Theorem (Basic Properties of Differentiability)

Let f, g be functions with common domain D, let x_0 be a cluster point of D. Let c be a fixed real number. Suppose that f and g are differentiable at x_0.

c) (Product Rule) Then the product $f \cdot g$ is differentiable at x_0 and

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0).$$

Exercise.

Write the proof using the linear transformation characterization of differentiability.

Theorem (Basic Properties of Differentiability)

Let f, g be functions with common domain D, let x_0 be a cluster point of D. Let c be a fixed real number. Suppose that f and g are differentiable at x_0.

d) (Reciprocal Rule) If in addition $g(x_0) \neq 0$, then $1/g$ is differentiable at x_0 and

$$(1/g)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}.$$

Exercise.

Write the proof using the linear transformation characterization of differentiability.

Future Homework.

Theorem (Basic Properties of Differentiability)

Let f, g be functions with common domain D, let x_0 be a cluster point of D. Let c be a fixed real number. Suppose that f and g are differentiable at x_0.

e) (Quotient Rule) If in addition $g(x_0) \neq 0$, then f/g is differentiable at x_0 and

$$(f/g)'(x_0) = \frac{f'(x_0)g(_0) - g'(x_0)f(x_0)}{(g(x_0))^2}.$$

Exercise.

Write the proof using the ~~linear transformation~~ characterization of differentiability.

This result follows by combining the product rule with the reciprocal rule:

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' \quad \text{(product rule)}$$

$$= f'\cdot\frac{1}{g} + f\cdot\left(\frac{-g'}{g^2}\right) \quad \text{(reciprocal rule)}$$

$$= \frac{f'g - g'f}{g^2}. \quad \square$$

We continue with applications of the Linear Characterization of Differentiability Theorem by proving the following theorem.

Theorem (Chain Rule)

Let g be a function with domain D_g and f a function whose domain is contained in the range of g. Let x_0 be a cluster point of D_g and $g(x_0)$ a cluster point of D_f. If g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composition $f \circ g$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

Exercise.

Write the proof.

Theorem (Chain Rule) Let g be a function with domain D_g and f a function whose domain D_f is contained in the range of g. Let x_0 be a cluster point of D_g and $g(x_0)$ a cluster point of D_f. If g is differentiable at x_0 and f is differentiable at $g(x_0)$, then $f\circ g$ is differentiable at x_0 and

$$(f\circ g)'(x_0) = f'(g(x_0))\,g'(x_0).$$

Proof Since f is differentiable at $g(x_0)$, there exists a function ε_f such that

$$① \qquad f(g(x_0)+\tilde h) = f(g(x_0)) + f'(g(x_0))\tilde h + \varepsilon_f(\tilde h)\,\tilde h,$$

where $\varepsilon_f(\tilde h)\to 0$ as $\tilde h \to 0$.

Since g is differentiable at x_0, there exists a function ε_g such that

$$② \qquad g(x_0+h) = g(x_0) + g'(x_0)h + \varepsilon_g(h)\cdot h,$$

where $\varepsilon_g(h)\to 0$ as $h\to 0$.

Let $\tilde h := g'(x_0)h + \varepsilon_g(h)h$. Note that $\tilde h \to 0$ as $h\to 0$.

Using first ② and then ①, we get

$$\begin{aligned}
(f\circ g)(x_0+h) - (f\circ g)(x_0) &= f(g(x_0+h)) - f(g(x_0))\\
&= f\big(g(x_0) + \underbrace{g'(x_0)h + \varepsilon_g(h)h}_{\tilde h}\big) - f(g(x_0))\\
&= \cancel{f(g(x_0))} + f'(g(x_0))\tilde h + \varepsilon_f(\tilde h)\tilde h - \cancel{f(g(x_0))}\\
&= f'(g(x_0))\big(g'(x_0)h + \varepsilon_g(h)h\big) + \varepsilon_f(\tilde h)\big(g'(x_0)h + \varepsilon_g(h)h\big)
\end{aligned}$$

$$= f'(g(x_0))g'(x_0)h + h\left[f'(g(x_0))\,\varepsilon_g(h) + \varepsilon_f(\tilde{h})g'(x_0) + \varepsilon_f(\tilde{h})\,\varepsilon_g(h)\right].$$

The result follows from the fact that the circled quantity goes to 0 as $h \to 0$. $\square$