the theorem says.
- The hypotheses are similar to the hypotheses of Rolle's Theorem. Do you see how to reduce the proof to an application of Rolle's Theorem?
- The idea is to subtract off something from f so that the result would follow immediately from Rolle's Theorem. What to subtract off?
- Should be a straight line which agrees with f at a and b. What is the equation of that line?

Exercise.

Write the proof of the Mean Value Theorem.

Theorem (Mean Value Theorem) Let $f \in C[a,b]$. Assume also f is differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - \left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a)\right].$$

Then $g \in C[a,b]$, g is differentiable on (a,b), and $g(a) = 0 = g(b)$. Thus by Rolle's Theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. This says

$$0 = g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right),$$

which means $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\square$

Application of Rolle's Theorem and the Mean Value Theorem

Theorem

Let f be differentiable on an open interval $I = (a, b)$.

(i) If $f' \equiv 0$ on I, then f is constant.

(ii) If $f' \equiv g'$ on I, then f and g differ by a constant function.

(iii) If $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I.

(iv) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.

Exercise.

a) Write the proof of the above theorem. Each of them follow easily if the right results are used.

b) Prove that the converse of (iii) is true. Note that you don't need any powerful tools to prove this.

c) Show that the converse of (iv) is not true in general. Do so by supplying an appropriate counterexample.

(i) Let $a < x_1 < x_2 < b$. By the Mean Value Thm, there exists $c \in (x_1, x_2)$ such that $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$. Since $f'(c) = 0$, it follows $f(x_1) = f(x_2)$. Thus f is constant.

(ii) Apply (i) to $f - g$. Then $(f-g)' = f' - g' \equiv 0$

So $f - g$ is constant by (i)

(iii) Let $a < x_1 < x_2 < b$. By MVT, $\exists c \in (x_1, x_2)$ s.t. $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0$. Thus $f(x_2) \geq f(x_1)$.

So f is increasing.